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New formalism for two-photon quantum optics. I. Quadrature phases and squeezed states

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This paper introduces a new formalism for analyzing two-photon devices (e.g., parametric amplifiers and phase-conjugate mirrors), in which photons in the output modes are created or destroyed two at a time. The key property of a two-photon device is that it excites pairs of output modes independently. Thus our new formalism deals with two modes at a time; a continuum multimode description can be built by integrating over independently excited pairs of modes. For a pair of modes at frequencies $\Omega\pm\epsilon$, we define (i) *quadrature-phase amplitudes,* which are complex-amplitude operators for modulation at frequency ϵ of waves "$\cos[\Omega(t-x/c)]$" and "$\sin[\Omega(t-x/c)]$" and (ii) *two-mode squeezed states,* which are the output states of an ideal two-photon device. The quadrature-phase amplitudes and the two-mode squeezed states serve as the building blocks for our formalism; their properties and their physical interpretation are extensively investigated.

I. INTRODUCTION AND OVERVIEW

In this and the accompanying paper we introduce a new formalism for analyzing a particular class of nonlinear optical devices—devices that we call two-photon devices. The light produced by any optical system is an excitation of various modes of the electromagnetic field; the defining feature of a two-photon device is that its output light is generated by the simultaneous emission of two photons into two of the output modes. Examples of two-photon devices include parametric amplifiers, where the simultaneously excited output modes are called the signal and the idler, and phase-conjugate mirrors (four-wave mixers), where the output modes are the transmitted and reflected waves.

Two-photon devices can produce, in principle, special states of the electromagnetic field called squeezed states[1] or two-photon coherent states.[2] Squeezed states[3,4] have manifestly nonclassical properties; they might find application in low-noise optical communications[5–7] and in high-precision interferometric measurements.[1,8,9] Experiments to generate squeezed states and to investigate their properties are now underway in several laboratories.[10–12]

Two-photon devices are to be contrasted with one-photon devices, such as the laser, in which photons are emitted into the output modes one at a time. The analytical tools of quantum optics were developed to describe and analyze one-photon processes; thus they are designed to analyze situations in which the modes of the electromagnetic field are excited independently. These tools are, in general, not adequate for analyzing two-photon devices, because a two-photon device excites modes in pairs, instead of singly. This series of papers develops a new set of analytical tools, which are suited to the description and analysis of two-photon devices. A brief, preliminary account of our work can be found in Ref. 13.

To motivate our approach, we start by reviewing briefly the formalism of one-photon optics. This review is heuristic, with emphasis on the features that tailor the formalism to the description of one-photon processes; in particular, we treat the electromagnetic field classically, ignoring its quantum-mechanical commutation relations. Consider a beam of light produced by a one-photon device, and idealize the beam as a plane wave with a particular linear polarization. The electric field can be written as the sum of positive- and negative-frequency parts:

$$E(x,t)=E^{(+)}(x,t)+E^{(-)}(x,t)\ , \tag{1.1}$$

where

$$\begin{aligned} E^{(+)}(x,t) &\equiv \int_{\mathscr{F}} \frac{d\omega}{2\pi} E(\omega) e^{-i\omega(t-x/c)}\ , \\ E^{(-)} &= (E^{(+)})^*\ . \end{aligned} \tag{1.2}$$

Here $E(\omega)$ is the complex amplitude of the plane-wave mode at (positive) frequency ω, and the integration runs over the bandwidth $\mathscr{F}$ of interest. That the photons in the beam are created one at a time means that the fluctuations in the electric field are due to random emission of single photons which have various frequencies and phases. As a result, the fluctuations at different frequencies are independent, and the fluctuations at each frequency are distributed randomly in phase. The mathematical embodiment of these two statements is

$$\langle \Delta E(\omega)\Delta E(\omega')\rangle=0\ , \tag{1.3a}$$

$$\langle \Delta E(\omega)\Delta E^*(\omega')\rangle=\frac{b}{2c}\mathscr{S}(\omega)2\pi\delta(\omega-\omega')\ , \tag{1.3b}$$

where $\Delta E(\omega)\equiv E(\omega)-\langle E(\omega)\rangle$, $\mathscr{S}(\omega)$ is the flux spectral density of the electric field fluctuations (dimensions of energy per area), and b is a units-dependent constant (e.g.,

$b \equiv 4\pi$ in cgs Gaussian units). In Eqs. (1.3) and throughout this first section, brackets denote a classical statistical average. The noise produced by a one-photon device is conveniently characterized by a single function of frequency $\mathscr{S}(\omega)$, derived from the second moments of the complex amplitudes. Equivalent to Eqs. (1.3) is the more compact statement that the electric field has *time-stationary* (TS) *noise*; i.e., the variance of the electric field is constant:

$$\frac{c}{b}\langle[\Delta E(x,t)]^2\rangle = \int_{\mathscr{I}} \frac{d\omega}{2\pi}\mathscr{S}(\omega) \tag{1.4}$$

$[\Delta E(x,t) \equiv E(x,t) - \langle E(x,t)\rangle]$.

Implicit in this discussion of TS noise is the assumption, made throughout this paper, that the noise is Gaussian, so that second moments are sufficient to characterize it. An important consequence of Gaussian TS noise, which does not hold for TS noise in general, is that the modes at different frequencies are statistically independent [Eqs. (1.3)]. The restriction to Gaussian noise will be lifted in a future paper (paper III of this series), where the relations among Gaussian noise, TS noise, and statistically independent modes will be considered.

The key property of a one-photon device is that its output consists of independently excited modes with TS noise. In terms of constructing a formalism, this property has two crucial consequences, which can be thought of as the cornerstones of one-photon optics: (i) one can deal with one plane-wave mode at a time, building a continuum multimode description by integrating over independently excited single modes; (ii) the natural variable to characterize the excitation of each mode is its complex amplitude $E(\omega)$.

One is now in a position to identify the fundamental "building blocks" of one-photon optics. Specialize to a single mode at frequency ω. The natural quantum-mechanical operator for the mode is its annihilation operator

$$a(\omega) \equiv (2cA_q/b\hbar\omega)^{1/2}E(\omega) , \tag{1.5}$$

which is just the mode's complex amplitude rewritten in "units" of square root of the number of quanta per root Hz. (A_q is an appropriate "quantization area" transverse to the propagation direction.) The natural quantum states for the mode are the coherent states[14]—the states generated from the vacuum by an ideal one-photon process (e.g., a classical current distribution radiating into the vacuum). The coherent states are eigenstates of the annihilation operator; thus they have the sharpest complex amplitude permitted by quantum mechanics. The formalism of one-photon optics is founded firmly on the annihilation operator as the fundamental operator and on the coherent states as the fundamental quantum states.

Real one-photon devices do not exhibit ideal behavior. Describing their nonideal behavior requires consideration of the complicated interaction of the light with atomic systems and of the effects of losses and their associated fluctuations. One approach to analyzing the light produced by a real one-photon device is to derive an equation for the evolution of the reduced density operator (quantum state) of the electromagnetic field. This equation, which is called the master equation, is generally a complicated operator equation not directly amenable to analysis. A powerful technique for rendering the master equation more tractable is to convert it into an equivalent c-number partial differential equation—a Fokker-Planck equation—for the evolution of a quasiprobability distribution (QPD). A QPD is a rigorous and complete representation of a density operator (i.e., it contains all the quantum statistics associated with the density operator), but it retains the appearance and some of the interpretation of a classical probability distribution.

The definition and interpretation of the QPD's used in one-photon optics ("one-photon QPD's") are intimately related to the use of the annihilation operator and the coherent states as the fundamental building blocks.[15–17] More than one QPD is associated with a given quantum state, each QPD corresponding to a different way of ordering the creation and annihilation operators. For a single mode of the electromagnetic field, each one-photon QPD is a function of a complex number μ, which is a c-number analog of the mode's annihilation operator. The expectation value of a suitably ordered product of creation and annihilation operators is calculated using the appropriate QPD as though it were a classical probability distribution. The one-photon QPD's are powerful tools for analyzing real one-photon devices, but based as they are on the annihilation operator and the coherent states, they are tools designed specifically for one-photon processes and are not necessarily suited to the analysis of two-photon devices. For example, one of the most useful and most used one-photon QPD's is the Glauber-Sudarshan P function,[18,19,14] which reproduces the normally ordered statistics of a and $a^\dagger$; this QPD does not exist as a well-behaved distribution for the squeezed states that can be produced by two-photon devices.[20]

Our philosophy has been that a new task requires new tools. The first step is to identify new operators and new quantum states, which are suited to the description of two-photon processes; this task is carried out exhaustively in papers I and II of this series. The second step is to use these operators and states to define "two-photon QPD's" that can be used to analyze real two-photon devices; this task will be tackled in paper III.

To simplify the introduction of our formalism, consider as an example a parametric amplifier, the prototype for all two-photon devices. In a paramp an intense laser beam at frequency 2Ω—the pump beam—illuminates a suitable nonlinear medium. The nonlinearity couples the pump beam to other modes of the electromagnetic field in such a way that a pump photon at frequency 2Ω can be annihilated to create "signal" and "idler" photons at frequencies $\Omega \pm \epsilon$ and, conversely, signal and idler photons can be annihilated to create a pump photon. Thus the light produced by a paramp consists of pairs of simultaneously emitted photons which excite pairs of modes at frequencies $\Omega \pm \epsilon$. In general, the modes in each pair have correlated complex amplitudes [i.e., $\langle \Delta E(\Omega+\epsilon)\Delta E(\Omega-\epsilon)\rangle \neq 0$; cf. Eq. (1.3a)]. This fact tells one immediately that the formalism of one-photon optics must be abandoned; the correlations produced by two-photon processes cannot be described in terms of independently excited single modes.

The electric field at the output of a paramp has the same form as Eq. (1.1); the difference lies in the correlation between the modes in each pair. It is useful to rewrite the field by factoring out the time dependence at frequency Ω. Define (real) quadrature phases $E_1(x,t)$ and $E_2(x,t)$ by

$$E^{(\pm)} \equiv \tfrac{1}{2}(E_1 \pm iE_2)e^{\mp i\Omega(t-x/c)} ; \tag{1.6}$$

E_1+iE_2 is the complex amplitude of the electric field, defined with respect to the carrier frequency Ω. In terms of the quadrature phases, the electric field is given by

$$E(x,t)=E_1(x,t)\cos[\Omega(t-x/c)] + E_2(x,t)\sin[\Omega(t-x/c)] ; \tag{1.7}$$

thus, E_1 and E_2 describe modulation of waves "$\cos[\Omega(t-x/c)]$" and "$\sin[\Omega(t-x/c)]$." The quadrature phases can be written in terms of their Fourier components:

$$E_m(x,t)= \int_{\mathscr{R}} \frac{d\epsilon}{2\pi}[E_m(\epsilon)e^{-i\epsilon(t-x/c)} + E_m^*(\epsilon)e^{i\epsilon(t-x/c)}], \quad m=1,2 . \tag{1.8}$$

Here the integral runs over a suitable set $\mathscr{R}$ of (positive) modulation frequencies ϵ, and

$$E_1(\epsilon)=E(\Omega+\epsilon)+E^*(\Omega-\epsilon) , \tag{1.9a}$$

$$E_2(\epsilon)=-iE(\Omega+\epsilon)+iE^*(\Omega-\epsilon) . \tag{1.9b}$$

The Fourier component $E_1(\epsilon)$ $[E_2(\epsilon)]$ is a complex amplitude for modulation at frequency ϵ of a wave $\cos[\Omega(t-x/c)]$ $(\sin[\Omega(t-x/c)])$. Now consider the emission of a pair of photons at frequencies $\Omega\pm\epsilon$. The conventional view is that these photons excite a pair of modes that are sidebands of the carrier frequency Ω; an equally good alternative view is that they excite directly a modulation at frequency ϵ of a wave at frequency Ω. Roughly speaking, if the phases of the two photons are such that $E(\Omega+\epsilon)=E^*(\Omega-\epsilon)$, then they excite $E_1(\epsilon)$; if the phases are such that $E(\Omega+\epsilon)=-E^*(\Omega-\epsilon)$, then they excite $E_2(\epsilon)$. Our message is that two-photon optics should be formulated in a different language from one-photon optics. In one-photon optics attention focuses on the electric field $E(x,t)$ and its Fourier components $E(\omega)$; emission of a photon excites a mode at a particular frequency. In two-photon optics attention shifts to the quadrature phases $E_1(x,t)$ and $E_2(x,t)$ and their Fourier components $E_1(\epsilon)$ and $E_2(\epsilon)$; emission of a pair of photons excites one of the quadrature phases at a particular modulation frequency.

With this new language in hand, the discussion of natural variables for two-photon optics is just a translation of the preceding review of one-photon optics. The fluctuations in the quadrature phases are due to random emission of pairs of photons, which excite the quadrature phases at various modulation frequencies with various phases [phase in this context is the phase of the (complex) Fourier component $E_1(\epsilon)$ or $E_2(\epsilon)$]. As a result, the fluctuations at different modulation frequencies are independent, and the fluctuations at each modulation frequency are distributed randomly in phase. This means that the quadrature phases have time-stationary noise—a kind of noise that we call *time-stationary quadrature-phase* (TSQP) *noise*.[13,21] For Gaussian noise the conditions for TSQP noise are

$$\langle \Delta E_m(\epsilon)\Delta E_n(\epsilon')\rangle = 0 , \tag{1.10a}$$

$$\langle \Delta E_m(\epsilon)\Delta E_n^*(\epsilon')\rangle = \frac{b}{c}\mathscr{S}_{mn}(\epsilon)2\pi\delta(\epsilon-\epsilon') , \tag{1.10b}$$

where $m,n=1,2$, $\Delta E_m(\epsilon)\equiv E_m(\epsilon)-\langle E_m(\epsilon)\rangle$, and $\mathscr{S}_{mn}(\epsilon)=\mathscr{S}_{nm}^*(\epsilon)$ is the flux spectral-density matrix for the quadrature-phase fluctuations [dimensions of energy per area; cf. Eqs. (1.3)]. Equivalent to Eqs. (1.10) is the time independence of the covariance matrix of the quadrature phases:

$$\frac{c}{2b}\langle \Delta E_m(x,t)\Delta E_n(x,t)\rangle = \int_{\mathscr{R}} \frac{d\epsilon}{2\pi}\mathrm{Re}[\mathscr{S}_{mn}(\epsilon)] \tag{1.11}$$

[$m,n=1,2$; $\Delta E_m(x,t)\equiv E_m(x,t)-\langle E_m(x,t)\rangle$; "Re" denotes "the real part of"]. Unlike TS noise, TSQP noise allows the quadratures to carry different amounts of noise ($\mathscr{S}_{11}\neq\mathscr{S}_{22}$), and it allows them to have a nonvanishing time-stationary correlation [$\mathrm{Re}(\mathscr{S}_{12})\neq 0$]. This means that the variance of the electric field is not, in general, constant:

$$\frac{c}{b}\langle[\Delta E(x,t)]^2\rangle = \int_{\mathscr{R}} \frac{d\epsilon}{2\pi}\{\mathscr{S}_{11}+\mathscr{S}_{22}+(\mathscr{S}_{11}-\mathscr{S}_{22})\cos[2\Omega(t-x/c)] + 2\mathrm{Re}(\mathscr{S}_{12})\sin[2\Omega(t-x/c)]\} \tag{1.12}$$

[cf. Eq. (1.4)]. Equations (1.11) and (1.12) can be interpreted as saying that the fluctuations in the electric field are not distributed randomly in phase, where phase is here defined relative to frequency Ω.

The key property of a two-photon device is that its output consists of independently excited pairs of modes with TSQP noise. This property is the reason that two-photon optics is formulated more conveniently in terms of the quadrature phases and their Fourier components than in terms of the electric field and its Fourier components. The consequences of this property, and the cornerstones of two-photon optics, are the following: (i) one can deal with one pair of modes, i.e., one modulation frequency, at a time, building a continuum multimode description by integrating over independently excited pairs of modes; (ii) the natural variables for each pair of modes are the Fourier components $E_1(\epsilon)$ and $E_2(\epsilon)$.

We can now identify the fundamental building blocks for two-photon optics. Specialize to a pair of modes at frequencies $\Omega\pm\epsilon$. The natural quantum-mechanical operators for the modes are the quadrature-phase amplitudes $a_1(\epsilon)$ and $a_2(\epsilon)$,[13,21] defined by

$$\alpha_1(\epsilon) \equiv \left[\frac{cA_q}{b\hbar\Omega} \right]^{1/2} E_1(\epsilon) = \left[\frac{\Omega+\epsilon}{2\Omega} \right]^{1/2} a(\Omega+\epsilon) + \left[\frac{\Omega-\epsilon}{2\Omega} \right]^{1/2} a^\dagger(\Omega-\epsilon) , \tag{1.13a}$$

$$\alpha_2(\epsilon) \equiv \left[\frac{cA_q}{b\hbar\Omega} \right]^{1/2} E_2(\epsilon) = -i \left[\frac{\Omega+\epsilon}{2\Omega} \right]^{1/2} a(\Omega+\epsilon) + i \left[\frac{\Omega-\epsilon}{2\Omega} \right]^{1/2} a^\dagger(\Omega-\epsilon) \tag{1.13b}$$

[Eqs. (1.5) and (1.9)]. The quadrature-phase amplitudes are simply rescaled versions of $E_1(\epsilon)$ and $E_2(\epsilon)$—rescaled to be in units of square root of the number of quanta, referred to the carrier frequency Ω, per root Hz. The natural quantum states are the two-mode squeezed states[8,13,21]—the states generated from (two-mode) coherent states by an ideal two-photon device (e.g., an ideal paramp, with undepleted classical pump and no losses). The two-mode squeezed states have TSQP noise, and they have, in general, unequal amounts of noise in the two quadratures ($\mathscr{S}_{11} \neq \mathscr{S}_{22}$). The present paper (paper I) focuses on the properties and the significance of the quadrature-phase amplitudes and the two-mode squeezed states; the goal is to achieve a good physical understanding of these fundamental entities. The accompanying paper (paper II) develops a mathematical formalism suited to manipulating the quadrature-phase amplitudes and the two-mode squeezed states, and it uses the formalism to write their properties in a compact form. With its emphasis on physical interpretation, this first paper omits many mathematical details, which are filled in by paper II.

These building blocks of two-photon optics have been used to construct new two-photon quasiprobability distributions.[13] More than one two-photon QPD is associated with a given (two-mode) quantum state, each QPD corresponding to a different way of ordering the quadrature-phase amplitudes and their Hermitian conjugates. Since the two-photon QPD's are written in a language tailored to the description of two-photon processes, we think they will be valuable tools for analyzing nonideal behavior of two-photon devices. A future paper (paper III) will describe in detail the new operator orderings and the two-photon QPD's.

In this paper Sec. II deals with a couple of minor notational issues; Sec. III reviews briefly the building blocks of one-photon optics; Sec. IV introduces the quadrature-phase amplitudes and the two-mode squeezed states, with emphasis on the physical significance of the quadrature-phase amplitudes; Sec. V considers in detail TSQP noise for the case of Gaussian noise; Sec. VI discusses uncertainty principles for the quadrature-phase amplitudes; Sec. VII lists important properties of the two-mode squeezed states; finally, Sec. VIII specializes our work to the previously explored degenerate limit ($\epsilon=0$). An appendix treats uncertainty principles for non-Hermitian operators. Throughout the remainder of this paper we use units with $\hbar=c=1$.

II. NOTATIONAL ISSUES

For convenience we have adopted a notation that sometimes sacrifices precision for ease in use. To minimize confusion that might arise from our preference for convenience, we consider here a couple of notational issues.

Throughout our discussion of two-photon optics, we find that each physical quantity is most conveniently represented by its operator in a particular picture. For example, the creation and annihilation operators are most conveniently written in the Schrödinger picture (SP); field quantities, such as the electric field and the quadrature phases, are most conveniently written in the usual interaction picture (IP), in which all the free time dependence is incorporated in the operators; and the quadrature-phase amplitudes are most conveniently written in an interaction picture that we call the modulation picture (MP), which we define and discuss in Sec. IV. As a result, we have acquired the habit of mixing in the same equation various operators written in different pictures. This habit has the potential to cause confusion, which we seek to avoid by adhering strictly to the following procedure. For each physical quantity, the corresponding operators in different pictures are denoted differently. As each physical quantity is introduced in Secs. III and IV, we define its operator in a particular picture by a picture-consistent equation, i.e., an equation in which all operators are written in the same picture. The operators corresponding to the same physical quantity in other pictures are then defined as they are needed. The appropriate picture for a picture-consistent equation is indicated by writing SP, MP, or IP in parentheses next to the equation; of course, a picture-consistent equation retains the same form when all operators are transformed to another picture.

As an illustration of this procedure, consider a plane electromagnetic wave with a particular linear polarization, which propagates in the x direction. In the SP the creation and annihilation operators for the plane-wave mode at frequency ω are denoted by $a^\dagger(\omega)$ and $a(\omega)$; they satisfy the continuum commutation relations

$$[a(\omega), a^\dagger(\omega')] = 2\pi\delta(\omega-\omega') . \tag{2.1}$$

The electric field operator in the SP is given by

$$E(x) \equiv E^{(+)}(x) + E^{(-)}(x) \quad \text{(SP)} , \tag{2.2a}$$

$$E^{(+)}(x) \equiv \int_0^\infty \frac{d\omega}{2\pi} (b\omega/2A_q)^{1/2} a(\omega) e^{i\omega x} = [E^{(-)}(x)]^\dagger \quad \text{(SP)} , \tag{2.2b}$$

where $E^{(+)}(x)$ and $E^{(-)}(x)$ are the SP positive- and negative-frequency parts of the field, A_q is a suitable quantization area, and b is the units-dependent constant introduced in Eqs. (1.3). In the IP the electric field operator is given by

$$E(x,t) \equiv e^{iH_C t} E(x) e^{-iH_C t} \equiv E^{(+)}(x,t) + E^{(-)}(x,t) \ , \quad (2.3a)$$

$$\begin{aligned} E^{(+)}(x,t) &\equiv e^{iH_C t} E^{(+)}(x) e^{-iH_C t} \\ &= \int_0^\infty \frac{d\omega}{2\pi} (b\omega/2A_q)^{1/2} a(\omega) e^{-i\omega(t-x)} \\ &= [E^{(-)}(x,t)]^\dagger \end{aligned} \quad (2.3b)$$

[cf. Eqs. (1.1), (1.2), and (1.5)], where

$$H_C \equiv \int_0^\infty \frac{d\omega}{2\pi} \omega a^\dagger(\omega) a(\omega) \quad (2.4)$$

is the free Hamiltonian for the continuum of modes, and where we use the fact that the IP form of the annihilation operator for a particular mode has the harmonic time dependence of the mode, i.e., $e^{iH_C t} a(\omega) e^{-iH_C t} = a(\omega) e^{-i\omega t}$.

A second notational issue concerns the way we use the symbol Δ. In general, we use Δ to designate the difference between a quantity and its mean value. Thus, for a quantum-mechanical operator R, ΔR is defined to be the operator

$$\Delta R \equiv R - \langle R \rangle \ . \quad (2.5)$$

For a Hermitian operator B this notation allows the variance (squared uncertainty) of B to be written as $\langle (\Delta B)^2 \rangle$ ($\equiv \langle B^2 \rangle - \langle B \rangle^2$); we always write the variance in this form. For a general, possibly non-Hermitian operator R, a fundamental quantity in our analysis is the mean-square uncertainty in R, by which we mean the sum of the variances of the Hermitian real and imaginary parts of R [$\mathrm{Re}(R) \equiv \frac{1}{2}(R + R^\dagger)$; $\mathrm{Im}(R) \equiv -\frac{1}{2} i (R - R^\dagger)$]. To define and write the mean-square uncertainty compactly, we use three shorthand notations: (i) for two operators R and S, the subscript "sym" denotes a symmetrically ordered product, i.e.,

$$(RS)_{\mathrm{sym}} \equiv \tfrac{1}{2}(RS + SR) \ ; \quad (2.6)$$

(ii) the expectation value of a symmetrically ordered product is written

$$\langle (RS)_{\mathrm{sym}} \rangle \equiv \langle RS \rangle_{\mathrm{sym}} \ ; \quad (2.7)$$

(iii) $|\Delta R|^2$ denotes the operator

$$|\Delta R|^2 \equiv (\Delta R \, \Delta R^\dagger)_{\mathrm{sym}} = \tfrac{1}{2}(\Delta R \, \Delta R^\dagger + \Delta R^\dagger \Delta R) \ . \quad (2.8)$$

These shorthands allow us to write the mean-square uncertainty as

$$\langle |\Delta R|^2 \rangle = \langle \Delta R \, \Delta R^\dagger \rangle_{\mathrm{sym}} = \langle R R^\dagger \rangle_{\mathrm{sym}} - |\langle R \rangle|^2 \ . \quad (2.9)$$

For a Hermitian operator the mean-square uncertainty is the variance; our notation is consistent because $|\Delta B|^2 = (\Delta B)^2$ if $B = B^\dagger$.

III. REVIEW OF ONE-PHOTON OPTICS

We turn now to a brief review of one-photon optics, briefer even than the review in Sec. I, but rigorous quantum-mechanically. Consider the light produced by a one-photon device such as a laser. As is discussed in Sec. I, one can specialize to a single (discrete) plane-wave mode with frequency ω; a continuum multimode description is built by integrating over independently excited single modes. The mode's creation and annihilation operators in the SP are denoted by $a^\dagger$ and a, which satisfy the usual (discrete) commutation relation

$$[a, a^\dagger] = 1 \ . \quad (3.1)$$

We introduce an "electric field operator" for the mode, which is denoted in the SP by

$$E(x) \equiv E^{(+)}(x) + E^{(-)}(x) \quad (\mathrm{SP}) \ , \quad (3.2a)$$

$$E^{(+)}(x) \equiv (\omega/2)^{1/2} a e^{i\omega x} = [E^{(-)}(x)]^\dagger \quad (\mathrm{SP}) \quad (3.2b)$$

[cf. Eqs. (2.2)]. In the IP the single-mode electric field operator becomes

$$E(x,t) \equiv e^{iH_S t} E(x) e^{-iH_S t} = E^{(+)}(x,t) + E^{(-)}(x,t) \ , \quad (3.3a)$$

$$E^{(+)}(x,t) = (\omega/2)^{1/2} a e^{-i\omega(t-x)} = [E^{(-)}(x,t)]^\dagger \quad (3.3b)$$

[cf. Eqs. (2.3)], where

$$H_S \equiv \omega a^\dagger a \quad (\mathrm{SP}) \quad (3.4)$$

is the free Hamiltonian for a single mode.

Our motivation for introducing the single-mode electric field operators of Eqs. (3.2) and (3.3) is that we want to be able to calculate the statistics of fieldlike quantities associated with a single (discrete) plane-wave mode. The normalization of the electric field for a single plane-wave mode is somewhat arbitrary, so we have simply made a convenient choice that leaves our results uncluttered by irrelevant constants. The $\omega^{1/2}$ in Eqs. (3.2b) and (3.3b) is the obligatory factor of root frequency that accompanies the annihilation operator [cf. Eqs. (2.2b) and (2.3b)]; it gives the single-mode electric field units of square root of energy. The $2^{-1/2}$ in Eqs. (3.2b) and (3.3b) is chosen for convenience.

The natural states for describing the output of a one-photon device can be identified by considering the Hamiltonian for an ideal one-photon process:

$$H = H_S - i g^*(t) a e^{i\omega t} + i g(t) a^\dagger e^{-i\omega t} \quad (\mathrm{SP}) \ . \quad (3.5)$$

Here $g(t) \equiv g$ is an arbitrary complex function of time. The interaction part of this Hamiltonian creates and destroys photons one at a time; the process is ideal because it is characterized by a c-number function $g(t)e^{-i\omega t}$, which can be regarded as a classical generalized force acting on the mode. The Hamiltonian (3.5) describes a classical current distribution radiating into the mode of interest.[14,16] The SP unitary evolution operator $U(t,0)$ corresponding to the Hamiltonian (3.5) is[16,22]

$$\begin{aligned} U(t,0) &= e^{-ih(t)} e^{-iH_S t} D(a,\gamma) \\ &= e^{-ih(t)} D(a, \gamma e^{-i\omega t}) e^{-iH_S t} \ , \end{aligned} \quad (3.6a)$$

$$\gamma \equiv \gamma(t) \equiv \int_0^t g(t') dt' \ , \quad (3.6b)$$

$$h(t) \equiv \tfrac{1}{2} i \int_0^t (\gamma^* g - \gamma g^*) dt' \ . \quad (3.6c)$$

In Eq. (3.6a)

$$D(a,\mu) \equiv \exp(\mu a^\dagger - \mu^* a) \quad (3.7)$$

is the (unitary) *single-mode displacement operator,*[14] so named because of the important property[14]

$$D^\dagger(a,\mu)aD(a,\mu)=a+\mu \ . \tag{3.8}$$

The natural states for one-photon optics are those generated from the vacuum state $|0\rangle$ by an ideal one-photon process. These states, which are called (single-mode) *coherent states,*[14] are defined by

$$|\mu\rangle_{\rm coh}\equiv D(a,\mu)|0\rangle \ . \tag{3.9}$$

A coherent state is an eigenstate of the annihilation operator with complex eigenvalue μ:

$$a|\mu\rangle_{\rm coh}=\mu|\mu\rangle_{\rm coh} \tag{3.10}$$

[Eq. (3.8)]. The coherent states lie at the very core of one-photon optics; their properties have been extensively investigated.[14,16]

The natural variable for one-photon optics is the annihilation operator a, which is simply a complex-amplitude operator for the mode, written in units of square root of the number of quanta. The reason the annihilation operator is natural is that the states of interest in one-photon optics have *time-stationary* (TS) *noise.* To see what TS noise means, let the initial state of the mode be the density operator ρ. The noise associated with an arbitrary state ρ is completely characterized by the "noise moments" of a and $a^\dagger$, where by noise moments we mean moments of $\Delta a\equiv a-\langle a\rangle$ and $\Delta a^\dagger$ [Eq. (2.5)]. In this paper we consider only the lowest-order noise, which is described by the second-order noise moments

$$\langle(\Delta a)^2\rangle\equiv{\rm tr}[\rho(\Delta a)^2]=\langle a^2\rangle-\langle a\rangle^2 \ , \tag{3.11a}$$

$$\begin{aligned}\langle|\Delta a|^2\rangle&\equiv{\rm tr}[\rho(\Delta a\Delta a^\dagger)_{\rm sym}]\\&=\langle aa^\dagger\rangle_{\rm sym}-|\langle a\rangle|^2 \ .\end{aligned} \tag{3.11b}$$

The state ρ has (second-moment) TS noise if

$$\langle(\Delta a)^2\rangle=0 \tag{3.12}$$

[cf. Eqs. (1.3a) and (1.5)]; hence, for TS noise the lowest-order noise is described completely by the mean-square uncertainty $\langle|\Delta a|^2\rangle$ [Eq. (2.9); cf. Eqs. (1.3b) and (1.5)]. The physical content of Eq. (3.12) is that the noise in the single mode is distributed randomly in phase; thus TS noise can be characterized as random-phase noise or phase-insensitive noise. An immediate consequence of Eq. (3.12) is that the electric field has TS noise; i.e., if the mode undergoes free evolution (Hamiltonian H_S), the variance of the electric field (3.3a) is constant:

$$\langle[\Delta E(x,t)]^2\rangle=\omega\langle|\Delta a|^2\rangle \tag{3.13}$$

[cf. Eq. (1.4)].

It is useful to emphasize here why the annihilation operator is the natural variable for describing TS noise. Under free evolution (evolution operator e^{-iH_St}), the noise moment $\langle(\Delta a)^2\rangle$ acquires a harmonic time dependence $e^{-2i\omega t}$, whereas the mean-square uncertainty $\langle|\Delta a|^2\rangle$ remains constant. The essence of TS noise is that the time-dependent noise moment $\langle(\Delta a)^2\rangle$ vanishes, so that the lowest-order noise is described by the time-independent moment $\langle|\Delta a|^2\rangle$. These considerations are the key to generalizing the notion of TS noise to noise moments of arbitrarily high order. The definition (3.12) considers only the lowest-order noise moments, the justification being an implicit assumption of Gaussian noise. The general definition of TS noise, which will be given explicitly and discussed in paper III, requires that all the time-dependent noise moments of a and $a^\dagger$ vanish, so that the noise is completely characterized by the time-independent noise moments. *This, then, is the reason the annihilation operator is the natural variable for one-photon optics: the TS noise produced by one-photon devices is completely characterized by the time-independent noise moments of a and $a^\dagger$.*

The commutator $[a,a^\dagger]=1$ enforces an uncertainty principle,

$$\langle|\Delta a|^2\rangle\geq\tfrac{1}{2}|\langle[a,a^\dagger]\rangle|=\tfrac{1}{2} \ . \tag{3.14}$$

[This and other uncertainty principles for non-Hermitian operators are derived and discussed in the Appendix; see Eq. (A9).] The lower limit in Eq. (3.14) is the half-quantum of zero-point noise. A coherent state $|\mu\rangle_{\rm coh}$ has mean complex amplitude $\langle a\rangle=\mu$ and has TS noise with $\langle|\Delta a|^2\rangle=\frac{1}{2}$; it can be thought of as a classical excitation of the mode contaminated by zero-point noise.

The fundamental building blocks for one-photon optics are the annihilation operator and the coherent states. Although the coherent states arise from a consideration of ideal one-photon devices, they and the annihilation operator have been used to define quasiprobability distributions,[15–17,20] which are powerful tools for analyzing the nonideal behavior of real one-photon devices. Quasiprobability distributions will be considered in detail in a future paper (paper III).

IV. BUILDING BLOCKS OF TWO-PHOTON OPTICS

Attention shifts now to a discussion of the natural variables and natural quantum states for two-photon optics. As is made clear in Sec. I, one can analyze the light produced by a two-photon device by specializing to a pair of (discrete) plane-wave modes with frequencies $\Omega\pm\epsilon$, where Ω is a carrier frequency and $\epsilon<\Omega$ is a modulation frequency; a continuum multimode description is built by integrating over independently excited pairs of modes (i.e., integrating over ϵ). In optical applications it is always true that $\epsilon\ll\Omega$. The annihilation operators for the two modes in the SP are denoted by a_+ and a_-; they satisfy the usual (discrete) commutation relations

$$[a_+,a_-]=[a_+,a_-^\dagger]=0 \ , \tag{4.1a}$$

$$[a_+,a_+^\dagger]=[a_-,a_-^\dagger]=1 \ . \tag{4.1b}$$

The free Hamiltonian for the two modes is given by

$$\begin{aligned}H_0&=(\Omega+\epsilon)a_+^\dagger a_++(\Omega-\epsilon)a_-^\dagger a_-\\&=H_R+H_M \quad ({\rm SP}) \ ,\end{aligned} \tag{4.2a}$$

$$H_R\equiv\Omega(a_+^\dagger a_++a_-^\dagger a_-) \quad ({\rm SP}) \ , \tag{4.2b}$$

$$H_M\equiv\epsilon(a_+^\dagger a_+-a_-^\dagger a_-) \quad ({\rm SP}) \ . \tag{4.2c}$$

We find it useful to split H_0 into two commuting pieces, H_R and H_M ($[H_R,H_M]=0$), which are the key to defining the pictures we use in our new formalism. In the usual *interaction picture* (IP), all the free time dependence is transferred from the states to the operators; the relation between operators (including density operators) in the IP and the SP is

$$R_{\rm IP}(t)\equiv e^{iH_0t}R_{\rm SP}(t)e^{-iH_0t}\ . \tag{4.3}$$

The *modulation picture*[13] (MP) is an interaction picture in which the free time dependence at the carrier frequency Ω is transferred from the states to the operators, the states retaining the remaining free time dependence at modulation frequency ϵ; operators in all three pictures are related by

$$R_{\rm MP}(t)\equiv e^{iH_Rt}R_{\rm SP}(t)e^{-iH_Rt}=e^{-iH_Mt}R_{\rm IP}(t)e^{iH_Mt}\ . \tag{4.4}$$

There is no reason why the two modes we consider need be plane-wave modes with the same polarization propagating in the same direction. Nonetheless, we assume they are so that we can introduce a "two-mode electric field operator," which in the IP is given by

$$E(x,t)\equiv E^{(+)}(x,t)+E^{(-)}(x,t)\quad(\text{IP})\ , \tag{4.5a}$$

$$E^{(+)}(x,t)=2^{-1/2}[(\Omega+\epsilon)^{1/2}a_+e^{-i(\Omega+\epsilon)(t-x)}+(\Omega-\epsilon)^{1/2}a_-e^{-i(\Omega-\epsilon)(t-x)}]\ , \tag{4.5b}$$

$$E^{(-)}(x,t)=[E^{(+)}(x,t)]^\dagger \tag{4.5c}$$

[cf. Eqs. (3.3) and subsequent discussion].

A. Two-mode squeezed states

Consider now the Hamiltonian for an ideal two-photon process:

$$H=H_0+i\kappa(t)[a_+a_-e^{-2i(\varphi-\Omega t)}-a_+^\dagger a_-^\dagger e^{2i(\varphi-\Omega t)}]\quad(\text{SP})\ . \tag{4.6}$$

Here $\kappa(t)$ is an arbitrary real function of time. The interaction part of this Hamiltonian creates or destroys a pair of photons in the two modes simultaneously; the process is ideal because it is characterized by a c-number function $\kappa(t)e^{2i(\varphi-\Omega t)}$. For convenience we choose this function to have a harmonic time dependence at frequency 2Ω, with fixed phase but time-varying amplitude. The Hamiltonian (4.6) describes, for example, an ideal parametric amplifier[23–26] with an undepleted classical pump, which has stable frequency 2Ω but whose amplitude varies in time. The unitary evolution operator for the Hamiltonian (4.6) is given by[26,22]

$$U(t,0)=e^{-iH_0t}S(\zeta,\varphi)=S(\zeta,\varphi-\Omega t)e^{-iH_0t}\ , \tag{4.7}$$

$$\zeta\equiv\zeta(t)\equiv\int_0^t\kappa(t')dt'\ , \tag{4.8}$$

where

$$S(r,\varphi)\equiv\exp[r(a_+a_-e^{-2i\varphi}-a_+^\dagger a_-^\dagger e^{2i\varphi})] \tag{4.9}$$

is the (unitary) *two-mode squeeze operator*.[13,21] The real number r is called the *squeeze factor*. The most important property of the two-mode squeeze operator is that

$$S(r,\varphi)a_\pm S^\dagger(r,\varphi)=a_\pm\cosh r+a_\mp^\dagger e^{2i\varphi}\sinh r\ , \tag{4.10}$$

a result which follows from Eq. (8.105) of Ref. 27.

To construct the natural states for two-photon optics, one begins with the *two-mode coherent states*[14]

$$|\mu_+,\mu_-\rangle_{\rm coh}\equiv D(a_+,\mu_+)D(a_-,\mu_-)|0\rangle \tag{4.11}$$

[cf. Eq. (3.9)], which are eigenstates of a_+ and a_- with eigenvalues μ_+ and μ_-, respectively. Formally, a two-mode coherent state is obtained by applying the *two-mode displacement operator*[14]

$$D(a_+,\mu_+)D(a_-,\mu_-)=\exp(\mu_+a_+^\dagger-\mu_+^*a_++\mu_-a_-^\dagger-\mu_-^*a_-) \tag{4.12}$$

to the vacuum state [cf. Eq. (3.7)]; physically, it could be created from the vacuum by an ideal one-photon process for each of the two modes. The natural states for two-photon optics are those generated from two-mode coherent states by the ideal two-photon process (4.6). Before defining these states, it is useful to define operators that we call *squeezed annihilation operators*. In the SP these operators have explicit time dependence and are defined by

$$\begin{aligned}\alpha_\pm(r,\varphi;t)&\equiv e^{-iH_Rt}S(r,\varphi)a_\pm S^\dagger(r,\varphi)e^{iH_Rt}\\&=a_\pm e^{i\Omega t}\cosh r+a_\mp^\dagger e^{-i\Omega t}e^{2i\varphi}\sinh r\quad(\text{SP})\end{aligned} \tag{4.13}$$

[Eq. (4.10)]; in the MP [Eq. (4.4)] the squeezed annihilation operators are constant and are given by

$$\begin{aligned}\alpha_\pm(r,\varphi)&\equiv\alpha_\pm(r,\varphi;0)=S(r,\varphi)a_\pm S^\dagger(r,\varphi)\\&=a_\pm\cosh r+a_\mp^\dagger e^{2i\varphi}\sinh r\ .\end{aligned} \tag{4.14}$$

The natural states for two-photon optics are the *two-mode squeezed states*,[8,13,21] which are defined by

$$\begin{aligned}|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r,\varphi)}&\equiv S(r,\varphi)|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{\rm coh}\\&=S(r,\varphi)D(a_+,\mu_{\alpha_+})D(a_-,\mu_{\alpha_-})|0\rangle\ .\end{aligned} \tag{4.15}$$

We label these states by the complex eigenvalues of $\alpha_\pm(r,\varphi)$:

$$\alpha_\pm(r,\varphi)|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r,\varphi)}=\mu_{\alpha_\pm}|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r,\varphi)} \tag{4.16}$$

[Eqs. (4.14) and (3.10)]. Using Eq. (4.10), one can write the two-mode squeezed states in the form

$$|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r,\varphi)}=D(a_+,\mu_+)D(a_-,\mu_-)S(r,\varphi)|0\rangle\ , \tag{4.17}$$

where

$$\mu_{\alpha_\pm}=\mu_\pm\cosh r+\mu_\mp^*e^{2i\varphi}\sinh r\ . \tag{4.18}$$

Two-mode squeezed states were introduced independently by Caves[21] in an analysis of quantum limits on the performance of linear amplifiers (see also Ref. 13) and by Unruh[8] in a quantum-mechanical analysis of an interferometer; they have also been considered formally by Barut and Girardello,[28] Perelomov,[29] and Milburn.[30] Properties of two-mode squeezed states are considered in Sec. VII.

Almost all previous work on squeezed states has dealt with the degenerate limit, in which the two modes we consider coalesce into one ($\epsilon=0$, $a_+=a_-$). Our attitude is that the degenerate limit is not very important in describing real two-photon devices, because it is merely the $\epsilon=0$ boundary for a more realistic and more general multimode description. The degenerate limit can, however, play a useful heuristic role, so we consider it in some detail in Sec. VIII.

B. Quadrature-phase amplitudes

It is useful to decompose the electric field into its (Hermitian) *quadrature phases* defined with respect to the carrier frequency Ω.[13,21] In the IP the quadrature phases are defined by

$$E_1(x,t)\equiv E^{(+)}(x,t)e^{i\Omega(t-x)}+E^{(-)}(x,t)e^{-i\Omega(t-x)} \quad \text{(IP)}\,, \tag{4.19a}$$

$$E_2(x,t)\equiv -iE^{(+}(x,t)e^{i\Omega(t-x)}+iE^{(-)}(x,t)e^{-i\Omega(t-z)} \quad \text{(IP)}\,, \tag{4.19b}$$

$$E^{(\pm)}(x,t)=\tfrac{1}{2}[E_1(x,t)\pm iE_2(x,t)]e^{\mp i\Omega(t-x)} \quad \text{(IP)} \tag{4.20}$$

[cf. Eq. (1.6)]. In terms of the quadrature phases the IP electric field operator (4.5a) becomes

$$E(x,t)=E_1(x,t)\cos[\Omega(t-x)] +E_2(x,t)\sin[\Omega(t-x)] \quad \text{(IP)} \tag{4.21}$$

[cf. Eq. (1.7)]; thus $E_1(x,t)$ and $E_2(x,t)$ describe modulation of waves "$\cos[\Omega(t-x)]$" and "$\sin[\Omega(t-x)]$." The quadrature phases (4.19) or their multimode analogs [Eq. (1.8)] have been used in multimode analyses of optical homodyning,[7] resonance fluorescence,[31–33] parametric amplification,[34–36] and four-wave mixing.[34]

For two modes the concept of (second-moment) TS noise means that each mode has (second-moment) TS noise [Eq. (3.12)] and that the two modes have zero second-order correlation [cf. Eqs. (1.3)]; these conditions imply that the electric field has constant variance. One says that, for TS noise, the noise in the electric field is distributed randomly in phase, where phase is defined relative to frequency Ω; equivalently, one can say that TS noise means that the noise in the electric field is divided equally between the quadrature phases.

A two-mode squeezed state does not, in general, have TS noise. The two modes have correlated noise, and the quadrature phases carry different amounts of noise. Thus, in two-photon optics it is convenient to describe the noise in terms of the quadrature phases. In particular, the natural variables are the (two-mode) *quadrature-phase amplitudes*,[13,21] which are simply the Fourier components of the quadrature phases, normalized to be in units of square root of the number of quanta referred to the carrier frequency Ω. In the SP the quadrature-phase amplitudes are explicitly time-dependent operators defined by

$$\alpha_1(t)\equiv\left[\frac{\Omega+\epsilon}{2\Omega}\right]^{1/2}a_+e^{i\Omega t} +\left[\frac{\Omega-\epsilon}{2\Omega}\right]^{1/2}a_-^\dagger e^{-i\Omega t} \quad \text{(SP)}\,, \tag{4.22a}$$

$$\alpha_2(t)\equiv -i\left[\frac{\Omega+\epsilon}{2\Omega}\right]^{1/2}a_+e^{i\Omega t} +i\left[\frac{\Omega-\epsilon}{2\Omega}\right]^{1/2}a_-^\dagger e^{-i\Omega t} \quad \text{(SP)}\,. \tag{4.22b}$$

Notice that the quadrature-phase amplitudes are not Hermitian. In the MP the quadrature-phase amplitudes are constant and are denoted by

$$\alpha_1\equiv e^{iH_Rt}\alpha_1(t)e^{-iH_Rt}=\alpha_1(0) =\left[\frac{\Omega+\epsilon}{2\Omega}\right]^{1/2}a_+ +\left[\frac{\Omega-\epsilon}{2\Omega}\right]^{1/2}a_-^\dagger\,, \tag{4.23a}$$

$$\alpha_2\equiv e^{iH_Rt}\alpha_2(t)e^{-iH_Rt}=\alpha_2(0) =-i\left[\frac{\Omega+\epsilon}{2\Omega}\right]^{1/2}a_+ +i\left[\frac{\Omega-\epsilon}{2\Omega}\right]^{1/2}a_-^\dagger \tag{4.23b}$$

[cf. Eqs. (1.13)]. We find it convenient to introduce the symbols

$$\lambda_\pm\equiv[(\Omega\pm\epsilon)/\Omega]^{1/2}\,, \tag{4.24}$$

so that Eqs. (4.23) and their inverse can be written in the compact forms

$$\alpha_1=2^{-1/2}(\lambda_+a_++\lambda_-a_-^\dagger)\,, \tag{4.25a}$$

$$\alpha_2=2^{-1/2}(-i\lambda_+a_++i\lambda_-a_-^\dagger)\,, \tag{4.25b}$$

$$\lambda_+a_+=2^{-1/2}(\alpha_1+i\alpha_2)\,, \tag{4.26a}$$

$$\lambda_-a_-=2^{-1/2}(\alpha_1^\dagger+i\alpha_2^\dagger)\,. \tag{4.26b}$$

In the IP the quadrature-phase amplitudes acquire a harmonic time dependence at the modulation frequency:

$$e^{iH_Mt}\alpha_me^{-iH_Mt}=e^{iH_0t}\alpha_m(t)e^{-iH_0t}=\alpha_me^{-i\epsilon t}\,, \quad m=1,2\,. \tag{4.27}$$

Using Eqs. (4.5), (4.19), and (4.26), one can write the quadrature phases in the form

$$E_m(x,t)=\Omega^{1/2}[\alpha_me^{-i\epsilon(t-x)}+\alpha_m^\dagger e^{i\epsilon(t-x)}]\,, \quad m=1,2 \tag{4.28}$$

which shows explicitly that α_m is a complex-amplitude operator at modulation frequency ϵ for $E_m(x,t)$ [i.e., it is the Fourier component of $E_m(x,t)$ at positive frequency ϵ]. In our notation the MP is the most convenient picture for writing a picture-consistent equation relating the quadrature phases to their amplitudes; the MP quadrature phases are denoted by

$$E_m(x)\equiv e^{-iH_Mt}E_m(x,t)e^{iH_Mt}, \quad m=1,2 \tag{4.29}$$

so that

$$E_m(x)=\Omega^{1/2}(\alpha_m e^{i\epsilon x}+\alpha_m^\dagger e^{-i\epsilon x}),\quad m=1,2 \quad (\mathrm{MP})\ . \tag{4.30}$$

The two-mode quadrature-phase amplitudes have the following (discrete) commutator algebra:

$$[\alpha_1,\alpha_1^\dagger]=[\alpha_2,\alpha_2^\dagger]=\epsilon/\Omega\ , \tag{4.31a}$$

$$[\alpha_1,\alpha_2]=0\ , \tag{4.31b}$$

$$[\alpha_1,\alpha_2^\dagger]=[\alpha_1^\dagger,\alpha_2]=i\ . \tag{4.31c}$$

These commutators enforce a set of uncertainty principles which we discuss in detail in Sec. VI.

All of our two-mode results thus far can easily be extended to a continuum description by using "continuum" quadrature-phase amplitudes and integrating over the positive modulation frequencies of interest [cf. Eqs. (1.8) and (1.13)]. The MP continuum quadrature-phase amplitudes[21] $\alpha_1(\epsilon)$ and $\alpha_2(\epsilon)$ are related to the continuum creation and annihilation operators [Eq. (2.1)] by Eqs. (1.13) [cf. Eqs. (4.23)]; they obey the commutation relations

$$[\alpha_1(\epsilon),\alpha_1(\epsilon')]=[\alpha_1(\epsilon),\alpha_2(\epsilon')]=[\alpha_2(\epsilon),\alpha_2(\epsilon')]=0\ , \tag{4.32a}$$

$$[\alpha_1(\epsilon),\alpha_1^\dagger(\epsilon')]=[\alpha_2(\epsilon),\alpha_2^\dagger(\epsilon')]=\frac{\epsilon}{\Omega}2\pi\delta(\epsilon-\epsilon')\ , \tag{4.32b}$$

$$[\alpha_1(\epsilon),\alpha_2^\dagger(\epsilon')]=[\alpha_1^\dagger(\epsilon),\alpha_2(\epsilon')]=i2\pi\delta(\epsilon-\epsilon')\ . \tag{4.32c}$$

The fundamental building blocks of two-photon optics are the quadrature-phase amplitudes and the two-mode squeezed states. In paper III these building blocks will be used to define new two-photon quasiprobability distributions.

C. Pictorial convention

As is made clear by Eqs. (4.19)–(4.21), $E_1(x,t)+iE_2(x,t)$ is a complex-amplitude operator for the two-mode electric field, defined with respect to frequency Ω. The choice of phase for this complex amplitude is arbitrary, so one can ask what happens under a change of phase. The unitary operator

$$R(\theta)\equiv\exp[-i\theta(a_+^\dagger a_+ + a_-^\dagger a_-)] \tag{4.33}$$

generates just such a phase change, i.e.,

$$\begin{aligned}R^\dagger(\theta)[E_1(x,t)+iE_2(x,t)]R(\theta)&\equiv E_1'(x,t)+iE_2'(x,t)\\&=[E_1(x,t)+iE_2(x,t)]e^{-i\theta}\ .\end{aligned} \tag{4.34}$$

We call $R(\theta)$ the *rotation operator* because the transformation (4.34) is a rotation of the complex amplitude. This rotation corresponds to a common phase change for the annihilation operators,

$$R^\dagger(\theta)a_\pm R(\theta)\equiv a_\pm'=a_\pm e^{-i\theta}\ , \tag{4.35}$$

and to a rotation of the quadrature-phase amplitudes

$$R^\dagger(\theta)\alpha_1 R(\theta)\equiv\alpha_1'=\alpha_1\cos\theta+\alpha_2\sin\theta\ , \tag{4.36a}$$

$$R^\dagger(\theta)\alpha_2 R(\theta)\equiv\alpha_2'=-\alpha_1\sin\theta+\alpha_2\cos\theta\ . \tag{4.36b}$$

Notice that $e^{-iH_R t}=R(\Omega t)$ [Eqs. (4.2b) and (4.33)]; thus the time dependence at the carrier frequency is simply a rotation of the complex amplitude.

One is now in a position to appreciate the importance of the MP. In two-photon optics one deals with the quadrature phases and their amplitudes as the fundamental quantities. The time dependence at frequency Ω is trivial and uninteresting; the important free time dependence is at the modulation frequency. One would like to formulate the theory in such a way that the trivial time dependence at Ω is suppressed. This goal is achieved in two steps: (i) one works in the MP, thereby transferring the time dependence at Ω from the states to the operators; (ii) one defines the fundamental operators—the quadrature phases and their amplitudes—so that they are constant in the MP. The second step requires defining the quadrature phases and the quadrature-phase amplitudes with explicit time dependences in the SP [Eqs. (4.19) and (4.22)], which then disappear in the MP [Eqs. (4.23) and (4.30)]. The effect of the above two steps is to transform frequency Ω to zero frequency, thereby removing it from the problem. In two-photon optics the MP in essence replaces the SP: in the MP the states carry the important time dependence, and the fundamental operators are constant.

With these remarks in mind, we introduce a set of conventions that we adhere to throughout the remainder of this paper and subsequent papers in this series. The creation and annihilation operators are always written in the SP (operators $a_\pm^\dagger,a_\pm$); expectation values of the creation and annihilation operators are evaluated using the SP density operator $\rho_{\mathrm{SP}}(t)$. The electric field and the quadrature phases are always written in the IP [operators $E(x,t)$, $E^{(\pm)}(x,t)$, $E_1(x,t)$, and $E_2(x,t)$; Eqs. (4.5) and (4.19)]; expectation values of these field quantities are evaluated using the IP density operator $\rho_{\mathrm{IP}}(t)$. Finally, the quadrature-phase amplitudes and the squeezed annihilation operators are always written in the MP [operators α_1, α_2, and $\alpha_\pm(r,\varphi)$; Eqs. (4.23) and (4.14)]; expectation values of these quantities are evaluated using the MP density operator $\rho_{\mathrm{MP}}(t)$. The *MP free evolution operator* we dignify by a special notation,

$$U_M(t)\equiv e^{-iH_M t}=\exp[-i\epsilon t(a_+^\dagger a_+ - a_-^\dagger a_-)] \tag{4.37}$$

[Eq. (4.2c)], because of the importance of the MP in our formalism. In the SP the free evolution operator is $e^{-iH_0 t}=R(\Omega t)U_M(t)$, and in the IP the free evolution operator is the identity operator.

D. Physical significance of quadrature-phase amplitudes

Throughout this subsection we are interested in expectation values of field quantities (the electric field and the quadrature phases) which are undergoing free evolution. Thus, in accordance with the conventions just described in Sec. IV C, all expectation values are evaluated with respect to the initial ($t=0$) state.

We turn now to a detailed discussion of the meaning of the quadrature-phase amplitudes. To understand their close connection to experiment, it is useful first to look closely at how the expectation values of α_1 and α_2 determine the classical behavior of the electromagnetic field.

By classical behavior we mean simply the free time evolution of the expectation value of the electric field at a particular point in space, say $x=0$. Equivalent information, but with the rapid time dependence at frequency Ω removed, is contained in the expectation value of the field's complex amplitude:

$$\langle E(0,t)\rangle = \mathrm{Re}[\langle E_1(0,t)+iE_2(0,t)\rangle e^{-i\Omega t}] \ . \tag{4.38}$$

For present purposes it is more convenient to deal with a dimensionless complex amplitude, which is defined in the IP by

$$\begin{aligned}\mathscr{E}_1(x,t)+i\mathscr{E}_2(x,t) &\equiv (2\Omega)^{-1/2}[E_1(x,t)+iE_2(x,t)] \\ &= (2/\Omega)^{1/2}E^{(+)}(x,t)e^{i\Omega(t-x)} \ .\end{aligned} \tag{4.39}$$

This dimensionless complex amplitude is related to the annihilation operators by

$$\begin{aligned}\mathscr{E}_1(x,t)+i\mathscr{E}_2(x,t) = \lambda_+ a_+ e^{-i\epsilon(t-x)} \\ +\lambda_- a_- e^{i\epsilon(t-x)}\end{aligned} \tag{4.40}$$

[Eqs. (4.5b) and (4.24)], and its components, dimensionless (Hermitian) quadrature phases, can be written as

$$\begin{aligned}\mathscr{E}_m(x,t) &\equiv (2\Omega)^{-1/2}E_m(x,t) \\ &= 2^{-1/2}(\alpha_m e^{-i\epsilon(t-x)}+\alpha_m^\dagger e^{i\epsilon(t-x)}) \ , \\ &\qquad m=1,2\end{aligned} \tag{4.41}$$

[Eq. (4.28)].

For the simple case of a two-frequency field, the classical behavior is specified by

$$\begin{aligned}\langle \mathscr{E}_1(0,t)+i\mathscr{E}_2(0,t)\rangle &= \lambda_+\langle a_+\rangle e^{-i\epsilon t}+\lambda_-\langle a_-\rangle e^{i\epsilon t} \\ &= \mathrm{Re}(2^{1/2}\langle \alpha_1\rangle e^{-i\epsilon t}) \\ &\quad + i\mathrm{Re}(2^{1/2}\langle \alpha_2\rangle e^{-i\epsilon t}) \ .\end{aligned} \tag{4.42}$$

Equation (4.42) says that the mean complex amplitude rotates about the origin, its tip tracing out an ellipse, the "signal ellipse," during each modulation period $2\pi/\epsilon$. The classical behavior of the field can be pictured on a complex-amplitude diagram (Fig. 1). On a complex-amplitude plane one draws the signal ellipse, indicates the initial ($t=0$) complex amplitude by a vector whose tip lies on the signal ellipse, and shows the direction of rotation of the complex amplitude by arrows on the signal ellipse. Four pieces of information are required to specify the classical behavior: the two radii and the orientation of the signal ellipse, and the direction of the initial complex amplitude. Notice that the phase change (4.34) corresponds to rotating the axes of the complex-amplitude plane counterclockwise by an angle θ. Notice also that in the degenerate limit ($\epsilon=0$, $a_+=a_-$) the mean complex amplitude never changes; the signal ellipse collapses to a single point, which is just the unchanging complex amplitude of a single mode.

Simple though the representation in Fig. 1 may be, it is instructive to decompose the elliptical motion of the complex amplitude into even simpler parts. The obvious decomposition is in terms of the two Fourier components of the field, i.e., in terms of the mean complex amplitudes

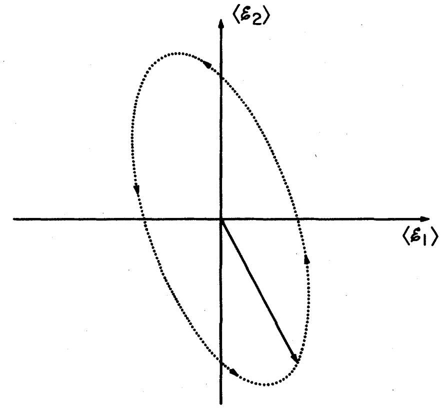

FIG. 1. Complex-amplitude diagram for the classical behavior of the electric field. The dotted ellipse is the signal ellipse traced out by the mean complex amplitude $\langle \mathscr{E}_1(0,t)+i\mathscr{E}_2(0,t)\rangle$ during each modulation period $2\pi/\epsilon$. Arrows on the signal ellipse show the direction of rotation of the complex amplitude. A vector indicates the initial ($t=0$) complex amplitude.

$\langle a_\pm\rangle$ of the two modes. In this decomposition [Eq. (4.42)], the mean complex amplitude is a sum of two vectors, $\lambda_+\langle a_+\rangle e^{-i\epsilon t}$, which rotates clockwise, and $\lambda_-\langle a_-\rangle e^{i\epsilon t}$, which rotates counterclockwise (see Fig. 2). The four classical pieces of information are given by the complex numbers $\langle a_+\rangle$ and $\langle a_-\rangle$, each of which specifies the (real) amplitude and phase of one of the modes.

The other useful decomposition is in terms of the quadrature-phase amplitudes:

$$\langle \mathscr{E}_m(0,t)\rangle = \mathrm{Re}(2^{1/2}\langle \alpha_m\rangle e^{-i\epsilon t}), \ \ m=1,2 \ . \tag{4.43}$$

In this decomposition the four required pieces of information are given by the complex numbers $\langle \alpha_1\rangle$ and $\langle \alpha_2\rangle$, each of which is a complex amplitude for one of the quadrature phases. To represent this decomposition graphically, one draws separate complex planes for the vectors $2^{1/2}\langle \alpha_1\rangle e^{-i\epsilon t}$ and $2^{1/2}\langle \alpha_2\rangle e^{-i\epsilon t}$. In each of these planes the vector $2^{1/2}\langle \alpha_m\rangle e^{-i\epsilon t}$ rotates clockwise, and its projection on the real axis gives $\langle \mathscr{E}_m(0,t)\rangle$ [Eq. (4.43); see Fig. 2]. These separate planes are phase planes for the quadrature phases; they show vividly how $\langle \alpha_m\rangle$ specifies the (real) amplitude and phase of $\langle \mathscr{E}_m(0,t)\rangle$.

Figure 2 shows, at four separate times, the complex-amplitude plane for $\langle \mathscr{E}_1(0,t)+i\mathscr{E}_2(0,t)\rangle$, together with the two decompositions discussed above. Such a diagram at any particular time (usually chosen to be $t=0$) contains all the information about the classical behavior of the field. In the next section we show how to include information about TSQP noise on such a diagram.

The physical significance of the quadrature-phase amplitudes can be demonstrated compellingly in two ways. The first is to consider their relation to amplitude and phase modulation. Superpose on the two-mode electric field (4.5) a strong, classical carrier wave at frequency Ω; let the carrier wave be given by $(2\Omega)^{1/2}B\cos[\Omega(t-x)]$, where B is real. The two modes at frequencies $\Omega\pm\epsilon$

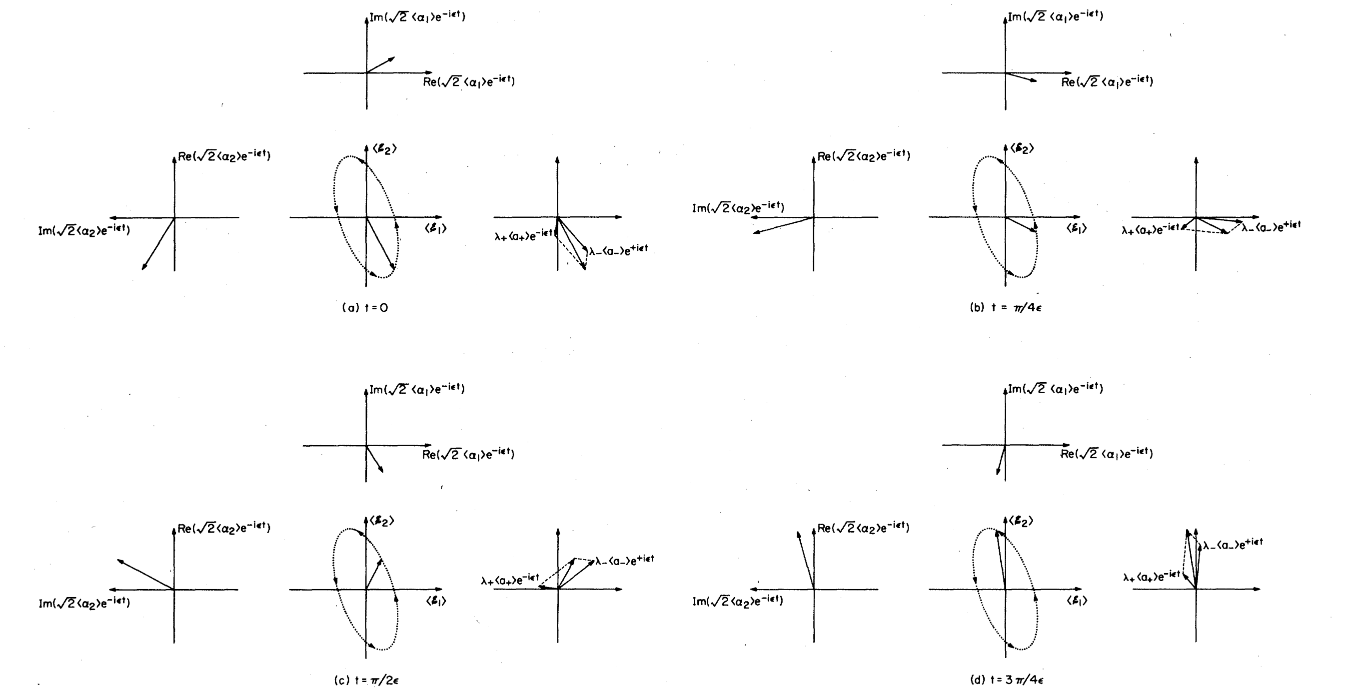

FIG. 2. Complex-amplitude diagrams at four times: (a) $t=0$, (b) $t=\pi/4\epsilon$, (c) $t=\pi/2\epsilon$, (d) $t=3\pi/4\epsilon$. At each time the central complex-amplitude diagram is the same as in Fig. 1, except that the vector indicates the mean complex amplitude at the appropriate time. To the right of the central diagram is a complex-amplitude plane which shows the decomposition of the mean complex amplitude into contributions from the two modes [Eq. (4.42)]. Above and to the left of the central diagram are phase planes for the quadrature phases. In the phase plane above the central diagram, a vector indicates the value of $2^{1/2}\langle\alpha_1\rangle e^{-i\epsilon t}$; its real part is $\langle\mathscr{E}_1(0,t)\rangle$ [Eq. (4.43)]. In the phase plane to the left a vector indicates the value of $2^{1/2}\langle\alpha_2\rangle e^{-i\epsilon t}$; its real part is $\langle\mathscr{E}_2(0,t)\rangle$.

represent sidebands of the carrier. The positive-frequency part of the total field is given by

$$\begin{aligned}\bar{E}^{(+)}(x,t)&=(\Omega/2)^{1/2}Be^{-i\Omega(t-x)}+E^{(+)}(x,t)\\&=\tfrac{1}{2}[(2\Omega)^{1/2}B+E_1(x,t)+iE_2(x,t)]e^{-i\Omega(t-x)}\end{aligned}\tag{4.44}$$

[Eq. (4.20)], corresponding to an electric field

$$\begin{aligned}\bar{E}(x,t)&=\bar{E}^{(+)}(x,t)+[\bar{E}^{(+)}(x,t)]^\dagger\\&=[(2\Omega)^{1/2}B+E_1(x,t)]\cos[\Omega(t-x)]\\&\quad+E_2(x,t)\sin[\Omega(t-x)]\ .\end{aligned}\tag{4.45}$$

In Eqs. (4.44) and (4.45) an overbar designates the total field, including both the carrier and the sidebands. Equation (4.45) shows that $E_1(x,t)$ modulates a wave that is in phase with the carrier—amplitude modulation of the carrier—and $E_2(x,t)$ modulates a wave that is 90° out of phase with the carrier—phase modulation of the carrier. Thus the quadrature-phase amplitudes are complex-amplitude operators for the amplitude and phase modulation. The expectation value of the total field's dimensionless complex amplitude is the sum of the constant amplitude B of the carrier and the modulated complex amplitude (4.42):

$$\begin{aligned}\langle\bar{\mathscr{E}}_1(0,t)+i\bar{\mathscr{E}}_2(0,t)\rangle&\equiv(2/\Omega)^{1/2}\langle\bar{E}^{(+)}(0,t)\rangle e^{i\Omega t}\\&=B+\langle\mathscr{E}_1(0,t)+i\mathscr{E}_2(0,t)\rangle\end{aligned}\tag{4.46}$$

[Eqs. (4.39) and (4.44)]. Thus the effect of the carrier on the complex-amplitude diagrams of Figs. 1 and 2 is to displace the signal ellipse a distance B along the real axis. The resulting complex-amplitude diagram makes clear that the oscillation of $\langle\mathscr{E}_1(0,t)\rangle$ is the amplitude-modulation signal and the oscillation of $\langle\mathscr{E}_2(0,t)\rangle$ is the phase-modulation signal. The separate planes for $2^{1/2}\langle\alpha_1\rangle e^{-i\epsilon t}$ and $2^{1/2}\langle\alpha_2\rangle e^{-i\epsilon t}$ are phase planes for the amplitude and phase modulation.

The second way to demonstrate the significance of the quadrature-phase amplitudes is to note their relation to ideal heterodyning. In heterodyne detection the two-mode field (4.21) is mixed with (multiplied by) a strong local-oscillator field at frequency Ω, and the result is filtered to pick out the Fourier component at frequency ϵ. If the local-oscillator field is proportional to $\cos[\Omega(t-x)]$ ($\sin[\Omega(t-x)]$) and if the mixing and filtering are ideal, then the output of the heterodyne detector is proportional to $E_1(x,t)$ [$E_2(x,t)$], and its complex amplitude is proportional to α_1 (α_2).[34] In terms of the complex-amplitude diagrams of Fig. 2, heterodyning picks out the oscillation of $\langle\mathscr{E}_1(0,t)\rangle$ [$\langle\mathscr{E}_2(0,t)\rangle$]; the separate plane for $2^{1/2}\langle\alpha_1\rangle e^{-i\epsilon t}$ ($2^{1/2}\langle\alpha_2\rangle e^{-i\epsilon t}$) is a phase plane for the heterodyned output.

At optical frequencies heterodyning is performed by combining the two-mode field (4.21) with a local-oscillator field at a beam splitter and then directing the combined field onto a photodetector; the mixing is a result of the photodetector's square-law response. Yuen and Shapiro[7] have analyzed optical heterodyning in detail. They assume $\epsilon \ll \Omega$ so they can neglect ϵ relative to Ω in factors like $\lambda_\pm = [(\Omega \pm \epsilon)/\Omega]^{1/2}$ [cf. Eqs. (4.25)]. In this approximation they find that ideal optical heterodyning does indeed produce an output whose complex amplitude is proportional to α_1.

The physical significance of the quadrature-phase amplitudes lies in their close connection to experimental techniques; they are the complex-amplitude operators for fields—the quadrature phases—that are directly accessible to measurement and experimental manipulation. The quadrature phases are accessible because they describe the physical process of putting amplitude and phase modulation on a carrier signal and because they are the quantities detected by phase-sensitive detection techniques such as heterodyning.

In place of the quadrature-phase amplitudes, one might be tempted to use operators[37] defined in the MP by

$$\beta_1 \equiv 2^{-1/2}(a_+ + a_-^\dagger) , \tag{4.47a}$$

$$\beta_2 \equiv 2^{-1/2}(-ia_+ + ia_-^\dagger) \tag{4.47b}$$

[cf. Eqs. (4.25)]. These operators have a simpler commutator algebra than α_1 and α_2:

$$[\beta_1,\beta_1^\dagger] = [\beta_2,\beta_2^\dagger] = [\beta_1,\beta_2] = 0 , \tag{4.48a}$$

$$[\beta_1,\beta_2^\dagger] = [\beta_1^\dagger,\beta_2] = i \tag{4.48b}$$

[cf. Eqs. (4.31)]; and under a unitary transformation generated by $S(r,0)$, they transform very simply:

$$S^\dagger(r,0)\beta_1 S(r,0) = \beta_1 e^{-r} , \tag{4.49a}$$

$$S^\dagger(r,0)\beta_2 S(r,0) = \beta_2 e^{r} \tag{4.49b}$$

[Eqs. (4.10) and (4.47)]. Despite these simple properties, β_1 and β_2 are not the natural variables for two-photon optics because they have no close connection to experimental techniques; they are not complex-amplitude operators for fields that can be measured. Shapiro and Wagner[37] have argued that β_1 or β_2 is the quantity detected by optical heterodyning. Their contention is based on Cook's claim[38] that photodetectors respond to "photon flux." The detailed analysis of Kimble and Mandel[39] does not support Cook's claim. Recent work by Yurke[49] indicates that α_1 or α_2, more nearly than β_1 or β_2, is the quantity detected by ideal optical heterodyning.

One can understand why β_1 and β_2 are not the natural variables—and at the same time understand the factors $\lambda_\pm$ which appear in the definition of α_1 and α_2 [Eqs. (4.25)]—by a simple units argument. The operators a_+ and $a_-^\dagger$ should not be added directly, as in Eqs. (4.47), because they have incompatible units; each has units of square root of the number of quanta, referred *to its own frequency*. Multiplication of a_+ by $(\Omega+\epsilon)^{1/2}$ and $a_-^\dagger$ by $(\Omega-\epsilon)^{1/2}$ converts the two quantities to common units of square root of energy; after this multiplication the two quantities may be added, as is done in the definitions of α_1 and α_2 [Eqs. (4.23)]. Division by $(2\Omega)^{1/2}$ then leaves α_1 and α_2 with dimensionless units of square root of the number of quanta, referred to the carrier frequency Ω. That α_1 and α_2 have these units is confirmed by writing the free Hamiltonian (4.2a) as

$$H_0 = \Omega[(\alpha_1\alpha_1^\dagger)_{\rm sym} + (\alpha_2\alpha_2^\dagger)_{\rm sym} - 1] . \tag{4.50}$$

Thus $(\alpha_1\alpha_1^\dagger)_{\rm sym} + (\alpha_2\alpha_2^\dagger)_{\rm sym} = (H_0+\Omega)/\Omega$ is the total energy, including the one quantum of zero-point energy, measured in units of the quantum at frequency Ω.

V. TIME-STATIONARY QUADRATURE-PHASE NOISE

A. Definition and discussion

The states encountered in two-photon optics—in particular, two-mode squeezed states—can have electric field noise that is not distributed randomly in phase, where phase is defined relative to Ω. This phase-sensitive noise is of a special sort, however, which we call *time-stationary quadrature-phase* (TSQP) *noise.*[13,21] The reason for the name is that the quadrature phases have time-stationary noise; this means that the natural variables to describe TSQP noise are the Fourier components of the quadrature phases, the quadrature-phase amplitudes.

To see what TSQP noise means, let ρ be the initial density operator for the pair of modes considered in Sec. IV. The noise associated with ρ can be characterized by the noise moments of α_1, α_2, $\alpha_1^\dagger$, and $\alpha_2^\dagger$. Just as we did for TS noise, we consider only the second-order noise moments—a specialization justified by the assumption that the noise is Gaussian; a complete description of TSQP and TS noise, based on all noise moments, will be presented in paper III. The state ρ is said to have (second-moment) TSQP noise if the quadrature-phase amplitudes have random-phase noise, i.e., if

$$\begin{aligned}\langle \Delta\alpha_m\,\Delta\alpha_n \rangle &\equiv {\rm tr}(\rho\,\Delta\alpha_m\,\Delta\alpha_n) \\ &= \langle \alpha_m\alpha_n \rangle - \langle \alpha_m \rangle \langle \alpha_n \rangle = 0 .\end{aligned} \tag{5.1}$$

where $m,n=1,2$ and $\Delta\alpha_m \equiv \alpha_m - \langle \alpha_m \rangle$ [cf. Eqs. (1.10a) and (1.13)]. In general, ten real numbers are required to specify all the second-order noise information, but the TSQP condition (5.1) eliminates six of those numbers. The remaining four numbers are contained in the "reduced" spectral-density matrix

$$\begin{aligned}\Sigma_{mn} &\equiv \langle \Delta\alpha_m\,\Delta\alpha_n^\dagger \rangle_{\rm sym} \\ &\equiv {\rm tr}[\rho(\Delta\alpha_m\,\Delta\alpha_n^\dagger)_{\rm sym}] = \langle \alpha_m\alpha_n^\dagger \rangle_{\rm sym} - \langle \alpha_m \rangle \langle \alpha_n^\dagger \rangle\end{aligned} \tag{5.2}$$

[cf. Eqs. (1.10b) and (1.13)], which is dimensionless (units of number of quanta at frequency Ω) and Hermitian:

$$\Sigma_{mn}^* = \Sigma_{nm} . \tag{5.3}$$

The spectral-density matrix, which has units of energy, is defined by

$$S_{mn} \equiv \Omega \Sigma_{mn} \ . \tag{5.4}$$

The diagonal elements of Σ_{mn} are simply the mean-square uncertainties in α_1 and α_2:

$$\Sigma_{mm} = \langle \, | \Delta\alpha_m |^2 \rangle, \ \ m=1,2 \ ; \tag{5.5}$$

the off-diagonal element $\Sigma_{12}=\Sigma_{21}^*$ is a complex correlation coefficient between the quadratures.

Under free evolution [MP evolution operator $U_M(t)$; Eq. (4.37)] the noise moments $\langle \Delta\alpha_m \, \Delta\alpha_n \rangle$ acquire a harmonic time dependence $e^{-2i\epsilon t}$, whereas the noise moments $\langle \Delta\alpha_m \, \Delta\alpha_n^\dagger \rangle_{\text{sym}}$ are constant [Eq. (4.27)]. Just as for TS noise, the vanishing of the time-dependent noise moments is the key to generalizing the notion of TSQP noise to moments of arbitrary order and also to understanding why the quadrature-phase amplitudes are suited to two-photon optics. The general definition of TSQP noise, which will be given explicitly in paper III, requires that all time-dependent noise moments of α_1, α_2, $\alpha_1^\dagger$, and $\alpha_2^\dagger$ vanish. *The quadrature-phase amplitudes are the natural variables for two-photon optics because the TSQP noise produced by two-photon devices is completely characterized by the time-independent noise moments of* α_1, α_2, $\alpha_1^\dagger$, *and* $\alpha_2^\dagger$.

It is often useful to have the TSQP condition (5.1) and the reduced spectral-density matrix (5.2) written in terms of creation and annihilation operators. The (second-moment) TSQP condition is equivalent to the following conditions on the second-order noise moments of the creation and annihilation operators:

$$\langle (\Delta a_\pm)^2 \rangle = 0 \ , \tag{5.6a}$$

$$\langle \Delta a_+ \, \Delta a_-^\dagger \rangle = 0 \tag{5.6b}$$

[Eqs. (4.26); Eq. (5.6a) means that for TSQP noise each mode by itself has random-phase noise]. The remaining second-order noise moments of the creation and annihilation operators are related to Σ_{mn}:

$$\begin{aligned}\lambda_+\lambda_- \langle \Delta a_+ \, \Delta a_- \rangle &= \tfrac{1}{2}(\Sigma_{11}-\Sigma_{22}) + \tfrac{1}{2}i(\Sigma_{12}+\Sigma_{21}) \\ &= \tfrac{1}{2}(\Sigma_{11}-\Sigma_{22}) + i\,\mathrm{Re}(\Sigma_{12}) \ ,\end{aligned} \tag{5.7a}$$

$$\begin{aligned}\lambda_\pm^2 \langle \, | \Delta a_\pm |^2 \rangle &= \tfrac{1}{2}(\Sigma_{11}+\Sigma_{22}) \mp \tfrac{1}{2}i(\Sigma_{12}-\Sigma_{21}) \\ &= \tfrac{1}{2}(\Sigma_{11}+\Sigma_{22}) \pm \mathrm{Im}(\Sigma_{12})\end{aligned} \tag{5.7b}$$

[Eqs. (4.26)]. Equations (5.7) can be recast in the form

$$\tfrac{1}{2}(\Sigma_{11}+\Sigma_{22}) = \tfrac{1}{2}(\lambda_+^2 \langle \, | \Delta a_+ |^2 \rangle + \lambda_-^2 \langle \, | \Delta a_- |^2 \rangle) \ , \tag{5.8a}$$

$$\tfrac{1}{2}(\Sigma_{11}-\Sigma_{22}) = \lambda_+\lambda_- \mathrm{Re}(\langle \Delta a_+ \, \Delta a_- \rangle) \ , \tag{5.8b}$$

$$\tfrac{1}{2}(\Sigma_{12}+\Sigma_{21}) = \mathrm{Re}(\Sigma_{12}) = \lambda_+\lambda_- \mathrm{Im}(\langle \Delta a_+ \, \Delta a_- \rangle) \ , \tag{5.8c}$$

$$-\tfrac{1}{2}i(\Sigma_{12}-\Sigma_{21}) = \mathrm{Im}(\Sigma_{12}) = \tfrac{1}{2}(\lambda_+^2 \langle \, | \Delta a_+ |^2 \rangle - \lambda_-^2 \langle \, | \Delta a_- |^2 \rangle) \ . \tag{5.8d}$$

Notice that for TSQP noise the time-dependent noise moment $\langle \Delta a_+ \, \Delta a_- \rangle$ (free time dependence $e^{-2i\Omega t}$) need not vanish. Since it must vanish for TS noise, Eqs. (5.7) imply that (second-moment) TSQP noise is (second-moment) TS noise if and only if

$$\Sigma_{11} = \Sigma_{22} = \tfrac{1}{2}(\lambda_+^2 \langle \, | \Delta a_+ |^2 \rangle + \lambda_-^2 \langle \, | \Delta a_- |^2 \rangle) \ , \tag{5.9a}$$

$$\Sigma_{12} = -\Sigma_{21} = \tfrac{1}{2}i(\lambda_+^2 \langle \, | \Delta a_+ |^2 \rangle - \lambda_-^2 \langle \, | \Delta a_- |^2 \rangle) \ . \tag{5.9b}$$

The reduced spectral-density matrix Σ_{mn} describes how the noise is distributed in phase, where phase is defined with respect to frequency Ω. There are two good ways of seeing this—ways that make clear the meaning of the four pieces of information in Σ_{mn}. The first way looks at the two-point correlation matrix of the dimensionless quadrature phases $\mathscr{E}_1(x,t)$ and $\mathscr{E}_2(x,t)$ [Eq. (4.41)],

$$\mathscr{K}_{mn}(\tau) \equiv \langle \Delta\mathscr{E}_m(x,t+\tau)\Delta\mathscr{E}_n(x,t) \rangle_{\text{sym}}, \ \ m,n=1,2 \tag{5.10}$$

which is a dimensionless, real matrix. If the two modes evolve freely, then

$$\mathscr{K}_{mn}(\tau) = \tfrac{1}{2}(\Sigma_{mn} e^{-i\epsilon\tau} + \Sigma_{nm} e^{i\epsilon\tau}) = \mathrm{Re}(\Sigma_{mn} e^{-i\epsilon\tau}) \tag{5.11}$$

[Eqs. (4.41), (5.1), and (5.2)]. The two-point correlation matrix also satisfies

$$\mathscr{K}_{mn}(-\tau) = \mathscr{K}_{nm}(\tau) \tag{5.12}$$

[Eq. (5.3)]. That the two-point correlation matrix depends on the time delay τ, but not on the retarded time $t-x$, is the essence of TSQP noise, and it is a direct consequence of the TSQP condition (5.1). Note, however, that TSQP noise does *not* mean that the two-point correlation matrix for the two-mode electric field depends only on the time delay τ; that condition is met only for TS noise [Eqs. (5.9)].

Consider now the zero time-delay ($\tau=0$) correlation matrix

$$\mathscr{K}_{mn} \equiv \mathscr{K}_{mn}(0) = \langle \Delta\mathscr{E}_m(x,t)\Delta\mathscr{E}_n(x,t) \rangle_{\text{sym}} = \mathrm{Re}(\Sigma_{mn}) \ , \tag{5.13}$$

which is just the symmetric covariance matrix of the dimensionless quadrature phases [cf. Eq. (1.11)]. If the noise is distributed randomly in phase, then $\mathscr{K}_{mn}$ is a multiple of the unit matrix. The covariance matrix $\mathscr{K}_{mn}$ contains three of the pieces of information in Σ_{mn}. Two pieces of information are contained in the diagonal elements

$$\mathscr{K}_{mm} = \langle [\Delta\mathscr{E}_m(x,t)]^2 \rangle = \Sigma_{mm} = \langle \, | \Delta\alpha_m |^2 \rangle, \ \ m=1,2 \tag{5.14}$$

which give the (constant) variances of $\mathscr{E}_1(x,t)$ and $\mathscr{E}_2(x,t)$, and the third piece is contained in the off-diagonal element

$$\mathscr{K}_{12} = \mathscr{K}_{21} = \langle \Delta\mathscr{E}_1(x,t)\Delta\mathscr{E}_2(x,t) \rangle_{\text{sym}} = \mathrm{Re}(\Sigma_{12}) \ , \tag{5.15}$$

which is a correlation coefficient for $\mathscr{E}_1(x,t)$ and $\mathscr{E}_2(x,t)$. These three pieces of information characterize the noise in the following way: the overall scale of the noise is set by $\tfrac{1}{2}(\Sigma_{11}+\Sigma_{22})$, which is the average noise in the quadratures [Eq. (5.14)] or the average noise in the two modes [Eq. (5.8a)]; the extent to which the noise is not distributed randomly in phase is specified by $\frac{1}{2}(\Sigma_{11}-\Sigma_{22})$ and $\mathrm{Re}(\Sigma_{12})$. The roles of these quantities are immediately apparent in the variance of the electric field

$$\langle[\Delta E(x,t)]^2\rangle=\Omega\{\Sigma_{11}+\Sigma_{22}+(\Sigma_{11}-\Sigma_{22})\cos[2\Omega(t-x)] +2\,\mathrm{Re}(\Sigma_{12})\sin[2\Omega(t-x)]\} \quad (5.16)$$

[Eqs. (4.21), (4.39), and (5.13); cf. Eqs. (1.12) and (3.13)]. A nonrandom distribution of noise in phase corresponds to a time-dependent electric field variance. The quantities that describe a nonrandom distribution, $\frac{1}{2}(\Sigma_{11}-\Sigma_{22})$ and $\mathrm{Re}(\Sigma_{12})$, are related to $\langle\Delta a_+\Delta a_-\rangle$ [Eqs. (5.8b) and (5.8c)]; if the electric field noise is not distributed randomly in phase, the two modes must be correlated.

The fourth piece of information in Σ_{mn} shows up in the time-delayed ($\tau\neq0$) correlation between the dimensionless quadrature phases. Specifically, for $\tau=\pi/2\epsilon$, the two-point correlation matrix becomes an antisymmetric matrix

$$\begin{aligned}\overline{\mathscr{K}}_{mn}&\equiv\mathscr{K}_{mn}(\pi/2\epsilon)\\&=\langle\Delta\mathscr{E}_m(x,t+\pi/2\epsilon)\Delta\mathscr{E}_n(x,t)\rangle_{\mathrm{sym}}\\&=\mathrm{Im}(\Sigma_{mn})\,.\end{aligned} \quad (5.17)$$

The diagonal elements of $\overline{\mathscr{K}}_{mn}$ vanish. This result one expects for TSQP noise; it says that for each quadrature phase the noise at a particular time is uncorrelated with the noise a quarter cycle later. In contrast, the off-diagonal element of $\overline{\mathscr{K}}_{mn}$ need not vanish. It gives the fourth piece of information in Σ_{mn}:

$$\begin{aligned}\overline{\mathscr{K}}_{12}=-\overline{\mathscr{K}}_{21}&=\langle\Delta\mathscr{E}_1(x,t+\pi/2\epsilon)\Delta\mathscr{E}_2(x,t)\rangle_{\mathrm{sym}}\\&=\mathrm{Im}(\Sigma_{12})\,.\end{aligned} \quad (5.18)$$

This result is a bit mysterious; it says that the noise in one quadrature at a particular time is correlated with the noise in the other quadrature a quarter cycle later. The explanation lies in the definitions of α_1 and α_2 [Eqs. (4.25)]. A fluctuation in the upper mode a_+ corresponds to identical fluctuations in α_1 and α_2, but the fluctuation in α_1 lags that in α_2 by a quarter cycle; this produces a positive contribution to $\overline{\mathscr{K}}_{12}$. Similarly, a fluctuation in the lower mode a_- corresponds to a fluctuation in α_1 that leads the fluctuation in α_2 by a quarter cycle; this produces a negative contribution to $\overline{\mathscr{K}}_{12}$. Thus $\overline{\mathscr{K}}_{12}$ should be related to the difference in noise in the two modes, an inference confirmed by Eqs. (5.8d) and (5.18), which show that

$$\overline{\mathscr{K}}_{12}=\mathrm{Im}(\Sigma_{12})=\tfrac{1}{2}(\lambda_+^2\langle|\Delta a_+|^2\rangle-\lambda_-^2\langle|\Delta a_-|^2\rangle)\,.$$

The second way of investigating the meaning of Σ_{mn} is to look at how it transforms under a rotation (phase change) of the complex amplitude of the electric field [rotation produced by $R(\theta)$; Eqs. (4.33)–(4.36)]. Recalling that a rotation produces a common phase change of the annihilation operators [Eq. (4.35)], one sees from Eq. (5.7b) that $\frac{1}{2}(\Sigma_{11}+\Sigma_{22})$ and $-\frac{1}{2}i(\Sigma_{12}-\Sigma_{21})=\mathrm{Im}(\Sigma_{12})$ are invariant under rotations; these quantities have nothing to do with the differential distribution of noise in phase. Similarly, one sees from Eq. (5.7a) that $\frac{1}{2}(\Sigma_{11}-\Sigma_{22})$ and $\frac{1}{2}(\Sigma_{12}+\Sigma_{21})=\mathrm{Re}(\Sigma_{12})$ transform as

$$\begin{aligned}&\tfrac{1}{2}(\Sigma'_{11}-\Sigma'_{22})+\tfrac{1}{2}i(\Sigma'_{12}+\Sigma'_{21})\\&\quad=e^{-2i\theta}[\tfrac{1}{2}(\Sigma_{11}-\Sigma_{22})+\tfrac{1}{2}i(\Sigma_{12}+\Sigma_{21})]\;;\end{aligned} \quad (5.19)$$

these quantities characterize precisely the extent to which the noise is not distributed randomly in phase.

B. Complex-amplitude diagrams

One can add information about TSQP noise to the complex-amplitude diagrams in Figs. 1 and 2. Start with the complex-amplitude diagram ($t=0$) in Fig. 1, which describes the classical behavior of the electric field. To add information about TSQP noise, draw an "error ellipse" centered at the tip of the initial complex-amplitude vector (Fig. 3). The error ellipse displays the information

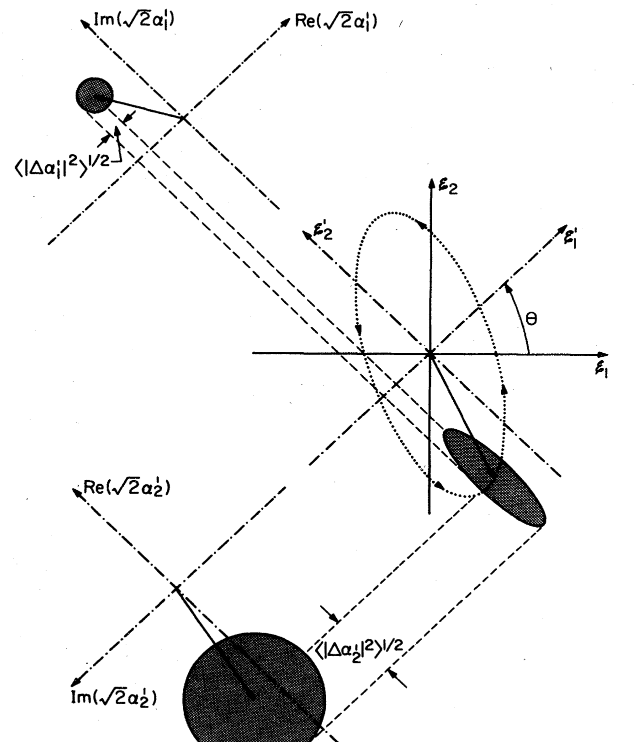

FIG. 3. Standard complex-amplitude diagram for TSQP noise. The behavior of the mean complex amplitude $\langle\mathscr{E}_1(0,t)+i\mathscr{E}_2(0,t)\rangle$ is shown, as in Fig. 1, by a dotted signal ellipse and an initial ($t=0$) complex-amplitude vector. The quadrature-phase noise is depicted by a shaded error ellipse. The principal axes of the error ellipse are the eigendirections of the covariance matrix $\mathscr{K}_{mn}$ [Eq. (5.13)], and the principal radii are the square roots of the eigenvalues of $\mathscr{K}_{mn}$. The complex-amplitude diagram shows rotated (primed) axes that lie along the principal axes of the error ellipse. With respect to the rotated axes the covariance matrix $\mathscr{K}'_{mn}$ is diagonal, its diagonal elements $\mathscr{K}'_{mm}=\langle[\Delta\mathscr{E}'_m(x,t)]^2\rangle=\Sigma'_{mm}=\langle|\Delta\alpha'_m|^2\rangle$ giving the squares of the principal radii. Separate phase planes are drawn for the rotated quadrature phases (cf. Fig. 2). In each a vector indicates the initial ($t=0$) value of $2^{1/2}\langle\alpha'_m\rangle$, and a shaded error circle, with radius $\langle|\Delta\alpha'_m|^2\rangle^{1/2}$, depicts the noise in the quadrature phase.

contained in the covariance matrix $\mathcal{K}_{mn}=\mathrm{Re}(\Sigma_{mn})$ [Eq. (5.13)]: its principle axes are the eigendirections of $\mathcal{K}_{mn}$, and its principle radii are the square roots of the eigenvalues of $\mathcal{K}_{mn}$. It is convenient to rotate the axes of the complex-amplitude plane counterclockwise by an angle θ [rotation defined by Eq. (4.34)] so that the new (primed) axes are parallel to the principal axes of the error ellipse (see Fig. 3), i.e., so that the covariance matrix is diagonal with respect to the new axes. The angle θ is obtained from

$$\tfrac{1}{2}(\Sigma_{11}-\Sigma_{22})+\tfrac{1}{2}i(\Sigma_{12}+\Sigma_{21}) = -\tfrac{1}{2}[(\Sigma_{11}-\Sigma_{22})^2+(\Sigma_{12}+\Sigma_{21})^2]^{1/2}e^{2i\theta} ,$$

where $0\le\theta<\pi$ [Eq. (5.19)]. The diagonal elements of the rotated covariance matrix are given by $\mathcal{K}'_{mm}=\langle[\Delta\mathcal{E}'_m(x,t)]^2\rangle=\Sigma'_{mm}=\langle\,|\Delta\alpha'_m\,|^2\rangle$; their square roots—the uncertainties in the rotated quadrature phases—are the principal radii of the error ellipse. The error ellipse is a convenient way to show graphically the nonrandom distribution of noise in phase.

Figure 3 also shows separate phase planes for the rotated quadrature phases (cf. Fig. 2). In each phase plane a vector indicates the initial expectation value of $2^{1/2}\alpha'_m$. The noise in each quadrature phase is depicted by an "error circle," which is centered at the tip of the vector $2^{1/2}\langle\alpha'_m\rangle$ and whose radius is the root-mean-square uncertainty in α'_m. That one uses a circle expresses the fact that the quadrature phases have time-stationary (random-phase) noise; i.e., the uncertainties in the Hermitian real and imaginary parts of $2^{1/2}\alpha'_m$ are the same, and they are equal to the root-mean-square uncertainty $\langle\,|\Delta\alpha'_m\,|^2\rangle^{1/2}$. Just as the projection of $2^{1/2}\langle\alpha'_m\rangle$ onto its real axis gives the associated component $\langle\mathcal{E}'_m(0,t)\rangle$ of the mean complex amplitude, so the projection of the error circle on the real axis gives the associated principal diameter of the error ellipse (Fig. 3).

We refer to Fig. 3 as the standard complex-amplitude diagram. The vectors in it are drawn at $t=0$, but a similar diagram could be constructed at any time. As time passes, the vector in each separate phase plane rotates clockwise with angular velocity ϵ, dragging its error circle with it; the projection of the vector and its error circle on the real axis describes the oscillation of the associated quadrature phase with constant variance. These projections can also be used to construct the mean complex amplitude $\langle\mathcal{E}_1(0,t)+i\mathcal{E}_2(0,t)\rangle$ and its error ellipse. The mean complex-amplitude vector rotates in the direction shown by the arrows. The error ellipse is dragged along as the mean complex-amplitude vector rotates, but it retains the same size, shape, and orientation—a consequence of TSQP noise.

The axes in Fig. 3 are somewhat loosely labeled by operators because the diagrams are supposed to indicate both the mean behavior and the fluctuations about the mean. The axes of the separate phase planes are labeled by the Hermitian real and imaginary parts of $2^{1/2}\alpha'_m$. Notice that the free time dependence $e^{-i\epsilon t}$ is not indicated explicitly as in Fig. 2. The reason is that this time dependence is implicit; the expectation values of α'_1 and α'_2 are evaluated in the MP (Sec. IV C), where they have the free time dependence $e^{-i\epsilon t}$. The standard complex-amplitude diagram can be put on a more rigorous footing after the two-photon quasiprobability distributions are introduced in a future paper (paper III). Then the axes can be labeled by variables of an appropriate quasiprobability distribution, and the error ellipse and the error circles become particular contours of the quasiprobability distribution.

TS noise is distributed randomly in phase ($\mathcal{K}_{mn}$ is a multiple of the unit matrix). In the complex-amplitude diagram in Fig. 3 this means that the error ellipse is a circle [$\Sigma_{11}=\Sigma_{22}$, $\mathrm{Re}(\Sigma_{12})=0$; Eqs. (5.9)] and the error circles in the separate phase planes have the same size. To go from TS noise to TSQP noise, one imagines "squeezing" the error circle of TS noise into the error ellipse that characterizes TSQP noise; noise is squeezed from one quadrature phase into the other so that the error circles in the separate phase planes have different sizes. The use of the term squeezed to describe a nonrandom distribution of noise in phase arose from this simple picture of a circle being squeezed into an ellipse. The term[40] was originally applied to the degenerate limit ($\epsilon=0$, $a_+=a_-$), where one draws complex-amplitude diagrams very much like the central diagram in Fig. 3. In the degenerate limit the noise is depicted by an error ellipse just as in Fig. 3, but the signal ellipse collapses to a point, which is the unchanging complex amplitude of a single mode (see, for example, Fig. 1 of Ref. 1). It should be emphasized that squeezing is a consequence of correlation between the two modes [Eqs. (5.8b) and (5.8c)]; each mode by itself has random-phase noise [Eq. (5.6a)].

The standard complex-amplitude diagram (Fig. 3) does not display all the information about the second-order noise. It shows graphically the three pieces of information in the covariance matrix $\mathcal{K}_{mn}=\mathrm{Re}(\Sigma_{mn})$ [Eq. (5.13)], but it does not include any information about $\mathrm{Im}(\Sigma_{12})$ [Eq. (5.18)]. This omission is really not very serious. The purpose of the standard complex-amplitude diagram is to depict the nonrandom distribution of noise in phase, which does not depend on $\mathrm{Im}(\Sigma_{12})$.

The relation of the standard complex-amplitude diagram to the behavior of the electric field and the quadrature phases is made clearer by the graphs in Fig. 4. Each part of Fig. 4 shows two complex-amplitude diagrams for a particular state of the field which has TSQP noise; one diagram is drawn at $t=0$ and the other at $t=\pi/2\epsilon$. The states depicted in Fig. 4 are special in two ways: (i) All the signal is carried by $\mathcal{E}_1(x,t)$, i.e., $\langle\mathcal{E}_2(x,t)\rangle=0$. Thus the signal ellipse collapses to a line along the $\mathcal{E}_1$ axis, and the mean electric field at $x=0$ is given by

$$\langle E(0,t)\rangle=\langle E_1(0,t)\rangle\cos(\Omega t) \tag{5.20}$$

[Eq. (4.21)]. (ii) The quadrature phases have zero second-order correlation, i.e., $\mathcal{K}_{12}=\mathrm{Re}(\Sigma_{12})=0$. Thus the principal axes of the error ellipse are parallel to the $\mathcal{E}_1$ and $\mathcal{E}_2$ axes, and the uncertainty in the electric field at $x=0$ is given by

$$\langle[\Delta E(0,t)]^2\rangle^{1/2}=(2\Omega)^{1/2}[\Sigma_{11}\cos^2(\Omega t) +\Sigma_{22}\sin^2(\Omega t)]^{1/2} \tag{5.21}$$

[Eq. (5.16)]. Figure 4(a) depicts a state with TS noise

$(\Sigma_{11}=\Sigma_{22})$, Fig. 4(b) depicts a state with less noise in $\mathscr{E}_1(x,t)$ than in $\mathscr{E}_2(x,t)$ $(\Sigma_{11}<\Sigma_{22})$, and Fig. 4(c) depicts a state with less noise in $\mathscr{E}_2(x,t)$ than in $\mathscr{E}_1(x,t)$ $(\Sigma_{22}<\Sigma_{11})$. Below the complex-amplitude diagrams in each part of Fig. 4 are graphs for the electric field $E(0,t)$ and the quadrature phases $E_1(0,t)$ and $E_2(0,t)$. The dark central line in each graph is the expectation value of the appropriate field, and the width of the shaded region is twice the uncertainty in the same quantity. The graph for $E_1(0,t)$ shows a sinusoidal oscillation at frequency ϵ with constant uncertainty $\langle[\Delta E_1(0,t)]^2\rangle^{1/2}=(2\Omega\Sigma_{11})^{1/2}$; this behavior is described by the projection on the real axis of the rotating vector $2^{1/2}\langle\alpha_1\rangle$ and its associated error circle. The graph for $E_2(0,t)$ shows a zero expectation value with constant uncertainty $\langle[\Delta E_2(0,t)]^2\rangle^{1/2}=(2\Omega\Sigma_{22})^{1/2}$; this behavior is described by the unchanging projection on the real axis of the error circle in the phase plane for $2^{1/2}\alpha_2$. In the graph for $E(0,t)$, the mean electric field is modulated at frequency ϵ [Eq. (5.20)], and the uncertainty oscillates as given by Eq. (5.21). Similar graphs for the behavior of the electric field have been drawn in the degenerate limit (see, for example, Fig. 2 of Ref. 1); the uncertainty oscillates just as in Eq. (5.21), but the mean electric field is unmodulated.

The graphs for $E_1(0,t)$ and $E_2(0,t)$ in Fig. 4 are closely related to the output of an ideal heterodyne detector and to amplitude and phase modulation of a carrier wave (see discussion in Sec. IV D). If the electric field in Fig. 4 is mixed with a local-oscillator wave proportional to $\cos[\Omega(t-x)]$, then the graph for $E_1(0,t)$ characterizes the heterodyned output at frequency ϵ, which has constant noise. If a strong classical carrier wave proportional to $\cos[\Omega(t-x)]$ is added to the electric field in Fig. 4, then the graph for $E_1(0,t)$ describes an amplitude-modulation signal with constant amplitude-modulation noise, and the graph for $E_2(0,t)$ describes a zero phase-modulation signal with constant phase-modulation noise. The differences among the three parts of Fig. 4 lie in the different ratios of amplitude-modulation noise to phase-modulation noise.

VI. UNCERTAINTY PRINCIPLES FOR QUADRATURE-PHASE AMPLITUDES

In this section we consider uncertainty principles that apply to the mean-square uncertainties in the quadrature-phase amplitudes. The analogous uncertainty principles

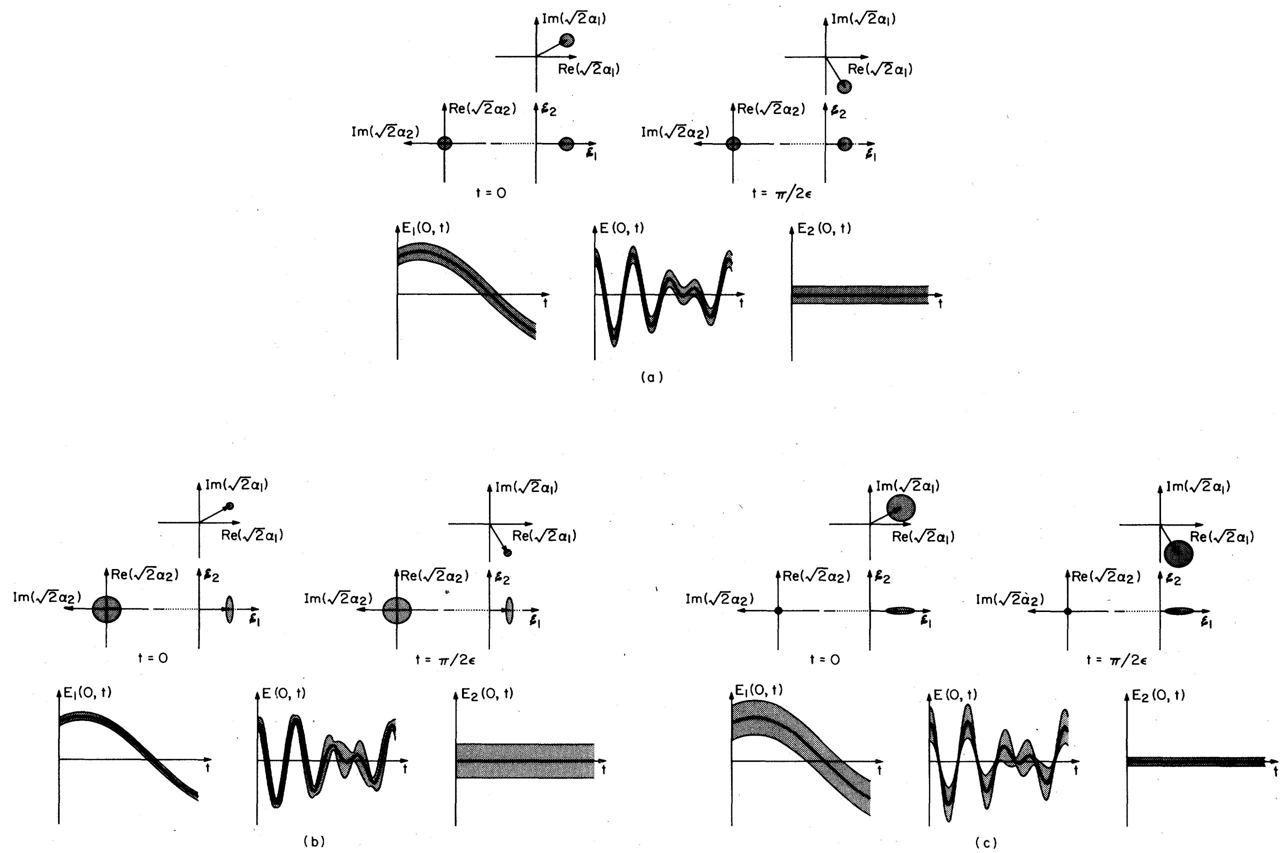

FIG. 4. Graphs of the electric field $E(0,t)$ and the quadrature phases $E_1(0,t)$ and $E_2(0,t)$ for three states with TSQP noise: (a) a state with TS noise; (b) a state with less noise in $\mathscr{E}_1(x,t)$ than in $\mathscr{E}_2(x,t)$; (c) a state with less noise in $\mathscr{E}_2(x,t)$ than in $\mathscr{E}_1(x,t)$. Above the graphs in each part are two complex-amplitude diagrams for the same state, one at $t=0$ and one at $t=\pi/2\epsilon$. In each graph the dark central line is the expectation value of the appropriate field quantity, and the width of the shaded region at any time is twice the uncertainty in the same quantity. See the text for further discussion.

for more general non-Hermitian operators are derived and discussed in the Appendix; here we simply apply the more general results to the particular case of α_1 and α_2.

The most important uncertainty principle[13,21] places a lower limit on the product of the root-mean-square uncertainties in α_1 and α_2:

$$\langle\, |\Delta\alpha_1|^2\rangle^{1/2}\langle\, |\Delta\alpha_2|^2\rangle^{1/2} \geq \tfrac{1}{2}\,|\langle[\alpha_1,\alpha_2^\dagger]\rangle\,| = \tfrac{1}{2} \tag{6.1}$$

[Eqs. (A16) and (4.31c)]. In terms of the spectral-density matrix S_{mn} [Eq. (5.4)], the uncertainty principle (6.1) becomes $S_{11}S_{22} \geq \frac{1}{4}\Omega^2$. It should be noted that Eq. (6.1) does not require an assumption of TSQP noise, but it does rely on the fact that α_1 and α_2 commute [Eq. (4.31b)]. Yurke and Denker[34,41] have considered an uncertainty principle similar to Eq. (6.1), but in terms of the multimode quadrature phases [Eq. (1.6)].

What is the meaning of the uncertainty principle (6.1)? The zero-point noise in each mode corresponds to half a quantum at the mode's frequency. In units of energy the combined zero-point noise in the two modes is $\frac{1}{2}(\Omega+\epsilon)+\frac{1}{2}(\Omega-\epsilon)=\Omega$, which amounts to one quantum at the carrier frequency. If

$$\langle\, |\Delta\alpha_1|^2\rangle = \langle\, |\Delta\alpha_2|^2\rangle = \tfrac{1}{2} \tag{6.2}$$

($S_{11}=S_{22}=\frac{1}{2}\Omega$), then each quadrature carries half of the one quantum of zero-point noise. The uncertainty principle (6.1) allows the uncertainty in one quadrature to be reduced below the level set by zero-point noise, but only at the expense of increasing the noise in the other quadrature above the zero-point level. Thus the uncertainty principle describes the squeezing referred to in Sec. V B: noise can be reduced below the zero-point level only by squeezing noise from one quadrature phase into the other.

Equation (6.1) is the two-mode analog of an uncertainty principle[2] that applies in the degenerate limit—$\epsilon=0$, $a_+=a_-=a$. This uncertainty principle, which is equivalent to the position-momentum uncertainty principle, is usually written in terms of a_1 and a_2, the Hermitian real and imaginary parts of $a=a_1+ia_2$:

$$\langle(\Delta a_1)^2\rangle^{1/2}\langle(\Delta a_2)^2\rangle^{1/2} \geq \tfrac{1}{2}\,|\langle[a_1,a_2]\rangle\,| = \tfrac{1}{4}\ . \tag{6.3}$$

Further discussion of the degenerate limit can be found in Sec. VIII.

Equality in Eq. (6.1) imposes very restrictive conditions on the state vector $|\Psi\rangle$; indeed, Eqs. (A27), specialized to the case $R=\alpha_1$ and $S=\alpha_2$, show that equality holds in Eq. (6.1) if and only if

$$(\Delta\alpha_1+i\,\Delta\alpha_2)\,|\Psi\rangle=0\ , \tag{6.4a}$$

$$(\Delta\alpha_1^\dagger+i\,\Delta\alpha_2^\dagger)\,|\Psi\rangle=0 \tag{6.4b}$$

[Eqs. (4.31), (A27c), and (A29)]. Plugging in the definitions (4.25) of α_1 and α_2, one finds that Eqs. (6.4) reduce to

$$\Delta a_\pm\,|\Psi\rangle=0\ . \tag{6.5}$$

Thus the only states that yield equality in Eq. (6.1) are the simultaneous eigenstates of a_+ and a_-, i.e., the two-mode coherent states (4.11).

In addition to the uncertainty principle (6.1), there is a separate uncertainty principle for each quadrature-phase amplitude:[13,21]

$$\langle\, |\Delta\alpha_m|^2\rangle \geq \tfrac{1}{2}\,|\langle[\alpha_m,\alpha_m^\dagger]\rangle\,| = \epsilon/2\Omega,\quad m=1,2 \tag{6.6}$$

[Eqs. (A9) and (4.31a)]. Equation (6.6) does not rely on an assumption of TSQP noise. Equality holds in Eq. (6.6) if and only if the state vector $|\Psi\rangle$ is an eigenstate of α_m, i.e.,

$$\Delta\alpha_m\,|\Psi\rangle=0 \tag{6.7}$$

[Eq. (A12a)]. Since $\epsilon<\Omega$, it is immediately apparent from Eq. (6.1) that it is impossible to find a state $|\Psi\rangle$ for which both $\langle\,|\Delta\alpha_1|^2\rangle$ and $\langle\,|\Delta\alpha_2|^2\rangle$ have the minimum value $\epsilon/2\Omega$. This means that there are no simultaneous eigenstates of α_1 and α_2.

What can one learn from the uncertainty principle (6.6)? For each quadrature it says that the minimum noise is a factor ϵ/Ω smaller than the level set by zero-point noise [Eq. (6.2)]. If one writes Eq. (6.6) in units of energy—$S_{mm}\geq\frac{1}{2}\epsilon$—one sees that *the minimum noise corresponds to half a quantum at the modulation frequency* ϵ. This suggests interpreting the minimum noise $\frac{1}{2}\epsilon$ as a sort of zero-point noise for the quadrature phases; we call it the *quadrature-phase zero-point noise.* This interpretation is strengthened by noting that the quadrature phase $E_m(x,t)$ is a "field operator" at frequency ϵ [Eq. (4.28)]. The variance of $2^{-1/2}E_m(x,t)$ for a state with TSQP noise,

$$\tfrac{1}{2}\langle[\Delta E_m(x,t)]^2\rangle=\Omega\langle\,|\Delta\alpha_m|^2\rangle\geq\tfrac{1}{2}\epsilon\ , \tag{6.8}$$

should be compared with the single-mode electric-field variance (3.13) for a state with TS noise, where the single mode has frequency $\omega=\epsilon$. In terms of energy the lower limit in Eq. (6.8), which is enforced by the quadrature-phase zero-point noise, is the same as the lower limit in Eq. (3.13), which is enforced by the ordinary zero-point noise at frequency $\omega=\epsilon$ [Eq. (3.14)]. Physically the quadrature-phase zero-point noise means the following: *if one chooses to work at modulation frequency* ϵ *about a high carrier frequency* Ω, *then the noise in one quadrature phase can be made as small as, but no smaller than, the minimum noise that one would encounter if working directly at the low frequency* ϵ.

The relation between the quadrature-phase amplitudes and the quadrature-phase zero-point noise is analogous to the relation between the creation and annihilation operators and the ordinary zero-point noise. The analogy becomes apparent if one writes the free Hamiltonian H_0 [Eq. (4.2a)] in terms of various operator orderings. Orderings of the creation and annihilation operators give expressions that involve the ordinary zero-point energy Ω:

$$(\Omega+\epsilon)(a_+a_+^\dagger)_{\rm sym}+(\Omega-\epsilon)(a_-a_-^\dagger)_{\rm sym}=H_0+\Omega\ , \tag{6.9a}$$

$$(\Omega+\epsilon)a_+^\dagger a_+ +(\Omega-\epsilon)a_-^\dagger a_- =H_0\ , \tag{6.9b}$$

$$(\Omega+\epsilon)a_+a_+^\dagger +(\Omega-\epsilon)a_-a_-^\dagger =H_0+2\Omega\ . \tag{6.9c}$$

Symmetric ordering [Eq. (6.9a)] yields the total energy, including the one quantum of zero-point energy, normal ordering [Eq. (6.9b)] yields the total energy minus the zero-point energy, and antinormal ordering [Eq. (6.9c)] yields the total energy plus the zero-point energy. Analogous orderings of the quadrature-phase amplitudes and their Hermitian conjugates involve the quadrature-phase zero-point energy ϵ ($\frac{1}{2}\epsilon$ from each quadrature):

$$\Omega[(\alpha_1\alpha_1^\dagger)_{\text{sym}}+(\alpha_2\alpha_2^\dagger)_{\text{sym}}]=H_0+\Omega\ , \tag{6.10a}$$

$$\Omega(\alpha_1^\dagger\alpha_1+\alpha_2^\dagger\alpha_2)=H_0+\Omega-\epsilon\ , \tag{6.10b}$$

$$\Omega(\alpha_1\alpha_1^\dagger+\alpha_2\alpha_2^\dagger)=H_0+\Omega+\epsilon\ . \tag{6.10c}$$

Symmetric ordering [Eq. (6.10a)] again yields the total energy. If one places $\alpha_1^\dagger$ and $\alpha_2^\dagger$ to the left of α_1 and α_2 [Eq. (6.10b)], an ordering which is analogous to ordinary normal ordering and which we call *quadrature-phase normal ordering,* then one obtains the total energy minus the quadrature-phase zero-point energy. Similarly, if one places α_1 and α_2 to the left of $\alpha_1^\dagger$ and $\alpha_2^\dagger$ [Eq. (6.10c)], an ordering which we call *quadrature-phase antinormal ordering,* then one obtains the total energy plus the quadrature-phase zero-point energy. These and other more general orderings for the quadrature-phase amplitudes will be considered in paper III.

One can also write an uncertainty principle for the operators β_1 and β_2 [Eqs. (4.47) and (4.48)]. Analogous to Eq. (6.1) is an uncertainty principle

$$\langle\,|\Delta\beta_1|^2\rangle^{1/2}\langle\,|\Delta\beta_2|^2\rangle^{1/2}\ge\tfrac{1}{2}\,|\langle[\beta_1,\beta_2^\dagger]\rangle|=\tfrac{1}{2}\ , \tag{6.11}$$

but there is no analog of Eq. (6.6); i.e., $\langle\,|\Delta\beta_m|^2\rangle$ can be made arbitrarily small.

VII. TWO-MODE SQUEEZED STATES

Two-mode squeezed states are the natural states for two-photon optics because they are the output states of an ideal two-photon device (see Sec. IV A). Here we discuss briefly the most important properties of two-mode squeezed states; our purpose is to show how they fit into the general framework developed in Secs. IV–VI. A more thorough investigation of their properties is undertaken in paper II.

A useful preliminary to the properties of two-mode squeezed states is a review of the most basic properties of two-mode coherent states [Eq. (4.11)]:

$$|\mu_+,\mu_-\rangle_{\text{coh}}\equiv D(a_+,\mu_+)D(a_-,\mu_-)\,|\,0\rangle\ . \tag{7.1}$$

Using the fact that $|\mu_+,\mu_-\rangle_{\text{coh}}$ is an eigenstate of a_+ and a_-, one can show, first, that the expectation values of the annihilation operators and the quadrature-phase amplitudes are given by

$$\langle a_\pm\rangle=\mu_\pm\ , \tag{7.2a}$$

$$\langle\alpha_1\rangle=\xi_1\equiv2^{-1/2}(\lambda_+\mu_++\lambda_-\mu_-^*)\ , \tag{7.2b}$$

$$\langle\alpha_2\rangle=\xi_2\equiv2^{-1/2}(-i\lambda_+\mu_++i\lambda_-\mu_-^*) \tag{7.2c}$$

[Eqs. (4.25)] and, second, that $|\mu_+,\mu_-\rangle_{\text{coh}}$ has TS noise—$\langle(\Delta a_\pm)^2\rangle=\langle\Delta a_+\Delta a_-^\dagger\rangle=\langle\Delta a_+\Delta a_-\rangle=0$ [Eqs. (5.6) and (5.9)]—with $\langle\,|\Delta a_+|^2\rangle=\langle\,|\Delta a_-|^2\rangle=\frac{1}{2}$—i.e.,

$$\Sigma_{mm}=\langle\,|\Delta\alpha_m|^2\rangle=\tfrac{1}{2},\quad m=1,2 \tag{7.3a}$$

$$\Sigma_{12}=-\Sigma_{21}=\tfrac{1}{2}i(\epsilon/\Omega) \tag{7.3b}$$

[Eq. (5.2)]. A two-mode coherent state can be regarded as a classical excitation of the two modes, contaminated by zero-point noise. The covariance matrix of the dimensionless quadrature phases [Eq. (5.13)] is a multiple of the unit matrix,

$$\mathscr{K}_{mn}=\text{Re}(\Sigma_{mn})=\tfrac{1}{2}\delta_{mn}\ , \tag{7.4}$$

which shows that the noise associated with a coherent state is distributed randomly in phase. In the standard complex-amplitude diagram (see Sec. V B and Fig. 3), these properties of a two-mode coherent state show up in the following ways: the error ellipse in the central complex-amplitude plane is a circle, the two error circles in the separate phase planes have the same size, and all three circles have radius $2^{-1/2}$. Notice that $\text{Im}(\Sigma_{12})=\epsilon/2\Omega$ does not vanish for a coherent state—a consequence of the fact that the energy associated with the zero-point noise is different for the two modes [see discussion surrounding Eq. (5.18)].

Turn now to the two-mode squeezed states defined by Eqs. (4.15), (4.17), and (4.18):

$$\begin{aligned}|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r,\varphi)}&\equiv S(r,\varphi)\,|\,\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{\text{coh}}\\&=S(r,\varphi)D(a_+,\mu_{\alpha_+})D(a_-,\mu_{\alpha_-})\,|\,0\rangle\\&=D(a_+,\mu_+)D(a_-,\mu_-)S(r,\varphi)\,|\,0\rangle\ ,\end{aligned} \tag{7.5}$$

$$\mu_{\alpha_\pm}=\mu_\pm\cosh r+\mu_\mp^*e^{2i\varphi}\sinh r\ . \tag{7.6}$$

When $r=0$ a two-mode squeezed state reduces to a two-mode coherent state. The unitary equivalence between the squeezed annihilation operators and the annihilation operators [Eq. (4.14)] provides an easy way to calculate first and second moments for a two-mode squeezed state; the moments of $a_\pm(r,\varphi)$ with respect to $|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r,\varphi)}$ are the same as the moments of $a_\pm$ with respect to $|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{\text{coh}}$. Using this approach, one can calculate the following expectation values for the two-mode squeezed state (7.5):

$$\langle a_\pm\rangle=\mu_\pm\ , \tag{7.7a}$$

$$\langle\alpha_m\rangle=\xi_m \tag{7.7b}$$

[cf. Eqs. (7.2)]. In addition, one can show that $|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r,\varphi)}$ has TSQP noise [Eq. (5.1) or Eqs. (5.6)] with

$$\langle\,|\Delta a_+|^2\rangle=\langle\,|\Delta a_-|^2\rangle=\tfrac{1}{2}\cosh(2r)\ , \tag{7.8a}$$

$$\langle\Delta a_+\,\Delta a_-\rangle=-\tfrac{1}{2}e^{2i\varphi}\sinh(2r)\ ; \tag{7.8b}$$

translated into the language of the reduced spectral-density matrix (5.2), Eqs. (7.8) become

$$\begin{aligned}\Sigma_{11}&=\langle\,|\Delta\alpha_1|^2\rangle\\&=\tfrac{1}{2}\cosh(2r)-\tfrac{1}{2}(1-\epsilon^2/\Omega^2)^{1/2}\sinh(2r)\cos(2\varphi)\ ,\end{aligned} \tag{7.9a}$$

$$\Sigma_{22}=\langle\,|\Delta\alpha_2|^2\rangle$$
$$=\tfrac{1}{2}\cosh(2r)+\tfrac{1}{2}(1-\epsilon^2/\Omega^2)^{1/2}\sinh(2r)\cos(2\varphi)\,, \tag{7.9b}$$

$$\Sigma_{12}=\Sigma_{21}^*=-\tfrac{1}{2}(1-\epsilon^2/\Omega^2)^{1/2}\sinh(2r)\sin(2\varphi)$$
$$+\tfrac{1}{2}i(\epsilon/\Omega)\cosh(2r) \tag{7.9c}$$

[Eqs. (5.8)]. For $r\neq 0$ a two-mode squeezed state does indeed display the nonrandom distribution of noise in phase which entitles it to be called squeezed. The standard complex-amplitude diagram looks like Fig. 3 with $\theta=\varphi$; the error ellipse has principal radii $2^{-1/2}[\cosh(2r)\mp(1-\epsilon^2/\Omega^2)^{1/2}\sinh(2r)]^{1/2}$, which also are the radii of the error circles in the separate phase planes.

An important subset of the two-mode squeezed states consists of those with $\varphi=0$. For this subset the reduced spectral-density matrix (7.9) becomes

$$\Sigma_{11}=\langle\,|\Delta\alpha_1|^2\rangle$$
$$=\tfrac{1}{2}e^{-2r}+\tfrac{1}{2}[1-(1-\epsilon^2/\Omega^2)^{1/2}]\sinh(2r)\,, \tag{7.10a}$$

$$\Sigma_{22}=\langle\,|\Delta\alpha_2|^2\rangle$$
$$=\tfrac{1}{2}e^{2r}-\tfrac{1}{2}[1-(1-\epsilon^2/\Omega^2)^{1/2}]\sinh(2r)\,, \tag{7.10b}$$

$$\Sigma_{12}=-\Sigma_{21}=\tfrac{1}{2}i(\epsilon/\Omega)\cosh(2r)\,. \tag{7.10c}$$

Letting $\varphi=0$ yields a diagonal covariance matrix $\mathscr{K}_{mn}=\mathrm{Re}(\Sigma_{mn})$, which means that the squeezing of the error ellipse in Fig. 3 occurs along the $\mathscr{E}_1$ and $\mathscr{E}_2$ axes or, equivalently, that the quadrature phases $E_1(x,t)$ and $E_2(x,t)$ have zero second-order correlation. The reduced spectral-density matrix for any squeezed state can be put in the form (7.10) by using rotated quadrature-phase amplitudes $\alpha_1'=\alpha_1\cos\varphi+\alpha_2\sin\varphi$ and $\alpha_2'=-\alpha_1\sin\varphi+\alpha_2\cos\varphi$ [Eqs. (4.36) and (5.19)]. Thus the subset defined by $\varphi=0$ is not so much a special case as it is a convenient choice of phase for defining the quadrature phases—a choice that puts the information about squeezing wholly into the diagonal elements of $\mathscr{K}_{mn}$. For $\varphi=0$ the product of the root-mean-square uncertainties in α_1 and α_2 is given by

$$\langle\,|\Delta\alpha_1|^2\rangle^{1/2}\langle\,|\Delta\alpha_2|^2\rangle^{1/2}=\tfrac{1}{2}[1+(\epsilon^2/\Omega^2)\sinh^2 r]^{1/2}\,. \tag{7.11}$$

In accordance with the proof in Sec. VI and the Appendix [Eq. (6.5)], the uncertainty product (7.11) achieves the minimum value of $\frac{1}{2}$ if and only if $r=0$ (provided $\epsilon\neq 0$).

Consider now what happens as the squeeze factor r increases from $r=0$; choose $\varphi=0$ for easy interpretation. For small r [$\cosh(2r)\ll\Omega/\epsilon$], the mean-square uncertainties in α_1 and α_2 are given approximately by

$$\langle\,|\Delta\alpha_1|^2\rangle\simeq\tfrac{1}{2}e^{-2r},\quad \langle\,|\Delta\alpha_2|^2\rangle\simeq\tfrac{1}{2}e^{2r}\,. \tag{7.12}$$

These mean-square uncertainties are the two-mode analog of the variances that apply in the degenerate limit [see Eq. (8.25)]. They show that $\langle\,|\Delta\alpha_1|^2\rangle$ is squeezed below the zero-point level; in accordance with the uncertainty principle (6.1), $\langle\,|\Delta\alpha_2|^2\rangle$ increases above the zero-point level. As long as $\cosh(2r)<\Omega/\epsilon$, $\langle\,|\Delta\alpha_1|^2\rangle$ continues to decrease as r increases, but it departs more and more from $\frac{1}{2}e^{-2r}$. When $r=r_0>0$, where

$$\cosh(2r_0)\equiv\Omega/\epsilon\,, \tag{7.13a}$$

$$\cosh r_0=(\Omega/2\epsilon)^{1/2}\lambda_+,\quad \sinh r_0=(\Omega/2\epsilon)^{1/2}\lambda_- \tag{7.13b}$$

[Eq. (4.24)], $\langle\,|\Delta\alpha_1|^2\rangle$ achieves the minimum possible value $\epsilon/2\Omega$ [Eq. (6.6)]; thus the state $|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r_0,0)}$ yields a classical excitation of the quadrature phase $E_1(x,t)$, contaminated only by quadrature-phase zero-point noise. Equation (6.7) guarantees that $|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r_0,0)}$ is an eigenstate of $\alpha_1=(\epsilon/\Omega)^{1/2}\alpha_+(r_0,0)$ [Eqs. (4.14), (4.25a), and (7.13b)]:

$$\alpha_1|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r_0,0)}=\xi_1|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r_0,0)}\,, \tag{7.14a}$$

$$\xi_1=(\epsilon/\Omega)^{1/2}\mu_{\alpha_+} \tag{7.14b}$$

[cf. Eq. (4.16)]. For $r>r_0$, $\langle\,|\Delta\alpha_1|^2\rangle$ increases as r increases.

The state $|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r_0,0)}$ belongs to a special class of two-mode squeezed states which we call *squashed states.*[13] The set of squashed states consists of the states $|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r_0,\varphi)}$ for all values of φ. The squashed state $|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r_0,\varphi)}$ is an eigenstate of the rotated quadrature-phase amplitude

$$\alpha_1'=\alpha_1\cos\varphi+\alpha_2\sin\varphi=(\epsilon/\Omega)^{1/2}e^{-i\varphi}\alpha_+(r_0,\varphi)$$

with eigenvalue $(\epsilon/\Omega)^{1/2}\mu_{\alpha_+}e^{-i\varphi}$ [Eqs. (4.14), (4.25), and (4.16)]; hence α_1' has the minimum mean-square uncertainty $\langle\,|\Delta\alpha_1'|^2\rangle=\epsilon/2\Omega$. In particular, $|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r_0,\pi/2)}$ is an eigenstate of $\alpha_2=-i(\epsilon/\Omega)^{1/2}\times\alpha_+(r_0,\frac{1}{2}\pi)$ with eigenvalue $\xi_2=-i(\epsilon/\Omega)^{1/2}\mu_{\alpha_+}$. Initially we hoped that the squashed states, as eigenstates of the quadrature-phase amplitudes, might play a fundamental role in two-photon optics, analogous to the role played by the eigenstates of the annihilation operator—the coherent states—in one-photon optics. Our initial hopes were quashed, however, by our inability to find any special role for the squashed states. In the formalism presented in this series of papers, therefore, the squashed states are on the same footing as all the other two-mode squeezed states.

The mean-square uncertainties in β_1 and β_2 [Eqs. (4.47)] for a two-mode squeezed state can be obtained from Eqs. (7.9) and (7.10) by setting $\epsilon=0$. In particular, for $\varphi=0$ one finds that

$$\langle\,|\Delta\beta_1|^2\rangle=\tfrac{1}{2}e^{-2r},\quad \langle\,|\Delta\beta_2|^2\rangle=\tfrac{1}{2}e^{2r}\,. \tag{7.15}$$

VIII. DEGENERATE LIMIT

A. Definition and discussion

We shift attention now to the degenerate limit of our two-mode formalism. By the degenerate limit we mean that the two modes we have dealt with coalesce into a single mode at frequency Ω. Taking this limit is not an entirely trivial task. An obvious first step is to set $\epsilon=0$, so *we assume* $\epsilon=0$ *throughout the remainder of this subsec-*

tion. This step alone, however, is not sufficient, because it leaves two degenerate, but distinct modes at frequency Ω, which have distinct annihilation operators a_+ and a_-. [Simply setting $\epsilon=0$ would describe, for example, the case where the two modes are plane waves of the same frequency traveling in different directions; see discussion preceding Eqs. (4.5).] To take the desired degenerate limit, one must somehow reduce the number of modes from the two original modes to one mode that corresponds to the coalescence of the two original modes; out of the four original degrees of freedom, one must pick two relevant degrees of freedom and discard the other two.

The key to picking the relevant degrees of freedom is to define new annihilation operators a and b, which are unitarily related to a_+ and a_-:

$$a \equiv 2^{-1/2}(a_+ + a_-), \quad b \equiv 2^{-1/2}(-ia_+ + ia_-) , \tag{8.1a}$$

$$a_\pm = 2^{-1/2}(a \pm ib) . \tag{8.1b}$$

The importance of these new operators becomes apparent when one writes the positive-frequency part of the two-mode electric field [Eq. (4.5b) with $\epsilon=0$] in terms of a and b:

$$E^{(+)}(x,t) = \Omega^{1/2} a e^{-i\Omega(t-x)} . \tag{8.2}$$

One sees that a is the annihilation operator for a plane-wave mode at frequency Ω; it contains the relevant degrees of freedom. In contrast, b does not appear in the electric field; it contains the irrelevant degrees of freedom. One can write the operators introduced in Sec. IV in terms of a and b. For example, the quadrature-phase amplitudes (4.25) become

$$\alpha_m = a_m + ib_m, \quad m=1,2 \tag{8.3}$$

where a_1, a_2, b_1, and b_2 are the Hermitian real and imaginary parts of a and b, i.e.,

$$a = a_1 + ia_2, \quad b = b_1 + ib_2 . \tag{8.4}$$

Thus, another way to characterize the relevant degrees of freedom at degeneracy is that they are the real parts of α_1 and α_2, whereas the irrelevant degrees of freedom are the imaginary parts. In terms of a and b the fundamental unitary operators become

$$U_M(t)\,|_{\epsilon=0} = 1 , \tag{8.5a}$$

$$R(\theta)\,|_{\epsilon=0} = \exp(-i\theta a^\dagger a)\exp(-i\theta b^\dagger b) , \tag{8.5b}$$

$$D(a_+,\mu_+)D(a_-,\mu_-)\,|_{\epsilon=0} = D(a,\mu)D(b,\gamma) , \tag{8.5c}$$

$$S(r,\varphi)\,|_{\epsilon=0} = \exp[\tfrac{1}{2}r(a^2e^{-2i\varphi} - a^{\dagger 2}e^{2i\varphi})] \times \exp[\tfrac{1}{2}r(b^2e^{-2i\varphi} - b^{\dagger 2}e^{2i\varphi})] \tag{8.5d}$$

[Eqs. (4.37), (4.33), (4.12), and (4.9)], where

$$\mu \equiv 2^{-1/2}(\mu_+ + \mu_-) , \tag{8.6a}$$

$$\gamma \equiv 2^{-1/2}(-i\mu_+ + i\mu_-) . \tag{8.6b}$$

Notice that Eq. (8.5a) implies that when $\epsilon=0$ the MP and IP are the same.

The two-mode Hilbert space factors into a tensor (direct) product of Hilbert spaces for the a mode and the b mode. The a-mode Hilbert space is the Hilbert space for the relevant mode at degeneracy. We let tr_b denote a trace over the irrelevant b-mode Hilbert space. We use a subscript a to denote a state vector that lies in the a-mode space or an operator that operates in the a-mode space; a subscript b performs the same role for the b-mode space.

One is now in a position to define the degenerate limit: one reduces the Hilbert space from the two-mode space to the a-mode space; for a state vector or an operator, one extracts a piece that lies in or operates in the a-mode space. To make these notions precise, consider a two-mode density operator ρ. We say that ρ has a degenerate limit if the a mode is independent of the irrelevant b mode, so that no matter what operation is performed on the b mode, the a-mode is unaffected. Hence, *a density operator ρ has a (unique) degenerate limit $\rho_a \equiv \mathrm{tr}_b(\rho)$ if $\rho=\rho_a\rho_b$; we denote this limit by*

$$\rho \underset{p}{\rightarrow} \rho_a . \tag{8.7}$$

Similarly, *a state vector $|\psi\rangle$ has a degenerate limit $|\psi_a\rangle$, denoted by*

$$|\psi\rangle \underset{p}{\rightarrow} |\psi_a\rangle , \tag{8.8}$$

if $|\psi\rangle = |\psi_a\rangle \otimes |\psi_b\rangle$; requiring that $|\psi_a\rangle$ be normalized makes this limit unique up to an arbitrary phase factor. The limits (8.7) and (8.8) have an obvious extension to unitary operators. *A unitary operator U has a degenerate limit U_a, denoted by*

$$U \underset{p}{\rightarrow} U_a , \tag{8.9}$$

if $U=U_aU_b$; requiring that U_a be unitary makes this limit unique up to an arbitrary phase factor. In Eqs. (8.7)–(8.9), the p under the arrow signifies that these are *product degenerate limits;* i.e., each requires that the relevant quantity factor into a product of an a-mode quantity times a b-mode quantity. The limits (8.7) and (8.9) could easily be extended to a product degenerate limit for arbitrary operators, but we have no need for such a generalization here. For present purposes the important properties of the product degenerate limit are that

$$U \underset{p}{\rightarrow} U_a, \quad \rho \underset{p}{\rightarrow} \rho_a \Longrightarrow U\rho U^\dagger \underset{p}{\rightarrow} U_a\rho_a U_a^\dagger , \tag{8.10a}$$

$$U \underset{p}{\rightarrow} U_a, \quad |\psi\rangle \underset{p}{\rightarrow} |\psi_a\rangle \Longrightarrow U|\psi\rangle \underset{p}{\rightarrow} U_a|\psi_a\rangle . \tag{8.10b}$$

For observable quantities or for non-Hermitian operators like the quadrature-phase amplitudes, a different degenerate limit is appropriate. Consider an arbitrary operator R. We say that *R has a sum degenerate limit R_a, denoted by*

$$R \underset{s}{\rightarrow} R_a , \tag{8.11}$$

if $R=R_a+R_b$. The motivation for this definition is that for a state ρ with a degenerate limit, R_a and R_b are uncorrelated. The sum degenerate limit (8.11) is defined only up to an arbitrary additive constant.

Having specified how to take degenerate limits, we now consider the limits of the two-mode quantities introduced in Sec. IV. We adopt the sensible convention that the limit of an SP operator is an SP operator, and the limit of an IP or a MP operator is an IP operator. For normalization purposes we define the sum degenerate limits of the IP two-mode electric field operator [Eqs. (4.5)] and the IP quadrature phases [Eqs. (4.19)] to be $2^{1/2}$ times a quantity denoted by the same symbol [e.g., $E^{(+)}(x,t) \underset{s}{\rightarrow} 2^{1/2}(\Omega/2)^{1/2} a e^{-i\Omega(t-x)} \equiv 2^{1/2}E^{(+)}(x,t)$; cf. Eq. (8.2)]; with this choice the degenerate limit of the IP two-mode electric field operator yields the IP single-mode electric field operator defined by Eqs. (3.3) with $\omega=\Omega$, and the relations between the electric field and the quadrature phases [Eqs. (4.19)—(4.21)] retain the same form in the degenerate limit. The sum degenerate limit of the SP annihilation operators [Eq. (8.1b)],

$$a_{\pm} \underset{s}{\rightarrow} 2^{-1/2} , \tag{8.12}$$

suggests defining the sum degenerate limit of the MP squeezed annihilation operators (4.14) in the following way:

$$\alpha_{\pm}(r,\varphi) \underset{s}{\rightarrow} 2^{-1/2}(a \cosh r + a^{\dagger} e^{2i\varphi} \sinh r) \equiv 2^{-1/2}\alpha(r,\varphi) . \tag{8.13}$$

The MP quadrature-phase amplitudes (4.25) have a Hermitian sum degenerate limit

$$\alpha_m \underset{s}{\rightarrow} a_m \equiv 2^{-1/2}x_m, \quad m=1,2 \tag{8.14}$$

[Eq. (8.3)].

The loss of two degrees of freedom at degeneracy erases the distinction between the quadrature phases and the quadrature-phase amplitudes: the IP quadrature phases, which are initially Hermitian operators with harmonic time dependence at frequency ϵ, become constant in the degenerate limit; the MP quadrature-phase amplitudes, which are initially (constant) complex-amplitude operators, become Hermitian in the degenerate limit. As a result, at degeneracy there are three Hermitian IP operators,

$$E_m(x,t)=(2\Omega)^{1/2}a_m=\Omega^{1/2}x_m , \tag{8.15}$$

all of which are constant and any of which could be called a quadrature phase or a quadrature-phase amplitude.[2] We prefer to give x_1 and x_2 the distinction of being the (degenerate) *quadrature-phase amplitudes,* because their relation to the annihilation operator has the same form as Eqs. (4.25) with $\epsilon=0$, i.e.,

$$x_1=2^{-1/2}(a+a^{\dagger}) , \tag{8.16a}$$

$$x_2=2^{-1/2}(-ia+ia^{\dagger}) , \tag{8.16b}$$

and because their commutator

$$[x_1,x_2]=i \tag{8.17}$$

enforces the same uncertainty principle as Eq. (6.1), i.e.,

$$\langle(\Delta x_1)^2\rangle^{1/2}\langle(\Delta x_2)^2\rangle^{1/2} \geq \tfrac{1}{2} \tag{8.18}$$

[cf. Eq. (6.3)].

The fundamental unitary operators introduced in Sec. IV [Eqs. (8.5)] have the following (unitary) product degenerate limits:

$$U_M(t) \underset{p}{\rightarrow} 1 , \tag{8.19a}$$

$$R(\theta) \underset{p}{\rightarrow} \exp(-i\theta a^{\dagger}a) , \tag{8.19b}$$

$$D(a_+,\mu_+)D(a_-,\mu_-) \underset{p}{\rightarrow} D(a,\mu), \quad \mu \equiv 2^{-1/2}(\mu_+ + \mu_-) , \tag{8.19c}$$

$$S(r,\varphi) \underset{p}{\rightarrow} \exp[\tfrac{1}{2}r(a^2e^{-2i\varphi}-a^{\dagger 2}e^{2i\varphi})] \equiv S_1(r,\varphi) . \tag{8.19d}$$

The MP free evolution operator $U_M(t)$ becomes the identity operator, the rotation operator $R(\theta)$ becomes a single-mode rotation operator, the two-mode displacement operator $D(a_+,\mu_+)D(a_-,\mu_-)$ becomes the single-mode displacement operator (3.7), and the two-mode squeeze operator $S(r,\varphi)$ becomes the *degenerate squeeze operator*[42,43] $S_1(r,\varphi)$. Under a unitary transformation generated by $S_1(r,\varphi)$ the annihilation operator a becomes the *squeezed annihilation operator* $\alpha(r,\varphi)$ [Eq. (8.13)]:

$$\alpha(r,\varphi)=S_1(r,\varphi)aS_1^{\dagger}(r,\varphi)=a \cosh r + a^{\dagger}e^{2i\varphi}\sinh r \tag{8.20}$$

[cf. Eq. (4.14)]. For $\varphi=0$ the degenerate squeeze operator transforms the quadrature-phase amplitudes according to

$$S_1^{\dagger}(r,0)x_1S_1(r,0)=x_1e^{-r} , \tag{8.21a}$$

$$S_1^{\dagger}(r,0)x_2S_1(r,0)=x_2e^{r} . \tag{8.21b}$$

The degenerate limits (8.19) can be applied to obtain the degenerate limits of the special states defined in Sec. IV. The product degenerate limit of a two-mode coherent state [Eq. (4.11)] is a single-mode coherent state [Eq. (3.9)]:

$$|\mu_+,\mu_-\rangle_{\text{coh}} \underset{p}{\rightarrow} |\mu\rangle_{\text{coh}}, \quad \mu \equiv 2^{-1/2}(\mu_+ + \mu_-) . \tag{8.22}$$

The product degenerate limit of a two-mode squeezed state [Eq. (4.15)] is a *degenerate squeezed state*[42,44,2] $|\mu_\alpha\rangle_{(r,\varphi)}$:

$$|\mu_{\alpha_+},\mu_{\alpha_-}\rangle_{(r,\varphi)} \underset{p}{\rightarrow} S_1(r,\varphi)D(a,\mu_\alpha)|0\rangle = D(a,\mu)S_1(r,\varphi)|0\rangle \equiv |\mu_\alpha\rangle_{(r,\varphi)} , \tag{8.23a}$$

$$\mu_\alpha \equiv 2^{-1/2}(\mu_{\alpha_+}+\mu_{\alpha_-})=\mu \cosh r + \mu^* e^{2i\varphi}\sinh r \tag{8.23b}$$

[Eqs. (4.17) and (4.18)]. A degenerate squeezed state is labeled by the eigenvalue of $\alpha(r,\varphi)$ [Eq. (8.20)]:

$$\alpha(r,\varphi)|\mu_\alpha\rangle_{(r,\varphi)}=\mu_\alpha|\mu_\alpha\rangle_{(r,\varphi)} \tag{8.24}$$

[cf. Eq. (4.16)]. The quadrature-phase amplitudes have the following variances in a degenerate squeezed state with $\varphi=0$:

$$\langle(\Delta x_1)^2\rangle=\tfrac{1}{2}e^{-2r}, \quad \langle(\Delta x_2)^2\rangle=\tfrac{1}{2}e^{2r} \tag{8.25}$$

[cf. Eqs. (7.12)].

B. Review of previous work

Degenerate squeezed states were introduced independently by Stoler[42,45] ("minimum-uncertainty packets") and Lu[43,44] ("new coherent states"), both of whom used the degenerate squeeze operator to generate squeezed states from coherent states. The first comprehensive treatment of squeezed states is due to Yuen,[2] who called them "two-photon coherent states" because of their generation by ideal two-photon processes. Yuen explored in detail the properties of degenerate squeezed states, and he discussed several physical mechanisms for generating them. In this series of papers we adopt Yuen's notational convention, which labels a degenerate squeezed state by the eigenvalue of the squeezed annihilation operator. Not long after Yuen's paper, Yuen, Shapiro,[5,7] and Machado Mata[6] developed the theory of optical communications using squeezed states. At about the same time Hollenhorst[40] introduced squeezed states into the theory of "quantum nondemolition measurements."[46] Hollenhorst coined the term squeezed and applied it to the degenerate squeeze operator (in Ref. 1 the term was extended in an obvious way to apply to the states themselves). Hollenhorst's work led to the realization[1] that squeezed states could be used to improve the sensitivity of laser interferometers used to detect gravitational waves. In the last few years there has been an explosion of interest in squeezed states.[3,4] Optical communications and high-precision measurements remain their primary potential applications, but interest is also fueled by a desire to investigate their nonclassical behavior.[3]

In unpublished work Yuen[47] has considered general multimode squeezed states. Yuen and Shapiro[7] and Milburn[30] have defined two-mode or multimode squeezed states, but the states they define are simply tensor (direct) products of degenerate squeezed states for each mode. There is a formal sense, realized by Lu[43] and pointed out explicitly by Milburn,[30] in which the two-mode squeezed states defined here can be regarded as a tensor product of two degenerate squeezed states. For any value of ϵ one can define the operators a and b of Eqs. (8.1), and one can write the two-mode displacement operator and the two-mode squeeze operator in terms of a and b as in Eqs. (8.5c) and (8.5d). Thus a two-mode squeezed state (4.17) factors into a tensor product of degenerate squeezed states for the "a mode" and the "b mode."

The difficulty with this description is that unless $\epsilon=0$ the operators a and b are not modal annihilation operators because they do not have a harmonic time dependence in the IP. The operators a_+ and a_-—not a and b—appear in a modal decomposition of the electromagnetic field. Formally, it is correct to describe a two-mode squeezed state as a product of degenerate squeezed states for the "a mode" and the "b mode," and this description does permit one to obtain properties of two-mode squeezed states directly from properties of degenerate squeezed states. Physically, however, this description is very misleading, because it can easily lead one to believe that the way to produce nondegenerate (wide-band) squeezing is to squeeze separately two different modes. In reality, wide-band squeezing does not result from separately squeezing different modes [see Eq. (5.6a)]; rather, it is a consequence of a special sort of correlation between two modes symmetrically placed about a carrier frequency [Eqs. (5.8b) and (5.8c)]. Such correlation is produced by ideal two-photon devices, and it is the feature that characterizes two-photon optics.

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APPENDIX: UNCERTAINTY PRINCIPLES FOR NON-HERMITIAN OPERATORS

In this Appendix we derive and discuss uncertainty principles that apply to the mean-square uncertainties of non-Hermitian operators. Our immediate objective is to derive the uncertainty principles for α_1 and α_2 which are given in Sec. VI. The derivations are more general,[48] however, than the special case of α_1 and α_2, because we do not restrict ourselves to operators with c-number commutators. Since the uncertainty principles for non-Hermitian operators are based on the uncertainty principles for their Hermitian real and imaginary parts, we begin by reviewing the standard uncertainty principle for two Hermitian operators. The notation we use here is introduced in Sec. II.

1. Two Hermitian operators

Consider two Hermitian operators B and C. They satisfy the ordinary uncertainty principle for the product of their uncertainties:

$$\langle(\Delta B)^2\rangle^{1/2}\langle(\Delta C)^2\rangle^{1/2}\geq \tfrac{1}{2}\,|\langle[B,C]\rangle|\ . \tag{A1}$$

The derivation of Eq. (A1) can be found in most quantum-mechanics textbooks (see, e.g., Chap. 8.6 of Ref. 27). Equality holds in Eq. (A1) if and only if the state vector $|\Psi\rangle$ is an eigenstate of a particular linear combination of B and C:

$$(\Delta B+i\beta\,\Delta C)\,|\Psi\rangle=0\ , \tag{A2a}$$

$$\begin{aligned}\beta &\equiv -i\frac{\langle[B,C]\rangle}{|\langle[B,C]\rangle|}\frac{\langle(\Delta B)^2\rangle^{1/2}}{\langle(\Delta C)^2\rangle^{1/2}}\\ &=2i\frac{\langle(\Delta B)^2\rangle}{\langle[B,C]\rangle}=-\tfrac{1}{2}i\frac{\langle[B,C]\rangle}{\langle(\Delta C)^2\rangle}\ .\end{aligned} \tag{A2b}$$

Notice that β is real because $\langle[B,C]\rangle$ is pure imaginary. Equality in Eq. (A1) implies that B and C have zero second-order correlation, i.e.,

$$\langle \Delta B\,\Delta C\rangle_{\rm sym}=\langle BC\rangle_{\rm sym}-\langle B\rangle\langle C\rangle=0\ . \tag{A3}$$

2. One non-Hermitian operator

Let R be a general, possibly non-Hermitian operator. We want to derive a lower limit for its mean-square uncertainty $\langle |\Delta R|^2\rangle$ [Eq. (2.9)]. An instructive approach is to consider its Hermitian real and imaginary parts $R_1 \equiv \mathrm{Re}(R) = \frac{1}{2}(R+R^\dagger)$ and $R_2 \equiv \mathrm{Im}(R) = -\frac{1}{2}i(R-R^\dagger)$, i.e.,

$$R = R_1 + iR_2 \ . \tag{A4}$$

It is useful to note the following relations among operators:

$$(\Delta R)^2 = (\Delta R_1)^2 - (\Delta R_2)^2 + 2i(\Delta R_1\, \Delta R_2)_{\mathrm{sym}} \ , \tag{A5}$$

$$|\Delta R|^2 = (\Delta R_1)^2 + (\Delta R_2)^2 \ , \tag{A6}$$

$$[R,R^\dagger] = -2i[R_1,R_2] \ . \tag{A7}$$

Notice that $\langle [R,R^\dagger]\rangle$ is real.

By noting that

$$\begin{aligned}\langle |\Delta R|^2\rangle &= \langle (\Delta R_1)^2\rangle + \langle (\Delta R_2)^2\rangle \\ &\geq 2\langle (\Delta R_1)^2\rangle^{1/2}\langle (\Delta R_2)^2\rangle^{1/2} \ ,\end{aligned} \tag{A8}$$

one can use the ordinary uncertainty principle (A1), applied to R_1 and R_2, to establish a lower limit for $\langle |\Delta R|^2\rangle$:

$$\langle |\Delta R|^2\rangle \geq \tfrac{1}{2}|\langle [R,R^\dagger]\rangle| = |\langle [R_1,R_2]\rangle| \ . \tag{A9}$$

This derivation makes clear that equality in Eq. (A9) is equivalent to each of the following: (i) R_1 and R_2 have equal uncertainties, which have the minimum-uncertainty product, i.e.,

$$\langle (\Delta R_1)^2\rangle = \langle (\Delta R_2)^2\rangle = \tfrac{1}{2}|\langle [R_1,R_2]\rangle| \ ; \tag{A10}$$

(ii) the state vector $|\Psi\rangle$ satisfies

$$\left[\Delta R_1 + i\frac{\langle [R,R^\dagger]\rangle}{|\langle [R,R^\dagger]\rangle|}\Delta R_2\right]|\Psi\rangle = 0 \tag{A11}$$

[Eqs. (A2) and (A7)]; (iii) the state vector $|\Psi\rangle$ satisfies

$$\Delta R|\Psi\rangle = 0 \ \text{ if } \ \langle [R,R^\dagger]\rangle \geq 0 \ , \tag{A12a}$$

$$\Delta R^\dagger|\Psi\rangle = 0 \ \text{ if } \ \langle [R,R^\dagger]\rangle \leq 0 \ . \tag{A12b}$$

Equality in Eq. (A9) implies

$$\langle (\Delta R)^2\rangle = 0 \ . \tag{A13}$$

The uncertainty principle (A9) can also be obtained directly without introducing R_1 and R_2. One writes the mean-square uncertainty in two ways which imply two lower limits:

$$\langle |\Delta R|^2\rangle = \langle \Delta R^\dagger \Delta R\rangle + \tfrac{1}{2}\langle [R,R^\dagger]\rangle \geq \tfrac{1}{2}\langle [R,R^\dagger]\rangle \ , \tag{A14a}$$

$$\langle |\Delta R|^2\rangle = \langle \Delta R\, \Delta R^\dagger\rangle - \tfrac{1}{2}\langle [R,R^\dagger]\rangle \geq -\tfrac{1}{2}\langle [R,R^\dagger]\rangle \ . \tag{A14b}$$

Equations (A14) imply the uncertainty principle (A9). If $\langle [R,R^\dagger]\rangle \geq 0$, then equality holds in Eq. (A14a) if and only if $\langle \Delta R^\dagger \Delta R\rangle = 0$, which is equivalent to Eq. (A12a); similarly, if $\langle [R,R^\dagger]\rangle \leq 0$, then equality holds in Eq. (A14b) if and only if $\langle \Delta R\, \Delta R^\dagger\rangle = 0$, which is equivalent to Eq. (A12b).

3. Two commuting non-Hermitian operators

Consider now two general,[48] possibly non-Hermitian operators R and S which commute:

$$[R,S] = 0 \ ; \tag{A15}$$

thus the important commutator is $[R,S^\dagger] = -[R^\dagger,S]^\dagger$. In analogy with the ordinary uncertainty principle (A1), one might expect $|\langle [R,S^\dagger]\rangle|$ to set a lower limit on the product of the root-mean-square uncertainties in R and S. Indeed, the main result of this subsection is that

$$\langle |\Delta R|^2\rangle^{1/2}\langle |\Delta S|^2\rangle^{1/2} \geq \tfrac{1}{2}|\langle [R,S^\dagger]\rangle| \ , \tag{A16}$$

an uncertainty principle that bears a striking resemblance to the ordinary uncertainty principle (A1).

The uncertainty principle (A16) is a consequence of the ordinary uncertainty principles for the real and imaginary parts of R and S. We therefore begin a proof of Eq. (A16) by introducing the Hermitian real and imaginary parts of R as in Eq. (A4) and by introducing the Hermitian real and imaginary parts of $e^{i\lambda}S$,

$$S_1 \equiv \tfrac{1}{2}(e^{i\lambda}S + e^{-i\lambda}S^\dagger), \ \ S_2 \equiv -\tfrac{1}{2}i(e^{i\lambda}S - e^{-i\lambda}S^\dagger) \ , \tag{A17a}$$

$$S = e^{-i\lambda}(S_1 + iS_2) \ , \tag{A17b}$$

where $e^{i\lambda}$ is an arbitrary phase factor. For different values of λ the operators S_1 and S_2 are different linear combinations of the real and imaginary parts of S, but the mean-square uncertainty in S is still given by

$$\langle |\Delta S|^2\rangle = \langle (\Delta S_1)^2\rangle + \langle (\Delta S_2)^2\rangle \tag{A18}$$

[cf. Eq. (A8)]. In what follows we derive lower limits on $\langle |\Delta R|^2\rangle^{1/2}\langle |\Delta S|^2\rangle^{1/2}$ which depend on λ, and we then choose λ to enforce the most stringent limit. Using Eq. (A15), one can derive the following commutators:

$$\begin{aligned}[R_1,S_1] = [R_2,S_2] &= \tfrac{1}{4}(e^{-i\lambda}[R,S^\dagger] + e^{i\lambda}[R^\dagger,S]) \\ &= \tfrac{1}{2}i\,\mathrm{Im}(e^{-i\lambda}[R,S^\dagger]) \ ,\end{aligned} \tag{A19a}$$

$$\begin{aligned}[R_1,S_2] = -[R_2,S_1] &= \tfrac{1}{4}i(e^{-i\lambda}[R,S^\dagger] - e^{i\lambda}[R^\dagger,S]) \\ &= \tfrac{1}{2}i\,\mathrm{Re}(e^{-i\lambda}[R,S^\dagger]) \ .\end{aligned} \tag{A19b}$$

The notation is made less cumbersome by introducing the symbols

$$r_j \equiv \langle (\Delta R_j)^2\rangle^{1/2} \geq 0, \ \ s_j \equiv \langle (\Delta S_j)^2\rangle^{1/2} \geq 0, \ \ j = 1,2 \ . \tag{A20}$$

The commutators (A19) enforce four uncertainty principles [Eq. (A1)],

$$r_1 s_1 \geq \tfrac{1}{4}c\,|\sin(\delta-\lambda)| \ , \tag{A21a}$$

$$r_2 s_2 \geq \tfrac{1}{4}c\,|\sin(\delta-\lambda)| \ , \tag{A21b}$$

$$r_1 s_2 \geq \tfrac{1}{4}c\,|\cos(\delta-\lambda)| \ , \tag{A21c}$$

$$r_2 s_1 \geq \tfrac{1}{4}c\,|\cos(\delta-\lambda)| \ , \tag{A21d}$$

where we define

$$\langle[R,S^\dagger]\rangle \equiv ce^{i\delta},\quad c\equiv|\langle[R,S^\dagger]\rangle|\ . \tag{A22}$$

Hence, the problem is to minimize

$$\begin{aligned}\langle|\Delta R|^2\rangle\langle|\Delta S|^2\rangle &= (r_1^2+r_2^2)(s_1^2+s_2^2)\\ &= |(r_1+ir_2)(s_1+is_2)|^2\ ,\end{aligned} \tag{A23}$$

subject to the constraints (A21). An easy way to do this is to write Eq. (A23) in two ways, which lead to two different lower limits:

$$\begin{aligned}\langle|\Delta R|^2\rangle\langle|\Delta S|^2\rangle &= (r_1s_1-r_2s_2)^2+(r_1s_2+r_2s_1)^2\\ &\geq \tfrac{1}{4}c^2\cos^2(\delta-\lambda)\ ,\end{aligned} \tag{A24a}$$

$$\begin{aligned}\langle|\Delta R|^2\rangle\langle|\Delta S|^2\rangle &= (r_1s_1+r_2s_2)^2+(r_1s_2-r_2s_1)^2\\ &\geq \tfrac{1}{4}c^2\sin^2(\delta-\lambda)\ .\end{aligned} \tag{A24b}$$

If one chooses $\lambda=\delta$ ($\lambda=\delta-\pi/2$), then Eq. (A24a) [Eq. (A24b)] implies the uncertainty principle (A16).

The operators S_1 and S_2 defined by Eqs. (A17) with $\lambda=\delta-\pi/2$ (or, equivalently, the operators S_2 and $-S_1$ defined by $\lambda=\delta$) bear a special relationship to R_1 and R_2. For $\lambda=\delta-\pi/2$, Eq. (A24b) shows that equality in Eq. (A16) is equivalent to each of the following two statements: (i) R_1, R_2, S_1, and S_2 satisfy

$$\langle(\Delta R_1)^2\rangle=\langle(\Delta R_2)^2\rangle,\quad \langle(\Delta S_1)^2\rangle=\langle(\Delta S_2)^2\rangle\ , \tag{A25a}$$

$$\begin{aligned}\langle(\Delta R_1)^2\rangle^{1/2}\langle(\Delta S_1)^2\rangle^{1/2} &= \langle(\Delta R_2)^2\rangle^{1/2}\langle(\Delta S_2)^2\rangle^{1/2}\\ &= \tfrac{1}{4}|\langle[R,S^\dagger]\rangle|\ ;\end{aligned} \tag{A25b}$$

(ii) the state vector $|\Psi\rangle$ satisfies

$$(\Delta R_1+i\gamma\,\Delta S_1)|\Psi\rangle=0\ , \tag{A26a}$$

$$(\Delta R_2+i\gamma\,\Delta S_2)|\Psi\rangle=0\ , \tag{A26b}$$

$$\gamma\equiv\langle|\Delta R|^2\rangle^{1/2}/\langle|\Delta S|^2\rangle^{1/2} \tag{A26c}$$

[Eqs. (A2)]. By taking appropriate linear combinations of Eqs. (A26a) and (A26b), one can show that equality holds in Eq. (A16) if and only if

$$(\Delta R+\gamma e^{i\delta}\Delta S)|\Psi\rangle=0\ , \tag{A27a}$$

$$(\Delta R^\dagger-\gamma e^{-i\delta}\Delta S^\dagger)|\Psi\rangle=0\ , \tag{A27b}$$

$$\begin{aligned}\gamma e^{i\delta} &\equiv \frac{\langle[R,S^\dagger]\rangle}{|\langle[R,S^\dagger]\rangle|}\,\frac{\langle|\Delta R|^2\rangle^{1/2}}{\langle|\Delta S|^2\rangle^{1/2}}\\ &= 2\frac{\langle|\Delta R|^2\rangle}{\langle[R,S^\dagger]\rangle^*}=\frac{1}{2}\frac{\langle[R,S^\dagger]\rangle}{\langle|\Delta S|^2\rangle}\end{aligned} \tag{A27c}$$

[Eqs. (A22) and (A26c); cf. Eqs. (A2)]. Equations (A27) do not depend on any special choice for λ. They can be used to show that equality in Eq. (A16) implies the following:

$$\langle(\Delta R)^2\rangle=\langle(\Delta S)^2\rangle=\langle\Delta R\,\Delta S\rangle=0\ , \tag{A28a}$$

$$\begin{aligned}\langle\Delta R\,\Delta S^\dagger\rangle_{\text{sym}} &= \tfrac{1}{2}\gamma e^{i\delta}\langle[S,S^\dagger]\rangle\\ &= \tfrac{1}{2}\gamma^{-1}e^{i\delta}\langle[R,R^\dagger]\rangle\ .\end{aligned} \tag{A28b}$$

A simple, but important consequence of Eqs. (A27c) and (A28b) is that equality in the uncertainty principle (A16) implies

$$\gamma^2=\frac{\langle|\Delta R|^2\rangle}{\langle|\Delta S|^2\rangle}=\frac{\langle[R,R^\dagger]\rangle}{\langle[S,S^\dagger]\rangle}\ , \tag{A29}$$

provided that $\langle[R,R^\dagger]\rangle\neq0\neq\langle[S,S^\dagger]\rangle$.

Equation (A28a) shows that equality in Eq. (A16) implies $\langle(\Delta R_1)^2\rangle=\langle(\Delta R_2)^2\rangle$ and $\langle(\Delta S_1)^2\rangle=\langle(\Delta S_2)^2\rangle$, regardless of the choice of λ. This tells one immediately that Eqs. (A25) are a consequence of equality in Eq. (A16), regardless of the choice of λ. Equally true is that Eqs. (A25) imply equality in Eq. (A16), regardless of the choice of λ. On the other hand, only for the special choices $\lambda=\delta-\pi/2$ and $\lambda=\delta$ (or their equivalents) are Eqs. (A25) equivalent to eigenvalue equations like Eqs. (A26), because only for these special choices is Eq. (A25b) a minimum-uncertainty product [cf. Eqs. (A21)]. Thus it is the eigenvalue equations (A26) that pick out the operators S_1 and S_2 defined by $\lambda=\delta-\pi/2$.

An alternative method of proving the uncertainty principle (A16) goes as follows. Choose for illustration $\lambda=\delta-\pi/2$; the problem is then to minimize $f(r_1,r_2,s_1,s_2)\equiv(r_1^2+r_2^2)(s_1^2+s_2^2)$ [Eq. (A23)], subject to the constraints $r_1s_1\geq\frac{1}{4}c$ and $r_2s_2\geq\frac{1}{4}c$ [Eqs. (A21a) and (A21b)]. As a first step, minimize f on the hypersurface $r_1r_2s_1s_2=K^2\geq c^2/16$, where K is a constant. The minimum value $f=4K^2$ can be found by using a Lagrange multiplier to enforce the hypersurface constraint; the minimum occurs when $r_1=r_2$, $s_1=s_2$, $r_1s_1=r_2s_2=K$. Now vary K to find the absolute minimum consistent with the constraints; the obvious answer is $K=\frac{1}{4}c$, which yields an absolute minimum value $f=\frac{1}{4}c^2$.

It should be remembered that the uncertainty principle (A16) is not the whole story, since it is based only on the commutator $[R,S^\dagger]$. It is quite possible that the constraint

$$\langle|\Delta R|^2\rangle\langle|\Delta S|^2\rangle\geq\tfrac{1}{4}|\langle[R,R^\dagger]\rangle||\langle[S,S^\dagger]\rangle|\ , \tag{A30}$$

which follows from the separate uncertainty principles for R and S [Eq. (A9)], provides a more stringent lower limit than Eq. (A16).

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