Cloud of virtual photons in the ground state of the hydrogen atom

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A spinless, nonrelativistic hydrogen atom coupled to an electromagnetic field is considered. The interaction is taken in the minimal-coupling form, and the ground state of the coupled system is obtained by straightforward perturbation theory. The form of the cloud of virtual photons surrounding the atom is studied through the quantum-mechanical average on this state of an appropriately defined coarse-grained energy-density (CGED) operator $W(\mathbf{r})$. The properties of $W(\mathbf{r})$ are studied in order to show that this operator can give a reliable description of the shape of the virtual photon cloud. The quantum-mechanical average of $W(\mathbf{r})$ is obtained analytically and exactly, and the CGED is shown to consist of an infinite number of spherically symmetric contributions, each originating from the set of virtual transitions induced by the vacuum fluctuations between the bare atomic ground state and all the bare atomic eigenstates belonging to a given subshell. This yields a shell and subshell structure for the virtual photon cloud, each of them characterized by a different behavior of the CGED as a function of the distance from the atom. The details of this structure are studied both analytically and numerically, and the results obtained are compared to those pertaining to virtual clouds in other fields of physics.

I. INTRODUCTION

The first theoretical evidence that the main contributions to quantum-electrodynamical Lamb shift in the atom-radiation interaction are essentially nonrelativistic was presented by Bethe in 1947, who was also able to explain mass renormalization in the same terms.[1] Relativity was only used by Bethe to provide an upper limit (cutoff) $mc^2/\hbar$ to the frequencies of the radiation field which contribute to the shift of the atomic levels in order to avoid uv divergences. This contribution formed the basis for successive developments in the theory of nonrelativistic quantum electrodynamics. Welton[2] produced a very elegant explanation of the Lamb shift as arising from the averaging of the Coulomb potential acting on a bound electron, due to the random zero-point quantum fluctuations of the vacuum. Starting from an apparently disconnected point of view, Van Hove[3] ascribed the divergences typical of quantum-field theory to persistent occurrence, around each particle, of clouds of virtual particles, and he related these cloud effects to the concepts of charge and mass renormalization, familiar in quantum electrodynamics. More recently Moses[4] was able to include retardation effects exactly in the calculation of matrix elements and transition probabilities for hydrogenic atoms, thereby showing the existence of a natural cutoff at frequencies $\sim c/a_0$, where c is the velocity of light and a_0 the Bohr radius. Au and Feinberg[5] used a Green's-function technique to investigate retardation effects in the photon emission and absorption, yielding support to Moses's conclusions on the existence of a low-frequency natural cutoff; their numerical calculations also show that the Lamb shift, as obtained by the introduction of an abrupt cutoff at c/a_0, does not differ much from that obtained by taking exactly into account retardation effects. Further support to Welton's model for the Lamb shift has come from the work of Dupont-Roc *et al.*,[6] while Davidovich and Nussenzveig have used Moses's results and Van Hove's resolvent approach to obtain a nonrelativistic expression for the Lamb shift;[7] although their contribution is aimed at an investigation of the fluorescence line shape, they discuss at length the nature of cloud effects in the atom-field interaction and they point out the importance of the so-called counter-rotating terms in the interaction Hamiltonian in relation with these persistent effects. Quite recently Bykov and Zadernovsky[8] have calculated the time-dependent field originating from spontaneous decay of an excited atom, thereby proving the retarded nature of this field, and Grotch[9] has revisited the theory of Lamb shift and mass renormalization in a retarded theory framework taking also into account the hitherto neglected $\boldsymbol{\sigma}\cdot\mathbf{B}$ interaction, thereby obtaining a 2.5% agreement with the best available theories of the Lamb shift. Finally, Dalibard *et al.* have succeeded in relating in a qualitatively satisfactory way mass renormalization and Lamb shift, respectively, to the self-reaction field of the electron and to the vacuum zero-point fluctuations.[10]

The above discussion, although very schematic and far from complete, serves to show that in spite of progress made, little research has been aimed at obtaining an explicit answer in quantitative terms for the shape of the virtual photon cloud in the region of space around an atom interacting with the electromagnetic field. That such a cloud should exist is more or less implicit in all of the work cited above and rather explicitly discussed by Davidovich and Nussenzweig;[7] moreover, some of the time-dependent properties of the field around an atom constitute the main objective of the paper by Bykov and Zadernovskii.[8] In the latter case, however, the interest is centered on the radiation that develops during a process of spontaneous emission, in which virtual and real contributions are difficult to sort out; it is clear, however, that if the total system (atom plus radiation field) is in its ground state, the contributions of real photons are absent and cloud effects are only due to virtual photons. A step in this direction has been performed[11] by sorting out in the total energy shift of the ground state of a hydrogen atom the contributions originating from the field of the cloud of virtual photons. Although this emphasizes the role of the cloud, the question of its space dependence has remained open.

On the other hand, the problem of the detailed form of the cloud of virtual particles dressing the sources of the field has aroused a persistent interest in several fields of physics. Peeters and Devreese[12] have calculated the polarization charge density induced by an electron in a polar semiconductor for arbitrary strength of the electron-phonon coupling, temperature, and external magnetic field, obtaining an appealing and physically transparent picture of the polaron. Thomas, Theberge, and Miller[13] have incorporated chiral invariance in the MIT (Massachusetts Institute of Technology) bag model for the nucleon, and have developed an approximately linear coupling theory of the coupling between the three-quark bag model and the pion field. Thereafter they obtain the charge densities for the virtual pion field surrounding the bag and the proton and neutron charge densities, in good agreement with the available experimental results. Power and Thirunamachandran[14] have considered the problem of the electromagnetic field around a neutral atom. Working in the Heisenberg representation, they obtain solutions of the equations of motion for the field operators, valid up to terms quadratic in the atom-field coupling constant which they have used to calculate the rate of energy transfer between two atoms and intermolecular potential energies, directly related to Van der Waals forces. Although their work is not directly concerned with the shape of the virtual photon cloud, it is likely that their technique could be used also to obtain information of this sort.

The aim of this paper is to present a detailed calculation for the shape of the cloud of virtual photons surrounding a hydrogen atom in the ground state of the coupled system (atom plus photon field). We shall develop our theory in a nonrelativistic, minimal-coupling framework, and we shall assume a spinless electron. We use straightforward second-order perturbation theory to obtain the perturbed (or dressed) ground state of the system, and we perform our calculation taking full account of retardation effects, in such a way that we do not need any phenomenological frequency cutoff. In order to describe the shape of the cloud of virtual photons, we calculate the quantum-mechanical average on the perturbed ground state of a field operator $W(r)$, which we call the coarse-grained energy density (CGED), whose properties shall be defined later on in the course of this paper. In fact, $W(\mathbf{r})$ does not yield directly the density of virtual photons in space, since, as we shall see, it is rather related to the space distribution of the energy carried by each photon in space. Our choice of $W(\mathbf{r})$ to represent the virtual photon cloud has been based primarily on the fact that the mentioned quantum-mechanical averages can be performed exactly, which is a distinct advantage over other operators which may be more directly related to the density of photons,[15] but whose quantum-mechanical averages are in practice more difficult to obtain exactly. Finally, we shall discuss the results obtained by our approach, elucidating various aspects of the structure that we have obtained for our representation of the virtual photon distribution.

II. THE ELECTROMAGNETIC FIELD

Assuming periodic boundary conditions on the surface of a cubic volume V of space, the usual normal-mode expansion for the transverse part of the electromagnetic field yields

$$\mathbf{A}_\perp(\mathbf{r}) \equiv \mathbf{A}_\perp^+(\mathbf{r})+\mathbf{A}_\perp^-(\mathbf{r}) = \sum_{\mathbf{k},j}\left[\frac{2\pi\hbar c^2}{\omega_k V}\right]^{1/2} \mathbf{e}_{\mathbf{k}j}(\alpha_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{r}}+\alpha_{\mathbf{k}j}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}) ,$$

$$\mathbf{E}_\perp(\mathbf{r}) \equiv \mathbf{E}_\perp^+(\mathbf{r})+\mathbf{E}_\perp^-(\mathbf{r}) = i\sum_{\mathbf{k},j}\left[\frac{2\pi\hbar \omega_k}{V}\right]^{1/2} \mathbf{e}_{\mathbf{k}j}(\alpha_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{r}}-\alpha_{\mathbf{k}j}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}) , \tag{2.1}$$

$$\mathbf{B}_\perp(\mathbf{r}) \equiv \mathbf{B}_\perp^+(\mathbf{r})+\mathbf{B}_\perp^-(\mathbf{r}) = -i\sum_{\mathbf{k},j}\left[\frac{2\pi\hbar \omega_k}{V}\right]^{1/2} (\mathbf{e}_{\mathbf{k}j}\times\hat{\mathbf{k}})(\alpha_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{r}}-\alpha_{\mathbf{k}j}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}) .$$

Here ω_k is the frequency of the vacuum normal modes of wave vector $\mathbf{k}$ ($\hat{\mathbf{k}}\equiv\mathbf{k}/k$) and polarization vector $\mathbf{e}_{\mathbf{k}j}$, and where the α's are the Bose creation and annihilation operators for the $(\mathbf{k},j)$ normal mode. Consequently, the unperturbed Hamiltonian of the field is of the form

$$\mathscr{H}_f = \sum_{\mathbf{k},j}\hbar\omega_k\alpha_{\mathbf{k}j}^\dagger\alpha_{\mathbf{k}j} . \tag{2.2}$$

In the continuum limit the expressions (2.1) and (2.2) take the form[16]

$$\mathbf{A}_\perp(\mathbf{r}) = \frac{\hbar^{1/2}}{2\pi}c\sum_j\int\frac{1}{\sqrt{\omega_k}}\mathbf{e}_{\mathbf{k}j}(\alpha_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{r}}+\alpha_{\mathbf{k}j}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}})d^3k ,$$

$$\mathbf{E}_\perp(\mathbf{r}) = i\frac{\hbar^{1/2}}{2\pi}\sum_j\int\sqrt{\omega_k}\mathbf{e}_{\mathbf{k}j}(\alpha_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{r}}-\alpha_{\mathbf{k}j}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}})d^3k , \tag{2.3}$$

$$\mathbf{B}_\perp(\mathbf{r}) = -i\frac{\hbar^{1/2}}{2\pi}\sum_j\int\sqrt{\omega_k}(\mathbf{e}_{\mathbf{k}j}\times\hat{\mathbf{k}}) \times(\alpha_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{r}}-\alpha_{\mathbf{k}j}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}})d^3k ,$$

and

$$\mathscr{H}_f = \sum_j\int\hbar\omega_k\alpha_{\mathbf{k}j}^\dagger\alpha_{\mathbf{k}j}\,d^3k . \tag{2.4}$$

It is possible to show, using the well-known solutions of the vector Helmholtz equation[17] and the properties of spherical harmonics,[18] that a unitary transformation exists leading from the continuum-limit representation of the electromagnetic field in terms of operators $\alpha_{\mathbf{k}j}$, to the spherical-wave representation in terms of operators $\alpha(k,\lambda,l,m)$, where $\lambda=\mathscr{E},\mathscr{M}$ refers to the electric or magnetic component of the spherical partial wave of angular momentum l and z projection m. In the continuum limit the commutation rules for the field operators in the two representations are

$$[\alpha_{\mathbf{k}j},\alpha^{\dagger}_{\mathbf{k}'j'}]=\delta(\mathbf{k}-\mathbf{k}')\delta_{jj'},\quad [\alpha_{\mathbf{k}j},\alpha_{\mathbf{k}'j'}]=0\ , \tag{2.5}$$

$$[\alpha(k,\lambda,l,m),\alpha^{\dagger}(k',\lambda',l',m')]=\delta(k-k')\delta_{\lambda\lambda'}\delta_{ll'}\delta_{mm'}\ , \tag{2.6}$$

$$[\alpha(k,\lambda,l,m),\alpha(k',\lambda',l',m')]=0\ .$$

In terms of the new operators, the fields are expressed as

$$\begin{aligned}\mathbf{A}_{\perp}(\mathbf{r})=2\sqrt{\hbar c}\sum_{l,m}\int_0^{\infty}\Big\{&j_l(kr)\mathbf{Y}_{lm0}(\theta,\varphi)\alpha(k,\lambda=\mathscr{M},l,m)\\&+\frac{1}{\sqrt{2l+1}}[\sqrt{l}\,j_{l+1}(kr)\mathbf{Y}_{lm+}(\theta,\varphi)-\sqrt{l+1}\,j_{l-1}(kr)\mathbf{Y}_{lm-}(\theta,\varphi)]\,\alpha(k,\lambda=\mathscr{E},l,m)\Big\}k^{1/2}dk+\text{H.c.}\ ,\\ \mathbf{E}_{\perp}(\mathbf{r})=i2\sqrt{\hbar c}\sum_{l,m}\int_0^{\infty}\Big\{&j_l(kr)\mathbf{Y}_{lm0}(\theta,\varphi)\alpha(k,\lambda=\mathscr{M},l,m)\\&+\frac{1}{\sqrt{2l+1}}[\sqrt{l}\,j_{l+1}(kr)\mathbf{Y}_{lm+}(\theta,\varphi)-\sqrt{l+1}\,j_{l-1}(kr)\mathbf{Y}_{lm-}(\theta,\varphi)]\,\alpha(k,\lambda=\mathscr{E},l,m)\Big\}k^{3/2}dk+\text{H.c.}\ ,\\ \mathbf{B}_{\perp}(\mathbf{r})=-i2\sqrt{\hbar c}\sum_{l,m}\int_0^{\infty}\Big\{&j_l(kr)\mathbf{Y}_{lm0}(\theta,\varphi)\alpha(k,\lambda=\mathscr{E},l,m)\\&-\frac{1}{\sqrt{2l+1}}[\sqrt{l}\,j_{l+1}(kr)\mathbf{Y}_{lm+}(\theta,\varphi)-\sqrt{l+1}\,j_{l-1}(kr)\mathbf{Y}_{lm-}(\theta,\varphi)]\\&\times\alpha(k,\lambda=\mathscr{M},l,m)\Big\}k^{3/2}dk+\text{H.c.}\ ,\end{aligned} \tag{2.7}$$

where $\mathbf{E}$ and $\mathbf{B}$ have been obtained from $\mathbf{A}$ in the Coulomb gauge and where $j_l(kr)$ are spherical Bessel functions, while $\mathbf{Y}_{lmi}$ $(i=+,-,0)$ are spherical vector harmonics which differ by unessential phase and normalization factors from the usual ones.[17] Finally, the Hamiltonian in the spherical-wave representation is of the form

$$\mathscr{H}_f=\sum_{\lambda,l,m}\int\hbar\omega_k\alpha^{\dagger}(k,\lambda,l,m)\alpha(k,\lambda,l,m)dk\ . \tag{2.8}$$

III. THE ATOM-FIELD INTERACTION

As we shall see later on, the relevant part of the electron-field interaction in the minimal-coupling version is

$$\begin{aligned}\mathscr{H}_i=-\frac{e}{mc}\mathbf{A}_{\perp}(\mathbf{r})\cdot\mathbf{p}=-\frac{2e\hbar^{1/2}}{mc^{1/2}}\sum_{l,m}\int_0^{\infty}\Big\{&j_l(kr)\mathbf{Y}_{lm0}\cdot\mathbf{p}\alpha(k,\lambda=\mathscr{M},l,m)\\&+\frac{1}{\sqrt{2l+1}}[\sqrt{l}\,j_{l+1}(k,r)\mathbf{Y}_{lm+}(\theta,\varphi)\cdot\mathbf{p}-\sqrt{l+1}\,j_{l-1}(kr)\mathbf{Y}_{lm-}(\theta,\varphi)\cdot\mathbf{p}]\\&\times\alpha(k,\lambda=\mathscr{E},l,m)\Big\}k^{1/2}dk+\text{H.c.}\ ,\end{aligned} \tag{3.1}$$

where we have used (2.7). Our aim is to calculate the matrix elements of (3.1) between the ground state $|\Phi_{100},\{0\}\rangle$, where Φ_{100} is the ground $1s$ state of the hydrogen atom and $\{0\}$ is the photon vacuum, and an excited state $|\Phi_{NLM},1(k,\lambda,l,m)\rangle$ where N,L,M are the usual quantum numbers of an excited hydrogen state and $1(k,\lambda,l,m)$ represents a state of the field with a quantum of type λ (electric or magnetic) in the state k,l,m. We have

$$\Phi_{NLM}=R_{NL}(r)Y_L^M(\theta,\varphi)\ .$$

In particular

$$\Phi_{100}=\frac{1}{\sqrt{4\pi}}R_{10}(r)=\frac{2}{a_0^{3/2}}\frac{e^{-r/a_0}}{\sqrt{4\pi}}$$

and consequently

$$\mathbf{p}\,|\,\Phi_{100}\rangle = -i\hbar\nabla\,|\,\Phi_{100}\rangle = i\frac{\hbar}{a_0}\hat{\mathbf{r}}\,|\,\Phi_{100}\rangle \ . \tag{3.2}$$

After some algebraic manipulations it can be shown that the relevant matrix elements are

$$\langle \Phi_{NLM}, 1(k,l,\lambda,m)\,|\,\mathscr{H}_i\,|\,\Phi_{100},\{0\}\rangle = (-1)^M i^{-L-1} i\frac{e\hbar}{ma_0}\left[\frac{\hbar}{\pi c}\right]^{1/2} k^{-1/2}\sqrt{L(L+1)} \times \langle R_{NL}\,|\,r^{-1}j_L(kr)\,|\,R_{10}\rangle \delta_{\lambda,\mathscr{E}}\delta_{l,L}\delta_{-M,m} \ . \tag{3.3}$$

We now turn to the evaluation of the matrix element $\langle R_{NL}\,|\,r^{-1}j_L(kr)\,|\,R_{10}\rangle$ which can be done starting from the expressions for the radial part of the eigenfunctions of a bound state of the hydrogen atom[19] and by extensive use of integrations.[20] Since it is useful to distinguish the cases in which E_N belongs to the discrete or to the continuum part of the spectrum of the hydrogen atom, we shall consider the two cases separately. Then we have the following.

Discrete spectrum:

$$\langle R_{NL}\,|\,r^{-1}j_L(kr)\,|\,R_{10}\rangle = \frac{4}{a_0}\left[\frac{(N+L)!}{(N-L-1)!}\right]^{1/2}\frac{(2Nka_0)^L}{(2L+1)!!}\frac{[N^2(1+k^2a_0^2)-1]^{N-L-1}}{[(N+1)^2+N^2k^2a_0^2]^N} \times F(\tfrac{1}{2}(-N+L+1),\tfrac{1}{2}(-N+L+2);L+\tfrac{3}{2};-4N^2k^2a_0^2/[N^2(1+k^2a_0^2)-1]^2) \equiv f(N,L,k) \ . \tag{3.4}$$

Continuous spectrum:

$$\langle R_{qL}\,|\,r^{-1}j_l(kr)\,|\,R_{10}\rangle = \frac{2C_{qL}(2kq)^L}{(2L+1)!!a_0^{3/2}}\left[\frac{a_0^{-2}+k^2+q^2}{[a_0^{-1}+i(k+q)][a_0^{-1}-i(k-q)]}\right]^{-i/qa_0}(a_0^{-2}+k^2+q^2)^{-L-1} \times F(\tfrac{1}{2}[L+1+i(qa_0)^{-1}],\tfrac{1}{2}[L+2+i(qa_0)^{-1}];L+\tfrac{3}{2};[2kq/(a_0^{-2}+k^2+q^2)]^2) \equiv f(q,L,k) \ . \tag{3.5}$$

In these expressions F is the hypergeometric function, $q=\sqrt{2E}$ in atomic units, and C_{qL} is the usual normalization constant.[21]

IV. PERTURBATION THEORY

We set

$$-\frac{ie\hbar}{ma_0}\left[\frac{\hbar}{\pi c}\right]^{1/2} k^{-1/2}\sqrt{L(L+1)}\langle R_{NL}\,|\,r^{-1}j_L(kr)\,|\,R_{10}\rangle = \epsilon_{NL}(k) \tag{4.1}$$

so that (3.3) can be cast in the form

$$\langle \Phi_{NLM}, 1(k,\lambda,l,m)\,|\,\mathscr{H}_i\,|\,\Phi_{100},\{0\}\rangle = -(-1)^M(i)^{-L-1}\epsilon_{NL}(k)\delta_{\lambda,\mathscr{E}}\delta_{l,L}\delta_{m,-M} \ . \tag{4.2}$$

We remark that the notation used here for the principal quantum number is N, which we formally treat in this section as a discrete variable. If the intermediate state belongs to the continuum we should use q instead, and change the sums over N into appropriate integrals. Since no ambiguity arises here, we shall postpone this change of notation until Sec. VI.

The first-order correction to the unperturbed ground state Φ_{100} is

$$|\,1\rangle = -\int_0^\infty dk \sum_{\substack{N,L,M,\\ \lambda,l,m}} \frac{\langle \phi_{NLM}, 1(k,\lambda,l,m)\,|\,\mathscr{H}_i\,|\,\phi_{100},\{0\}\rangle}{E_N+\hbar\omega_k-E_0} \times |\,\phi_{NLM},1(k,\lambda,l,m)\rangle = \sum_{N,L,M}\int_0^\infty dk\,(-1)^M(i)^{-L-1}\frac{\epsilon_{NL}(k)}{\hbar(\omega_{N0}+\omega_k)} \times |\,\phi_{NLM},1(k,\lambda=\mathscr{E},L,-M)\rangle \ , \tag{4.3}$$

where $\hbar\omega_{N0}=E_N-E_0$. The second-order correction $|\,2\rangle$ to ϕ_{100} is of $O(e^2)$, and yields admixture with two-photon states of the form

$$|\,\phi_{N'L'M'},1(k_1,\lambda_1,l_1,m_1),1(k_2,\lambda_2,l_2,m_2)\rangle \tag{4.4}$$

plus a small part $|\,\phi_{100},\{0\}\rangle$. We are not going to write explicitly this correction, since the CGED operator $W(r)$ in which we are interested is a linear combination of terms of the form

$$\alpha^\dagger(k,\lambda,l,m)\alpha(k',\lambda',l',m') \ .$$

Thus there is no $O(e^2)$ contribution to $W(r)$ coming from $|\,2\rangle$. However, the second-order correction $|\,2\rangle$ is important for ensuring normalization of the perturbed ground state up to $O(e^2)$ terms because of its $|\phi_{100},\{0\}\rangle$ part. Therefore, although $|2\rangle$ does not contribute to the CGED in practice, the ground state normalized up to $O(e^2)$ is in fact $|\phi_{100},\{0\}\rangle+|1\rangle+|2\rangle$. This also explains why we have kept only the $\mathbf{A}\cdot\mathbf{p}$ part in $\mathscr{H}_i$, as in (3.1), since the first-order perturbation treatment of the $\mathbf{A}^2$ term would introduce two-photon, $O(e^2)$ corrections to the ground-state eigenvector, which would yield only $O(e^4)$ corrections to the CGED.

We shall also need in Sec. V the first-order correction to the ground state in the plane-wave representation, which we obtain from (2.1) and (3.1) as

$$\mathscr{H}_i=-\frac{e}{mc}\sum_{\mathbf{k}j}\left[\frac{2\pi\hbar c^2}{\omega_k V}\right]^{1/2}(\alpha_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{r}}+\alpha^{\dagger}_{\mathbf{k}j}e^{-i\mathbf{k}\cdot\mathbf{r}})\cdot\mathbf{e}_{\mathbf{k}j}\cdot\mathbf{p}\ , \tag{4.5}$$

$$|1\rangle=i\frac{e\hbar}{mca_0}\sum_{N,L,M,\mathbf{k},j}\left[\frac{2\pi\hbar c}{\omega_k V}\right]^{1/2}\times\frac{\langle\phi_{NLM},1(\mathbf{k},j)\,|\,e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{e}_{\mathbf{k}j}\cdot\hat{\mathbf{r}}\,|\,\Phi_{100},\{0\}\rangle}{E_N+\hbar\omega_k-E_0}\times|\Phi_{NLM},1(\mathbf{k},j)\rangle\ , \tag{4.6}$$

where symbol $1(\mathbf{k},j)$ indicates one photon in the state with wave vector $\mathbf{k}$ and polarization j.

The energy corrections obtained by second-order perturbation theory, with special reference to the role of the virtual photons in determining the Lamb shift, have been presented in a previous paper,[11] and consequently we shall not discuss them here.

V. THE COARSE-GRAINED ENERGY DENSITY

Expression (4.3) for the correction to the ground-state wave functions shows clearly that the hydrogen atom is surrounded by a cloud of virtual photons, which are continuously absorbed and reemitted in all ranges of frequency with a spectrum determined essentially by the atomic structure and by the time-energy uncertainty principle.

A conceptual problem arises in connection with the possibility of describing the shape of this cloud. In fact, according to the general theory of Newton and Wigner,[22] the photon does not satisfy the apparently simple conditions for localizability, while relaxing some of these conditions yields the possibility of defining a "coarse-grained localizability" within a volume V_1 whose linear dimensions cannot be reduced below the wavelength of the photon one is trying to localize. This has led to the introduction of a "coarse-grained photon density" defined in terms of the operator[23]

$$\rho(\mathbf{r},t)=\boldsymbol{\psi}^{\dagger}(\mathbf{r},t)\cdot\boldsymbol{\psi}(\mathbf{r},t)\ ,\qquad \boldsymbol{\psi}(\mathbf{r},t)=\frac{1}{V}\sum_{\mathbf{k},j}\mathbf{e}_{\mathbf{k}j}\alpha_{\mathbf{k}j}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\ . \tag{5.1}$$

One can then obtain the number of photons within V_1 as

$$n_{V_1,t}=\int_{V_1}\rho(\mathbf{r},t)d^3r \tag{5.2}$$

provided V_1 is smaller than the quantization volume V but larger than the largest wavelength in the photon field. Unfortunately, as discussed by Cook,[15] $\rho(\mathbf{r},t)$ as defined in (5.1) does not satisfy any simple continuity equation of the form

$$\frac{\partial}{\partial t}\rho+\nabla\cdot\mathbf{j}=0\ , \tag{5.3}$$

where $\mathbf{j}$ is a photon current density, which should be a desirable feature for a reasonable photon density operator. This difficulty was overcome[15] by introducing a pair of vector field operators (for an infinite quantization volume)

$$\boldsymbol{\psi}(\mathbf{r},t)=[2(2\pi)^3]^{-1/2}\sum_j\int\mathbf{e}_{\mathbf{k}j}\alpha_{\mathbf{k}j}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}d^3k\ ,\qquad \boldsymbol{\phi}(\mathbf{r},t)=[2(2\pi)^3]^{-1/2}\sum_j\int\hat{\mathbf{k}}\times\mathbf{e}_{\mathbf{k}j}\alpha_{\mathbf{k}j}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}d^3k\ . \tag{5.4}$$

By defining the photon density and the photon current density, respectively, as

$$\rho=\boldsymbol{\psi}^{\dagger}\cdot\boldsymbol{\psi}+\boldsymbol{\phi}^{\dagger}\cdot\boldsymbol{\phi}\ ,\qquad \mathbf{j}=c(\boldsymbol{\psi}^{\dagger}\times\boldsymbol{\phi}-\boldsymbol{\phi}^{\dagger}\times\boldsymbol{\psi})\ , \tag{5.5}$$

Cook obtained the continuity equation (5.3), and he was able to show that $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ are related to the positive energy parts of the transverse electric and magnetic fields which can be obtained from (2.3) as

$$\mathbf{E}^{+}(\mathbf{r})=\frac{i}{2\pi}\hbar^{1/2}\sum_j\int\sqrt{\omega_k}\,\mathbf{e}_{\mathbf{k}j}\alpha_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{r}}d^3k\ ,\qquad \mathbf{B}^{+}(\mathbf{r})=-\frac{i}{2\pi}\hbar^{1/2}\sum_j\int\sqrt{\omega_k}(\mathbf{e}_{\mathbf{k}j}\times\hat{\mathbf{k}})\alpha_{\mathbf{k}j}e^{i\mathbf{k}\cdot\mathbf{r}}d^3k\ , \tag{5.6}$$

where the transverse symbol has been dropped for convenience of notation. This relation, however, is rather complicated because it is integral and nonlocal, of the form

$$\boldsymbol{\psi}(\mathbf{r},t)=\int g(\mathbf{r}-\mathbf{r}')\mathbf{E}^{+}(\mathbf{r}',t)d^3r'\ ,\qquad \boldsymbol{\phi}(\mathbf{r},t)=\int g(\mathbf{r}-\mathbf{r}')\mathbf{B}^{+}(\mathbf{r}',t)d^3r'\ , \tag{5.7}$$

with the kernel $g(\mathbf{r})$ defined as

$$g(\mathbf{r})=-\frac{i}{(2\pi)^3}\int(4\pi\hbar\omega_k)^{-1/2}e^{-i\mathbf{k}\cdot\mathbf{r}}d^3k\ . \tag{5.8}$$

In Cook's theory, the operator representing the number of photons within a volume V_1 is also of the form (5.2), with the same limitations on V as in Mandel's theory.

The relation between the $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ operators in Cook's theory and the electromagnetic field operators leads us to investigate the possibility of representing the shape of the photon cloud dressing the atom by directly using electric and magnetic field operators, which refer to physically measurable quantities. Thus we use (5.6) and their Hermitian conjugates to define the operator

$$W(\mathbf{r})=\frac{1}{4\pi}(\mathbf{E}^{-}\cdot\mathbf{E}^{+}+\mathbf{B}^{-}\cdot\mathbf{B}^{+}) \tag{5.9}$$

which is related to the energy density of the electromagnetic field and which is obviously the counterpart of $\rho(\mathbf{r})$ in (5.5). It is interesting to compare (5.9) with the expression for the energy density of the field which, omitting zero-point energy terms, can be written

$$\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{B}^2) = W(\mathbf{r})+\frac{1}{8\pi}[(\mathbf{E}^+)^2+(\mathbf{E}^-)^2+(\mathbf{B}^+)^2+(\mathbf{B}^-)^2] . \tag{5.10}$$

In view of coarse graining with a cubic mesh of side L, integration of $(\mathbf{E}^+)^2$ and $(\mathbf{B}^+)^2$ over a cubic volume $V_1=L^3$ and use of (5.6) yields

$$\int_{V_1}[\mathbf{E}^+(\mathbf{r})]^2 d^3r = -\frac{\hbar}{4\pi^2}\sum_{j,j'}\int\int\sqrt{\omega_k\omega_{k'}}(\mathbf{e}_{\mathbf{k}j}\cdot\mathbf{e}_{\mathbf{k}'j'})\alpha_{\mathbf{k}j}\alpha_{\mathbf{k}'j'} \times\int_{L^3}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}d^3r\,d^3k\,d^3k' ,$$

$$\int_{V_1}[\mathbf{B}^+(\mathbf{r})]^2 d^3\mathbf{r} = -\frac{\hbar}{4\pi^2}\sum_{j,j'}\int\int\sqrt{\omega_k\omega_{k'}}(\mathbf{e}_{\mathbf{k}j}\times\hat{\mathbf{k}})(\mathbf{e}_{\mathbf{k}'j'}\times\hat{\mathbf{k}}')\alpha_{\mathbf{k}j}\alpha_{\mathbf{k}'j'} \times\int_{L^3}e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}d^3r\,d^3k\,d^3k' . \tag{5.11}$$

A slight modification of Mandel's argument[23] immediately leads to the conclusion that the largest contributions to the $\mathbf{k}$ and $\mathbf{k}'$ integrations in both expressions in (5.11) come from the pairs with $\mathbf{k}'\sim-\mathbf{k}$ for all $\mathbf{k}\gtrsim 2\pi/L$. On the other hand,

$$(\mathbf{e}_{\mathbf{k}j}\times\hat{\mathbf{k}})\cdot(-\mathbf{e}_{\mathbf{k}j}\times\hat{\mathbf{k}})=-\mathbf{e}_{\mathbf{k}j}\cdot\mathbf{e}_{\mathbf{k}j}$$

and these main contributions have opposite signs in the two integrals in (5.11). Consequently

$$\int_{V_1}[\mathbf{E}^\sigma(\mathbf{r})]^2 d^3\mathbf{r}\sim-\int_{V_1}[\mathbf{B}^\sigma(\mathbf{r})]^2 d^3r,\quad \sigma=\pm \tag{5.12}$$

and by an appropriate choice of the mesh size, only W in (5.10) yields sizeable contributions to the CGED. For this reason we call $W(r)$ the CGED of the field.

On the other hand, $\int_{V_1}W(\mathbf{r})d^3r$ can be interpreted as the energy carried by the photons and localized within volume V_1. To show this, it is convenient to start from a finite quantization volume V, to obtain $\mathbf{E}^\sigma$ and $\mathbf{B}^\sigma$ from (2.1) to introduce coarse graining with a mesh of side L ($L^3\ll V$), and then to consider the limit of large V. We obtain

$$\int_{V_1}W(\mathbf{r})d^3r=\frac{\hbar}{V_1}\sum_{j,j'}\sum_{\mathbf{k},\mathbf{k}'}\sqrt{\omega_k\omega_{k'}}\mathbf{e}_{\mathbf{k}j}\cdot\mathbf{e}_{\mathbf{k}'j'}\alpha^\dagger_{\mathbf{k}j}\alpha_{\mathbf{k}'j'} \times\int_{L^3}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}d^3r . \tag{5.13}$$

Using Mandel's argument again, we get

$$\int_{V_1}W(\mathbf{r})d^3r\sim\frac{L^3}{V}\sum_{\mathbf{k},j}\hbar\omega_k\alpha^\dagger_{\mathbf{k}j}\alpha_{\mathbf{k}j} \sim\frac{V_1}{V}\sum_j\int\hbar\omega_k\alpha^\dagger_{\mathbf{k}j}\alpha_{\mathbf{k}j}\,d^3k \tag{5.14}$$

for large V. Thus upon coarse graining the contributions to the energy density coming from correlations between photons with different $\mathbf{k}$ tend to vanish, and only diagonal pairs of operators of the form $\alpha^\dagger_{\mathbf{k}j}\alpha_{\mathbf{k}j}$ remain, which seems to support an interpretation of $\int W(\mathbf{r})d^3r$ in terms of the energy carried by the decorrelated photons within V_1.

Next we show that the CGED satisfies an appropriate continuity equation. Maxwell's equations for the transverse components of the classical field are[24]

$$\nabla\cdot\mathbf{E}^\sigma=\nabla\cdot\mathbf{B}^\sigma=\nabla\cdot\mathbf{j}^\sigma=0,\quad\sigma=\pm$$
$$\nabla\times\mathbf{E}^\sigma=-\frac{1}{c}\dot{\mathbf{B}}^\sigma,\quad\nabla\times\mathbf{B}^\sigma=\frac{1}{c}\dot{\mathbf{E}}^\sigma+\frac{4\pi}{c}\mathbf{j}^\sigma , \tag{5.15}$$

where $\mathbf{j}$ is the transverse current density. Using the general relation

$$\nabla\cdot(\mathbf{a}\times\mathbf{b})=\nabla\times\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\nabla\times\mathbf{b}$$

we obtain from (5.15)

$$\frac{\partial}{\partial t}(\mathbf{E}^-\cdot\mathbf{E}^+)=-4\pi(\mathbf{j}^-\cdot\mathbf{E}^++\mathbf{j}^+\cdot\mathbf{E}^-) -c\nabla\cdot(\mathbf{E}^+\times\mathbf{B}^-+\mathbf{E}^-\times\mathbf{B}^+) -(\mathbf{B}^-\cdot\dot{\mathbf{B}}^++\dot{\mathbf{B}}^-\cdot\mathbf{B}^+)$$

and consequently

$$\frac{\partial}{\partial t}W+\nabla\cdot\mathbf{S}=-q , \tag{5.16}$$

where we have set

$$\mathbf{S}=\frac{c}{4\pi}(\mathbf{E}^+\times\mathbf{B}^-+\mathbf{E}^-\times\mathbf{B}^+),\quad q=\mathbf{j}^-\cdot\mathbf{E}^++\mathbf{j}^+\cdot\mathbf{E}^- . \tag{5.17}$$

$\mathbf{S}$ is clearly the counterpart of Cook's photon current density,[15] and also the Poynting vector in the continuity equation for the total energy density, while q is the source term which closely corresponds the density of the rate of change of the kinetic energy.[25]

Further, we consider the continuity equation

$$\frac{\partial}{\partial t}\left[\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{B}^2)\right]=-\mathbf{j}\cdot\mathbf{E}-\nabla\cdot\left[\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\right] , \tag{5.18}$$

where all quantities are transverse, and which is directly obtainable from Maxwell's equations for the transverse field.[25] Integration of (5.18) over V_1 yields

$$\frac{\partial}{\partial t}\left[\int_{V_1}\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{B}^2)d^3r\right] =-\int_{V_1}\mathbf{j}\cdot\mathbf{E}\,d^3r-\oint\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}\cdot\mathbf{n}\,d\mathscr{S} , \tag{5.19}$$

where the last integral is over the surface of V_1. Introducing positive and negative frequency components, we get

$$\frac{\partial}{\partial t}\int_{V_1} W(\mathbf{r})d^3r+\sum_\sigma \frac{\partial}{\partial t}\left[\int_{V_1}[(\mathbf{E}^\sigma)^2+(\mathbf{B}^\sigma)^2]d^3r\right]$$
$$=-\int_{V_1} q\, d^3r-\oint S\cdot\mathbf{n}\,d\mathscr{S}-\sum_\sigma\int_{V_1}\mathbf{j}^\sigma\cdot\mathbf{E}^\sigma d^3r$$
$$-\sum_\sigma\oint\frac{c}{4\pi}\mathbf{E}^\sigma\times\mathbf{B}^\sigma\cdot\mathbf{n}\,d\mathscr{S}\ . \quad (5.20)$$

Using the same procedure as before, we find

$$\frac{\partial}{\partial t}[(\mathbf{E}^\sigma)^2]=-8\pi\mathbf{j}^\sigma\cdot\mathbf{E}^\sigma-2c\nabla\cdot(\mathbf{E}^\sigma\times\mathbf{B}^\sigma)-2\mathbf{B}^\sigma\cdot\dot{\mathbf{B}}^\sigma$$

from which

$$\frac{\partial}{\partial t}[(\mathbf{E}^\sigma)^2+(\mathbf{B}^\sigma)^2]=-8\pi\mathbf{j}^\sigma\cdot\mathbf{E}^\sigma-2c\nabla\cdot(\mathbf{E}^\sigma\times\mathbf{B}^\sigma)\ . \quad (5.21)$$

Integrating over V_1 and using (5.12) yields

$$-\int_{V_1}\mathbf{j}^\sigma\cdot\mathbf{E}^\sigma d^3r-\frac{c}{4\pi}\oint\mathbf{E}^\sigma\times\mathbf{B}^\sigma\cdot\mathbf{n}\,d\mathscr{S}\sim 0\ . \quad (5.22)$$

Substituting (5.12) and (5.22) into (5.20), we obtain

$$\frac{\partial}{\partial t}\int_{V_1} W(\mathbf{r})d^3r=-\int_{V_1} q\,d^3r-\oint\mathbf{S}\cdot\mathbf{n}\,d\mathscr{S} \quad (5.23)$$

which shows that the continuity equation for the CGED (5.16) can also be obtained by coarse graining from the well-established continuity equation (5.18) for the energy density.

A final point worth mentioning is that the CGED's at different points in space do not commute, in contrast to the usual energy density operator. In detail, using the plane-wave expansion (2.1), it can be shown that

$$[\mathbf{E}^-(\mathbf{r})\cdot\mathbf{E}^+(\mathbf{r}),\mathbf{E}^-(\mathbf{r}')\cdot\mathbf{E}^+(\mathbf{r}')]=\sum_{a,b}f_{ab}(\mathbf{r}-\mathbf{r}')[E_a^-(\mathbf{r})E_b^+(\mathbf{r}')-E_b^-(\mathbf{r}')E_a^+(\mathbf{r})]\ ,$$
$$[\mathbf{B}^-(\mathbf{r})\cdot\mathbf{B}^+(\mathbf{r}),\mathbf{B}^-(\mathbf{r}')\cdot\mathbf{B}^+(\mathbf{r}')]=\sum_{a,b}f_{ab}(\mathbf{r}-\mathbf{r}')[B_a^-(\mathbf{r})B_b^+(\mathbf{r}')-B_b^-(\mathbf{r}')B_a^+(\mathbf{r})]\ , \quad (5.24)$$
$$[\mathbf{E}^-(\mathbf{r})\cdot\mathbf{E}^+(\mathbf{r}),\mathbf{B}^-(\mathbf{r}')\cdot\mathbf{B}^+(\mathbf{r}')]=\sum_{a,b}h_{ab}(\mathbf{r}-\mathbf{r}')[E_a^-(\mathbf{r})B_b^+(\mathbf{r}')+B_b^-(\mathbf{r}')E_a^+(\mathbf{r})]\ ,$$
$$[\mathbf{B}^-(\mathbf{r})\cdot\mathbf{B}^+(\mathbf{r}),\mathbf{E}^-(\mathbf{r}')\cdot\mathbf{E}^+(\mathbf{r}')]=\sum_{a,b}h_{ab}(\mathbf{r}-\mathbf{r}')[E_a^-(\mathbf{r})B_b^+(\mathbf{r})+B_b^-(\mathbf{r})E_a^+(\mathbf{r}')]\ ,$$

where $a,b=x,y,z$ and

$$f_{ab}(\mathbf{r}-\mathbf{r}')=\frac{2\pi\hbar}{V}\sum_{\mathbf{k}}\omega_k\left[\delta_{ab}-\frac{k_ak_b}{k^2}\right]e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\ ,$$
$$h_{ab}(\mathbf{r}-\mathbf{r}')=-\frac{2\pi\hbar}{V}\sum_{\mathbf{k}}\omega_k\varepsilon_{abc}\frac{k_c}{k}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\ , \quad (5.25)$$

where ε_{abc} is the third-ranked antisymmetric unit tensor. Consequently

$$[W(\mathbf{r}),W(\mathbf{r}')]\equiv C(\mathbf{r},\mathbf{r}')\neq 0\ . \quad (5.26)$$

Substitution of (2.1) into (5.24) yields the expression for $C(\mathbf{r},\mathbf{r}')$ as a linear combination of terms

$$\alpha_{\mathbf{k}j}^\dagger\alpha_{\mathbf{k}'j'}-\alpha_{-\mathbf{k}'j'}^\dagger\alpha_{-\mathbf{k}j}\ . \quad (5.27)$$

Using

$$\mathbf{e}_{\mathbf{k}j}=\mathbf{e}_{\mathbf{k}j}^*=\mathbf{e}_{-\mathbf{k}j}$$

the quantum-mechanical average of each term (5.27) on the perturbed ground state

$$|\Phi_{100},\{0\}\rangle+|1\rangle$$

with $|1\rangle$ given by (4.6) can be easily seen to vanish. Consequently $\langle C(\mathbf{r},\mathbf{r}')\rangle$ on the perturbed ground state vanishes, and on the basis of the generalized Heisenberg uncertainty relations one finds

$$\Delta W(\mathbf{r})\Delta W(\mathbf{r}')\geq 0\ . \quad (5.28)$$

Relations of this kind have been shown by Deutsch[26] to be in general too weak to express properly the uncertainty principle. A more appropriate expression of this principle, strictly speaking valid for a discrete spectrum, has been proposed by Deutsch in the form

$$\mathscr{B}(W(\mathbf{r}),W(\mathbf{r}'))\geq 2\ln\frac{2}{1+\sup\{|\langle\phi(\mathbf{r})|\phi(\mathbf{r}')\rangle|\}}\ , \quad (5.29)$$

where $\mathscr{B}(A,B)$ is an irreducible lower bound in the uncertainty in the result of a simultaneous measurement of A and B, and where $|\phi(\mathbf{r})\rangle$ is an eigenfunction of operator $W(\mathbf{r})$. The exact eigenfunctions of $W(\mathbf{r})$ are not easy to find, but from (5.9) and (2.1) it is easy to see that both $W(\mathbf{r})$ and $W(\mathbf{r}')$ can be written as linear combinations of terms of the form $\alpha_{\mathbf{k}j}^\dagger\alpha_{\mathbf{k}'j'}$. Consequently a tensor product of coherent states for each mode $(\mathbf{k},j)$ in the field, each with a large number of photons, is almost an eigenstate of both $W(\mathbf{r})$ and $W(\mathbf{r}')$, corresponding to different eigenvalues in the two cases. Thus in (5.29)

$$\sup\{|\langle\phi(\mathbf{r})|\phi(\mathbf{r}')\rangle|\}\sim 1 \quad (5.30)$$

we find again

$$\mathscr{B}(W(\mathbf{r}),W(\mathbf{r}'))\geq 0\ . \quad (5.31)$$

Actually the situation might be more involved than this, since the eigenvalue spectrum of both $W(r)$ and $W(r')$ is continuous rather than discrete, and instead of Deutsch's approach one should use Partovi's generalization of this theory[27] to the continuum case. This, however, is clearly beyond the scope of this paper, and consequently we shall take result (5.31) as an indication that the limitations in the theory caused by the noncommutation of $W(r)$ and $W(r')$ are not very severe.

The results obtained in this section lead us to conclude that the CGED $W(r)$ is a reasonably well-defined physical quantity, capable of describing in a reliable way the main features of the photon distribution in a quantized electromagnetic field.

VI. SHELL STRUCTURE OF THE PHOTON CLOUD

We are now ready to calculate the space distribution of the CGED in the ground state of a hydrogen atom, which we take as

$$|\psi\rangle = |\phi_{100},\{0\}\rangle + |1\rangle \tag{6.1}$$

following the discussion of Sec. IV, with $|1\rangle$ given by (4.3). Using (5.9) we have

$$\langle\psi|W|\psi\rangle = \frac{1}{4\pi}\langle 1|(\mathbf{E}^{-}\cdot\mathbf{E}^{+}+\mathbf{B}^{-}\cdot\mathbf{B}^{+})|1\rangle , \tag{6.2}$$

where

$$\langle 1|\mathbf{E}^{-}\cdot\mathbf{E}^{+}|1\rangle = 4\hbar c \sum_{l,m,l',m'} \frac{1}{\sqrt{(2l+1)(2l'+1)}} \int_0^\infty \int_0^\infty [\sqrt{l}\, j_{l+1}(kr)\mathbf{Y}^*_{lm+}(\theta,\varphi) - \sqrt{l+1}\, j_{l-1}(kr)\mathbf{Y}^*_{lm-}(\theta,\varphi)]$$
$$\cdot[\sqrt{l'}\, j_{l'+1}(k'r)\mathbf{Y}_{l'm'+}(\theta,\varphi) - \sqrt{l'+1}\, j_{l'-1}(k'r)\mathbf{Y}_{l'm'-}(\theta,\varphi)]$$
$$\times\langle 1|\alpha^\dagger(k,\lambda=\mathscr{E},l,m)\alpha(k',\lambda=\mathscr{E},l',m')|1\rangle k^{3/2}(k')^{3/2} dk\, dk' \tag{6.3}$$

and

$$\langle 1|\mathbf{B}^{-}\cdot\mathbf{B}^{+}|1\rangle = 4\hbar c \sum_{l,m,l',m'} \int_0^\infty \int_0^\infty [j_l(kr)\mathbf{Y}^*_{lm0}(\theta,\varphi)]\cdot[j_{l'}(k'r)\mathbf{Y}_{l'm'0}(\theta,\varphi)]$$
$$\times\langle 1|\alpha^\dagger(k,\lambda=\mathscr{E},l,m)\alpha(k',\lambda=\mathscr{E},l',m')|1\rangle k^{3/2}(k')^{3/2} dk\, dk' . \tag{6.4}$$

On the basis of (4.3), we are thus led to evaluate the matrix element

$$\langle 1|\alpha^\dagger(k,\lambda=\mathscr{E},l,m)\alpha(k',\lambda=\mathscr{E},l',m')|1\rangle$$
$$= \sum_{\substack{N,L,M,\\ N',L',M'}} \int_0^\infty \int_0^\infty (-1)^{M+M'} i^{L-L'} \frac{\epsilon_{NL}(k_1)\epsilon^*_{N'L'}(k_2)}{\hbar^2(\omega_{N0}+\omega_{k_1})(\omega_{N'0}+\omega_{k_2})}$$
$$\times\langle\phi_{NLM},1(k,\lambda=\mathscr{E},l,-M)|\alpha^\dagger(k,\lambda=\mathscr{E},l,m)\alpha(k',\lambda=\mathscr{E},l',m')$$
$$\times|\phi_{N'L'M'},1(k_2,\lambda=\mathscr{E},L',-M')\rangle\times dk_1 dk_2 . \tag{6.5}$$

We have

$$\langle 1(k_1,\lambda=\mathscr{E},L,-M)|\alpha^\dagger(k,\lambda=\mathscr{E},l,m)\alpha(k',\lambda=\mathscr{E},l',m')|1(k_2,\lambda=\mathscr{E},L'-M')\rangle$$
$$= \sum_i \langle 1(k_1,\lambda=\mathscr{E},L,-M)|\alpha^\dagger(k,\lambda=\mathscr{E},l,m)|i\rangle\langle i|\alpha(k',\lambda=\mathscr{E},l',m')|1(k_2,\lambda=\mathscr{E},L',M)\rangle$$
$$= \langle 1(k,\lambda=\mathscr{E},L,-M)|\alpha^\dagger(k,\lambda=\mathscr{E},l,m)|\{0\}\rangle\langle\{0\}\alpha(k',\lambda=\mathscr{E},l',m')|1(k_2,\lambda=\mathscr{E},L',-M')\rangle$$
$$= \delta(k-k_1)\delta_{l,L}\delta_{m,-M}\delta(k'-k_2)\delta_{l',L'}\delta_{m',-M'} , \tag{6.6}$$

where $|i\rangle$ is a complete set of eigenstates of the field. Substituting (6.6) into (6.5) and taking into account the orthonormality of ϕ_{NLM}, we find

$$\langle 1|\alpha^\dagger(k,\lambda=\mathscr{E},l,m)\alpha(k',\lambda=\mathscr{E},l',m')|1\rangle = \sum_{N,L,M} \frac{\epsilon_{NL}(k)\epsilon^*_{NL}(k')}{\hbar^2(\omega_{N0}+\omega_k)(\omega_{N0}+\omega_{k'})}\delta_{l,L}\delta_{l',L}\delta_{m,-M}\delta_{m,-M'}$$
$$= \sum_N \frac{\epsilon_{NL}(k)\epsilon^*_{NL}(k')}{\hbar^2(\omega_{N0}+\omega_k)(\omega_{N0}+\omega_{k'})}\delta_{l,l'}\delta_{m,m'} . \tag{6.7}$$

Substituting (6.7) in (6.3) and (6.4), we obtain

$$\langle 1|\mathbf{E}^{-}\cdot\mathbf{E}^{+}|1\rangle = 4\hbar c \sum_{N,L,M} \frac{1}{2l+1} \int_0^\infty \int_0^\infty [\sqrt{l}\, j_{l+1}(kr)\mathbf{Y}^*_{lm+}(\theta,\varphi) - \sqrt{l+1}\, j_{l-1}(kr)\mathbf{Y}^*_{lm-}(\theta,\varphi)]$$
$$\cdot[\sqrt{l}\, j_{l+1}(k'r)\mathbf{Y}_{lm+}(\theta,\varphi) - \sqrt{l+1}\, j_{l-1}(k'r)\mathbf{Y}_{lm-}(\theta,\varphi)]$$
$$\times\frac{\epsilon_{NL}(k)\epsilon^*_{NL}(k')}{\hbar^2(\omega_{N0}+\omega_k)(\omega_{N0}+\omega_{k'})} k^{3/2}(k')^{3/2} dk\, dk' , \tag{6.8}$$

$$\langle 1|\mathbf{B}^{-}\cdot\mathbf{B}^{+}|1\rangle=4\hbar c\sum_{N,L,M}\int_0^\infty\int_0^\infty[j_l(kr)\mathbf{Y}^*_{lm0}(\theta,\varphi)]\cdot[j_l(k'r)\mathbf{Y}_{lm0}(\theta,\varphi)]\frac{\epsilon_{NL}(k)\epsilon^*_{NL}(k')}{\hbar^2(\omega_{N0}+\omega_k)(\omega_{N0}+\omega_{k'})}k^{3/2}(k')^{3/2}dk\,dk' \,. \tag{6.9}$$

Introducing appropriate sum rules we find

$$\langle 1|\mathbf{E}^{-}\cdot\mathbf{E}^{+}|1\rangle =\frac{\hbar c}{\pi}\sum_{N,L}\int_0^\infty\int_0^\infty[lj_{l+1}(kr)j_{l+1}(k'r)+(l+1)j_{l-1}(kr)j_{l-1}(k'r)]\frac{\epsilon_{Nl}(k)\epsilon^*_{Nl}(k')}{\hbar^2(\omega_{N0}+\omega_k)(\omega_{N0}+\omega_{k'})}k^{3/2}(k')^{3/2}dk\,dk' \,, \tag{6.10}$$

$$\langle 1|\mathbf{B}^{-}\cdot\mathbf{B}^{+}|1\rangle=\frac{\hbar c}{\pi}\sum_{N,l}(2l+1)\int_0^\infty\int_0^\infty j_l(kr)j_l(k'r)\frac{\epsilon_{Nl}(k)\epsilon^*_{Nl}(k')}{\hbar^2(\omega_{N0}+\omega_k)(\omega_{N0}+\omega_{k'})}k^{3/2}(k')^{3/2}dk\,dk' \,. \tag{6.11}$$

Using (3.4), (3.5), (4.1), and $\omega=ck$ in (6.10) we obtain

$$\langle 1|\mathbf{E}^{-}\cdot\mathbf{E}^{+}|1\rangle=\frac{\hbar^2e^2}{\pi^2m^2a_0^2c^4}\sum\!\!\!\!\!\!\int_E\sum_l l(l+1)\int_0^\infty\int_0^\infty[lj_{l+1}(\omega r/c)j_{l+1}(\omega' r/c)+(l+1)j_{l-1}(\omega r/c)j_{l-1}(\omega' r/c)] \times\frac{f(E,l,\omega)f^*(E,l,\omega')}{(\omega_{E0}+\omega)(\omega_{E0}+\omega')}\omega\omega'd\omega\,d\omega' \,, \tag{6.12}$$

where we have written $\sum\!\!\!\!\!\!\int_E$ (with $E=N$ or q) instead of $\sum_N$ to emphasize that the sum over intermediate atomic states becomes an integral in the region of the continuous spectrum. Factorizing the integrations in (6.12) we find

$$\langle 1|\mathbf{E}^{-}\cdot\mathbf{E}^{+}|1\rangle =\frac{\hbar^2e^2}{\pi^2m^2a_0^2c^4}\sum\!\!\!\!\!\!\int_E\sum_l l(l+1)\left[l\left|\int_0^\infty\frac{\omega j_{l+1}(\omega r/c)}{\omega_{E0}+\omega}f(E,l,\omega)d\omega\right|^2+(l+1)\left|\int_0^\infty\frac{\omega j_{l-1}(\omega r/c)}{\omega_{E0}+\omega}f(E,l,\omega)d\omega\right|^2\right] . \tag{6.13}$$

Proceeding in the same way, we obtain from (6.11)

$$\langle 1|\mathbf{B}^{-}\cdot\mathbf{B}^{+}|1\rangle=\frac{\hbar^2e^2}{\pi^2m^2a_0^2c^4}\sum\!\!\!\!\!\!\int_E\sum_l l(l+1)(2l+1)\left|\int_0^\infty\frac{\omega j_l(\omega r/c)}{\omega_{E0}+\omega}f(E,l,\omega)d\omega\right|^2 . \tag{6.14}$$

Unfortunately, the expressions in (6.13) and (6.14) cannot be summed exactly, and one is compelled to resort to approximations or to numerical computation. The main features of the photon cloud, however, can be preliminarily obtained by appropriate considerations on the form of (6.13) and (6.14), and it is these considerations that we shall try and develop next.

(i) First we remark that in (6.13) and (6.14) the contributions coming from the different intermediate atomic states are taken into account through the $\sum\!\!\!\!\!\!\int_E$, which runs over all possible energies including the continuous spectrum of the ionized atom, and through the l summations, which runs over all permitted angular momentum subshells belonging to the same E shell. At this stage of the theory, all these contributions add incoherently to give the total CGED, and consequently the total cloud of photons can be considered as if arising from an infinite set of two-level atoms, each consisting of the same $1s$ atomic ground level and of the excited lth subshell level of energy $\hbar\omega_{E0}$ above the ground state. Some of these ficititious two-level atoms, i.e., those belonging to different l subshells within the same E shell, have the same $\hbar\omega_{E0}$, and each of them contributes a spherically symmetric cloud; the shape of this cloud is different for different subshell, even if they belong to the same E shell and hence have the same ω_{E0}. We shall indicate by $\langle\mathbf{E}^{-}\cdot\mathbf{E}^{+}\rangle_{E,l}$ and by $\langle\mathbf{B}^{-}\cdot\mathbf{B}^{+}\rangle_{E,l}$ the electric and the magnetic contributions to the CGED coming from the E,l subshell, and shall occasionally consider this contribution independently of the others.

(ii) The E,l subshell contribution obviously depends on the amplitude of the form factor $f(E,l,\omega)$. We now prove that within the same E shell the dominant contribution comes from the $l=1$ subshell in both limits $\omega a_0/c\gg 1$ and $\omega a_0/c\ll 1$. We take $E\equiv N$ for simplicity, but the same kind of argument is also valid for $E\equiv q$. From (3.13) we have

$$\frac{f(N,1,\omega)}{f(N,l,\omega)} =\left[\frac{(N+1)!(N-l-1)!}{(N-2)!(N+l)!}\right]^{1/2}\frac{(2l+1)!!}{3} \times(2N\omega a_0/c)^{-l+1}[N^2(1+\omega^2a_0^2/c^2)-1]^{l+1} \times\frac{F(-\frac{1}{2}N+1,-\frac{1}{2}N+\frac{3}{2};\frac{5}{2};z)}{F(-\frac{1}{2}N+\frac{1}{2}(l+1),-\frac{1}{2}N+\frac{1}{2}(l+2);l+\frac{3}{2};z)} \,, \tag{6.15}$$

where

$$z=-\frac{4N^2\omega^2a_0^2/c^2}{[N^2(1+\omega^2a_0^2/c^2)-1]^2}\ .$$

Since $z \ll 1$ in both limits, we can approximate as 1 the ratio of the two hypergeometric function in (6.15). After some straightforward algebra and other obvious approximations, we find

$$\frac{f(N,1,\omega)}{f(N,l,\omega)}\sim\begin{cases}K(\omega a_0/c)^{l-1}, & \omega \gg c/a_0 \quad (6.16a)\\ K(\omega a_0/c)^{-l+1}, & \omega \ll c/a_0 \quad (6.16b)\end{cases}$$

where

$$K=\frac{2^{-l+1}}{3}(2l+1)!!\ .$$

Moreover, for $\omega \sim c/a_0$ we find from (6.15)

$$\frac{f(N,1,\omega)}{f(N,l,\omega)}\sim\frac{(2l+1)!!}{3}N^{2l+1}\ .$$

Thus we are led to conclude that within each shell the dominant contribution to the CGED comes from the $l=1$ subshell.

(iii) We now consider the E,l subshell contribution, and remark that the lifetime of a virtual photon of frequency ω emitted by the atom within this subshell is $\tau_\omega \sim (\omega_{E0}+\omega)^{-1}$. During this time the photon can reach the observation point at distance r from the atom only if $c\tau_\omega \gtrsim r$. Thus only photons of frequency $\omega \lesssim c/r-\omega_{E0}$ can contribute substantially to the CGED at r. Thus we make the ansatz that in integrals (6.13) and (6.14) the contribution of the integration in the range from $c/r-\omega_{E0}$ to infinity is negligible compared to that in the range from 0 to $c/r-\omega_{E0}$. Thus the CGED coming from the E,l subshell should decrease noticeably outside a sphere of radius $r_E \sim c/\omega_{E0}$. A first consequence of the ansatz is that, for any point inside a sphere of radius r sufficiently smaller than r_E, we can take the upper limits in integrals (6.13) and (6.14) to be approximately c/r, and coherently approximate

$$\frac{\omega}{\omega_{E0}+\omega}\sim 1 \tag{6.17}$$

in the integrand. Quite arbitrarily, we take $r<r_E/2$. As for the Bessel functions $j_{l'}(\omega r/c)$ (with $l'=l\pm1,l$) appearing in the same integrals, within the region $r \lesssim r_E/2$ their argument is <1; hence they can be approximated as

$$j_{l'}(z)\sim\frac{z^{l'}}{(2l'+1)!!},\quad r\lesssim r_E/2\ . \tag{6.18}$$

Moreover, for $r>a_0$ in the integrand of (6.13) and (6.14), we may take $\omega \ll c/a_0$, and we can approximate $\omega a_0/c \ll 1$. Thus we neglect both ka_0 and N^{-1} with respect to 1 and we neglect both k^2 and q^2 with respect to a_0^{-2}, thereby obtaining

$$f(E,l,\omega)\propto(\omega a_0/c)^l,\quad r>a_0\ . \tag{6.19}$$

Using (6.17), (6.18), and (6.19), we find

$$\int_0^\infty \frac{\omega j_{l'}(\omega r/c)}{\omega_{E0}+\omega}f(E,l,\omega)d\omega \sim \int_0^{c/r}\frac{\omega j_{l'}(\omega r/c)}{\omega_{E0}+\omega}f(E,l,\omega)d\omega \propto \frac{ca_0^2}{r^{l+1}}\ ,$$

$$a_0<r\lesssim r_E/2\ . \tag{6.20}$$

Substituting (6.20) into (6.13) and (6.14) we obtain a fairly large region of space, external to the atom and up to a distance of $\sim r_E/2$, in which the contribution of the E,l subshell to the CGED varies like r^{2l-2}. A second consequence of the ansatz is that within the spherical shell $r_E/2 \lesssim r \lesssim r_E$ we have to take the upper limit of integration in (6.13) and (6.14) to the $c/r-\omega_{E0}$. Within this region of r we can also approximate

$$\frac{\omega}{\omega_{E0}+\omega}\sim\frac{\omega}{\omega_{E0}},\quad r_E/2\lesssim r\lesssim r_E \tag{6.21}$$

and we can retain approximations (6.18) and (6.19) as valid. Therefore we obtain

$$\int_0^\infty \frac{\omega j_{l'}(\omega r/c)}{\omega_{E0}+\omega}f(E,l,\omega)d\omega \sim \int_0^{c/r-\omega_{E0}}\frac{\omega j_{l'}(\omega r/c)}{\omega_{E0}+\omega}f(E,l,\omega) \propto \left[\frac{r}{c}\right]^{l'}\left[\frac{a_0}{c}\right]^{l}\left[\frac{c}{r}-\omega_{E0}\right]^{l+l'+2},\quad r_E/2\lesssim r\lesssim r_E \tag{6.22}$$

whose leading term is of the form $c^2a_0^lr^{-l-2}$, since in this range $c/r>\omega_{E0}$. Thus we expect a CGED which varies like r^{-2l-4}. In view of the arbitrariness involved in the choice of $r_E/2$ as the radius of the surface dividing the two regions of space considered, we expect a rather smooth transition from the r^{-2l-2} behavior in the inner region to the r^{-2l-4} behavior in the outer region.

We can summarize the results of this section by saying that, on the basis of semiquantitative considerations, we expect that the total CGED can be described as consisting of a linear superposition of spherically symmetric contributions coming from each E,l atomic subshell, the most important of which for each E is that with $l=1$. In the range $a_0<r<r_E$, the r dependence of each contribution should change from r^{-2l-2} to r^{-2l-4} as we move outward from the atom. In particular the $l=1$ contributions should behave like r^{-4} in the inner region and like r^{-6} in the outer region. We shall see that these predictions are supported by the exact results of Sec. VII.

VII. THE 2P-SUBSHELL CONTRIBUTION

In order to check our semiquantitative predictions of Sec. VI, here we present the results of an exact calculation of the contribution to the CGED coming from the $1s \leftrightarrow 2p$ virtual transitions, which can be done analytically. From (3.13) we have

$$f(2,1,\omega)=\frac{16\sqrt{6}}{3}\frac{k}{(9+4k^2a_0^2)^2}\times F(0,\tfrac{1}{2};\tfrac{5}{2};-16k^2a_0^2/(3+4k^2a_0^2)^2)=\left[\frac{2}{3}\right]^{1/2}\frac{c^3}{a_0^4}\frac{\omega}{(\omega_c^2+\omega^2)^2}\,, \tag{7.1}$$

where $\omega_c=3c/2a_0$ plays the role of a high-frequency cutoff.[4] Using (7.1), the three integrals appearing in (6.13) and (6.14) can be written as

$$\int_0^\infty \frac{\omega^2 j_{l'}(\omega r/c)}{(\omega_{E0}+\omega)(\omega_c^2+\omega^2)^2}d\omega\equiv A_{l'1}+A_{l'2},\quad l'=0,1,2$$

$$A_{l'1}=\int_0^\infty \frac{\omega j_{l'}(\omega r/c)}{(\omega_c^2+\omega^2)^2}d\omega\,, \tag{7.2}$$

$$A_{l'2}=-\omega_{E0}\int_0^\infty \frac{\omega j_{l'}(\omega r/c)}{(\omega_{E_0}+\omega)(\omega_c^2+\omega^2)^2}d\omega\,.$$

After rather lengthy calculations and use of integral tables[20] we obtain, in terms of the exponential integral $\mathrm{Ei}(-x)=-E_1(x)$

$$A_{01}=\frac{1}{4\omega_c^2}\left[e^x\mathrm{Ei}(-x)\left[1-\frac{1}{x}\right]+e^{-x}\mathrm{Ei}(x)\left[1+\frac{1}{x}\right]\right],$$

$$A_{11}=\frac{\pi}{2\omega_c^2}\left[\frac{1}{x^2}-e^{-x}\left[\frac{1}{2}+\frac{1}{x}+\frac{1}{x^2}\right]\right],$$

$$A_{21}=\frac{1}{\omega_c^2}\left\{\frac{9}{2x^2}+\frac{1}{4}\left[e^x\mathrm{Ei}(-x)\left[-1+\frac{4}{x}+\frac{9}{x^2}+\frac{9}{x^3}\right]-e^{-x}\mathrm{Ei}(x)\left[1+\frac{4}{x}+\frac{9}{x^2}+\frac{9}{x^3}\right]\right]\right\},$$

$$\begin{aligned}A_{02}=&-\frac{\omega_{E0}/\omega_c}{4(\omega_{E0}^2+\omega_c^2)}\left\{\frac{\omega_{E0}}{\omega_c}\left[e^x\mathrm{Ei}(-x)\left[1-\frac{1}{x}\right]+e^{-x}\mathrm{Ei}(x)\left[1+\frac{1}{x}\right]\right]-\pi e^{-x}\right\}\\&-\frac{\omega_{E0}\omega_c}{(\omega_{E0}^2+\omega_c^2)^2}\left[\frac{\omega_{E0}}{\omega_c}\frac{1}{2x}[e^{-x}\mathrm{Ei}(x)-e^{-x}\mathrm{Ei}(-x)]-\frac{\pi}{2x}e^{-x}\right]-\frac{\omega_{E0}^2}{(\omega_{E0}^2+\omega_c^2)^2}[\mathrm{si}(z)y_0(z)+\mathrm{Ci}(z)j_0(z)]\,,\end{aligned}$$

$$\begin{aligned}A_{12}=&-\frac{\pi}{4x}\frac{\omega_{E0}^2/\omega_c^2}{\omega_{E0}^2+\omega_c^2}\left[\frac{2}{x}-e^{-x}\left[x+2+\frac{2}{x}\right]\right]-\frac{\pi}{2x^2}\frac{\omega_{E0}^2}{(\omega_{E0}^2+\omega_c^2)^2}[1-e^{-x}(1+x)]\\&+\frac{1}{4}\frac{\omega_{E0}/\omega_c}{\omega_{E0}^2+\omega_c^2}\left[-\frac{2}{x}+e^x\mathrm{Ei}(-x)\left[-1+\frac{1}{x}-\frac{1}{x^2}\right]+e^{-x}\mathrm{Ei}(x)\left[1+\frac{1}{x}+\frac{1}{x^2}\right]\right]\\&+\frac{1}{2x}\frac{\omega_{E0}\omega_c}{(\omega_{E0}+\omega_c^2)^2}\left[e^x\mathrm{Ei}(-x)\left[1-\frac{1}{x}\right]+e^{-x}\mathrm{Ei}(x)\left[1+\frac{1}{x}\right]\right]\\&-\frac{1}{x}\frac{\omega_c^2}{(\omega_{E0}^2+\omega_c^2)^2}\left[\frac{1}{x}\left[\frac{\pi}{2}-z[\mathrm{si}(z)y_0(z)+\mathrm{Ci}(z)j_0(z)]\right]-\frac{\omega_{E0}}{\omega_c}z[\mathrm{Ci}(z)y_0(z)-\mathrm{si}(z)j_0(z)]\right],\end{aligned}$$

$$\begin{aligned}A_{22}=&-\frac{\omega_{E0}/\omega_c}{\omega_{E0}^2+\omega_c^2}\left\{\frac{\omega_{E0}}{\omega_c}\left\{\frac{9}{2x^2}+\frac{1}{4}\left[e^x\mathrm{Ei}(-x)\left[-1+\frac{4}{x}-\frac{9}{x^2}+\frac{9}{x^3}\right]-e^{-x}\mathrm{Ei}(x)\left[1+\frac{4}{x}+\frac{9}{x^2}+\frac{9}{x^3}\right]\right]\right\}\right.\\&\left.-\frac{\pi}{2}\left[\frac{3}{x^3}-e^{-x}\left[\frac{3}{x^3}+\frac{3}{x^2}+\frac{3}{2x}+\frac{1}{2}\right]\right]\right\}\\&-\frac{\omega_{E0}\omega_c}{(\omega_{E0}^2+\omega_c^2)^2}\left\{\frac{\omega_{E0}}{\omega_C}\left[\frac{3}{x^2}+\frac{1}{2}e^x\mathrm{Ei}(-x)\left[\frac{1}{x}-\frac{3}{x^2}+\frac{3}{x^3}\right]-\frac{1}{2}e^{-x}\mathrm{Ei}(x)\left[\frac{1}{x}+\frac{3}{x^2}+\frac{3}{x^3}\right]\right]\right.\\&\left.\qquad-\frac{\pi}{2}\left[\frac{3}{x^3}-e^{-x}\left[\frac{1}{x}+\frac{3}{x^2}+\frac{3}{x^3}\right]\right]\right\}\\&-\frac{\omega_{E0}^2}{(\omega_{E0}^2+\omega_c^2)^2}\left[\left[\frac{3}{z^2}-1\right][\mathrm{si}(z)y_0(z)+\mathrm{Ci}(z)j_0(z)]+\frac{3}{z}[\mathrm{Ci}(z)y_0(z)-\mathrm{si}(z)j_0(z)]+\frac{3}{z^2}-\frac{3\pi}{2}\frac{1}{z^3}\right],\end{aligned} \tag{7.3}$$

where $x=\omega_c r/c$, $z=\omega_{E0} r/c$, $j_0(z)$, and $y_0(z)$ are spherical Bessel and Neumann functions and

$$\mathrm{si}(z)=-\frac{\pi}{2}+\int_0^z \frac{\sin z'}{z'}dz' ,$$
$$\mathrm{Ci}(z)=\gamma+\ln z+\int_0^z \frac{\cos z'-1}{z'}dz' \tag{7.4}$$

are the sine and cosine integrals,[28] γ being the Euler constant.

Expressions (7.3) are analytic and in closed form, but they are not easy to interpret in terms of more simple functions. Although it is evident from (7.2) that for each l'

$$|A_{l'2}|<|A_{l'1}| \tag{7.5}$$

one is compelled to resort to some approximations in order to have a feeling of the spatial behavior of the $1s$-$2p$ contribution to the CGED. Consequently we consider the region $x \gg 1$, we use the expansions

$$\mathrm{Ei}(x)=\frac{e^x}{x}\left[1+\frac{1}{x}+\frac{2}{x^2}+\cdots\right],$$
$$\mathrm{Ei}(-x)=\frac{-e^x}{x}\left[1-\frac{1}{x}+\frac{2}{x^2}-\cdots\right], \tag{7.6}$$

and we keep terms up to $O(x^4)$. Introducing the functions

$$f(z)=\mathrm{Ci}(z)\sin(z)-\mathrm{si}(z)\cos(z) ,$$
$$g(z)=-\mathrm{si}(z)\sin(z)-\mathrm{Ci}(z)\cos z , \tag{7.7}$$

we obtain from (7.3)

$$A_{01}\sim\frac{1}{\omega_c^2}\left[\frac{1}{x^2}+\frac{5}{2}\frac{1}{x^4}\right],\quad A_{11}\sim\frac{1}{\omega_c^2}\frac{\pi}{2}\frac{1}{x^2},\quad A_{21}\sim\frac{1}{\omega_c^2}\left[2\frac{1}{x^2}-\frac{29}{2}\frac{1}{x^4}\right],$$
$$A_{02}\sim-\left[\frac{\omega_{E0}^2/\omega_c^2}{\omega_{E0}^2+\omega_c^2}+\frac{\omega_{E0}^2}{(\omega_{E0}^2+\omega_c^2)^2}\right]\frac{1}{x^2}-\left[\frac{5}{2}\frac{\omega_{E0}^2/\omega_c^2}{\omega_{E0}^2+\omega_c^2}+\frac{2\omega_{E0}}{(\omega_{E0}^2+\omega_c^2)^2}\right]\frac{1}{x^4}-\frac{\omega_c\omega_{E0}}{(\omega_{E0}^2+\omega_c^2)^2}\frac{1}{x}f(z) ,$$
$$A_{12}\sim-\frac{1}{\omega_c^2}\frac{\pi}{2}\frac{1}{x^2}+2\left[\frac{\omega_{E0}/\omega_c}{\omega_{E0}^2+\omega_c^2}+\frac{\omega_c\omega_{E0}}{(\omega_{E0}^2+\omega_c^2)^2}\right]\frac{1}{x^3}+\frac{\omega_c^2}{(\omega_{E0}^2+\omega_c^2)^2}\left[\frac{\omega_{E0}}{\omega_c}\frac{1}{x}g(z)+\frac{1}{x^2}f(z)\right], \tag{7.8}$$
$$A_{22}\sim-\left[\frac{2\omega_{E0}^2/\omega_c^2}{\omega_{E0}^2+\omega_c^2}+\frac{2\omega_{E0}^2}{(\omega_{E0}^2+\omega_c^2)^2}+\frac{3\omega_c^2}{(\omega_{E0}^2+\omega_c^2)^2}\right]\frac{1}{x^2}+\frac{1}{\omega_c\omega_{E0}}\frac{1}{x^3}+\left[\frac{29}{2}\frac{\omega_{E0}^2/\omega_c^2}{\omega_{E0}^2+\omega_c^2}-\frac{8\omega_{E0}^2}{(\omega_{E0}^2+\omega_c^2)^2}\right]\frac{1}{x^4}$$
$$+\frac{\omega_c^2}{(\omega_{E0}^2+\omega_c^2)^2}\left[\frac{\omega_{E0}}{\omega_c}\frac{1}{x}f(z)-3\frac{1}{x^2}g(z)-\frac{3\omega_c}{\omega_{E0}}\frac{1}{x^3}f(z)\right],\quad r\gg 2a_0/3 .$$

Even form (7.8) for the coefficients is not very transparent and, although in view of (7.5) one can clearly see the x^{-2} dependence predicted in Sec. VI in the region ($x\gg1$, $z\ll1$), it is not easy to distinguish the x^{-3} dependence in the outer region of the sphere $r<r_E$. On the other hand, for $r\gg r_E$ it is $z\gg1$, and we may use expansions

$$f(z)=\frac{1}{z}\left[1-\frac{2!}{z^2}+\cdots\right],\quad g(z)=\frac{1}{z^2}\left[1-\frac{3!}{z^2}+\cdots\right], \tag{7.9}$$

which, after substitution into (7.8), yields

$$A_{02}\sim\frac{1}{\omega_c^2}\frac{1}{x^2}-\left[\frac{5}{2}\frac{\omega_{E0}^2/\omega_c^2}{\omega_{E0}^2+\omega_c^2}+\frac{2\omega_{E0}^2}{(\omega_{E0}^2+\omega_c^2)^2}-\frac{2\omega_c^4/\omega_{E0}^2}{(\omega_{E0}^2+\omega_c^2)^2}\right]\frac{1}{x^4} ,$$
$$A_{12}\sim-\frac{1}{\omega_c^2}\frac{\pi}{2}\frac{1}{x^2}+\frac{1}{\omega_c\omega_{E0}}\frac{2}{x^3} , \tag{7.10}$$
$$A_{22}\sim-\frac{1}{\omega_c^2}\frac{2}{x^2}+\frac{1}{\omega_c\omega_{E0}}\frac{3\pi}{2}\frac{1}{x^3}+\left[\frac{29}{2}\frac{\omega_{E0}^2/\omega_c^2}{\omega_{E0}^2+\omega_c^2}-8\frac{\omega_{E0}^2}{(\omega_{E0}^2+\omega_c^2)^2}-8\frac{\omega_c^4/\omega_{E0}^2}{(\omega_{E0}^2+\omega_c^2)^2}\right]\frac{1}{x^4}$$

from which it is easy to see the exact cancellation of the $O(x^{-2})$ terms taking place in each of the $A_{l'1}+A_{l'2}$ terms and leading to the $O(x^{-3})$ behavior at large distances from the atom.

In order to have a more complete picture of the photon cloud, we now resort to numerical calculations from the exact expressions (7.3). In Fig. 1 we plot the quantities

$$(A_{21}^2+2A_{01}^2)r^2\propto\langle\mathbf{E}^-\cdot\mathbf{E}^+\rangle_{2p}r^2 ,$$
$$3A_{11}^2x^2\propto\langle\mathbf{B}^-\cdot\mathbf{B}^+\rangle_{2p}r^2 \tag{7.11}$$

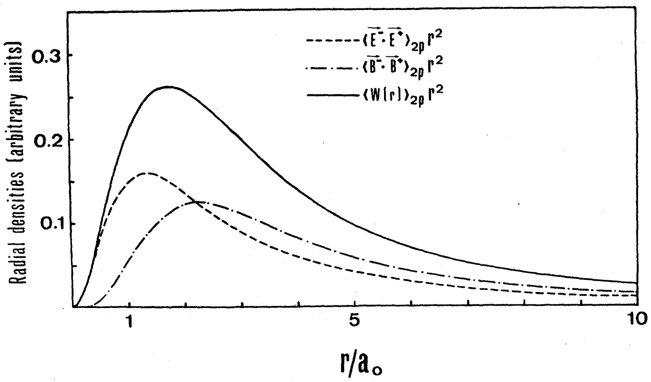

FIG. 1. The dashed lines refer to the $2p$ subshell electric and magnetic contributions to the CGED of the ground-state hydrogen atom. The solid line refers to the total $2p$ CGED. The scale is linear on both axes, and a_0 is the atomic Bohr radius.

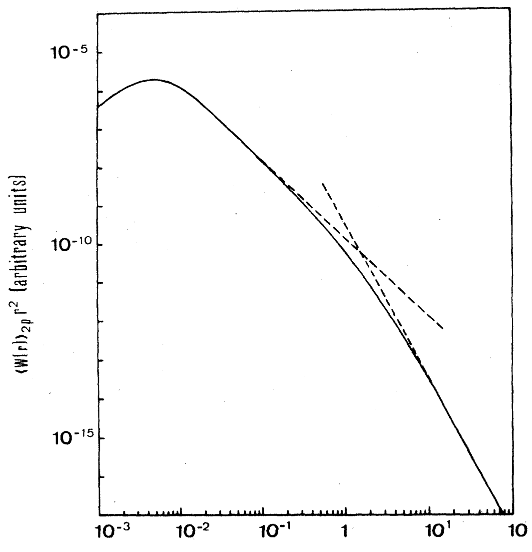

FIG. 2. The total $2p$ contribution to the CGED in the ground state of the hydrogen atom. The slopes of the tangents are -2 and -4, and they intersect at $z \simeq 1.6$. The scale is logarithmic on both axes, and $z=1$ corresponds to $r=c/\omega_{E0}$.

in the range $0<x<10$. Since in this range $A_{l'2}$ is negligible with respect to $A_{l'1}$, the two quantities in (7.11) are, respectively, proportional to the $2p$-subshell electric and magnetic contributions to the radial CGED, as is obvious from (6.13) and (6.14). The two densities are different, but the maxima of both are in the neighborhood of the Bohr radius and they display the same general behavior. In the same figure we also plot the total $2p$-subshell radial CGED

$$(A_{21}^2+2A_{01}^2+3A_{11}^2)x^2 \propto \langle W(\mathbf{r})\rangle_{2p} r^2 \tag{7.12}$$

as the sum of the two previous contributions. In Fig. 2 the quantity

$$[(A_{21}+A_{22})^2+2(A_{01}+A_{02})^2+3(A_{11}+A_{12})^2]z^2 \propto \langle W(\mathbf{r})\rangle_{2p} r^2 \tag{7.13}$$

is displayed in a doubly logarithmic scale in the range $10^{-3}<z<10^2$. The scale has been fixed by assuming $\omega_c/\omega_{E0}=548$. The change in slope around $z\sim 1$ is from -2 to -4, in agreement with the approximate estimate based on the ansatz of Sec. I, and it is seen to take place gradually over a broad range of z values. The position at which the tangents to the z^{-2} and z^{-4} parts of the curve cross is, however, a well-defined quantity which can be taken as a measure of the geometrical boundary of the $2p$ virtual photons subshell. This boundary can be analytically calculated as follows. In the z^{-2} region the main part of the CGED is obtained from the first three expressions for $A_{l'1}$ in (7.8) as

$$W_< \propto \left[6+\frac{3\pi^2}{4}\right]\frac{c^2}{\omega_c^8}\frac{1}{r^4} \tag{7.14}$$

while in the z^{-4} range we may take the $O(x^{-3})$ terms of $A_{l'2}$ in (7.10), yielding

$$W_> \propto \left[12+\frac{9\pi^2}{4}\right]\frac{c^6}{\omega_{E0}^2\omega_c^3}\frac{1}{r^6}\ . \tag{7.15}$$

The two straight lines representing (7.14) and (7.15) in a double-logarithmic scale intersect for $W_< = W_>$, that is, at

$$r_E=\left[\frac{12+9\pi^2/4}{6+3\pi^2/4}\right]\frac{c}{\omega_{E0}}\sim 1.6\frac{c}{\omega_{E0}}\ . \tag{7.16}$$

The semiquantitative treatment of Sec. VII leads us to conclude that the boundary of the E,l subshell was around $r_E \sim c/\omega_{E0}$. We see that this prediction does not compare too badly with (7.16).

As for the contributions coming from the other subshells with $l=1$ and different E, we remark that in both ranges $\omega a_0/c \gg 1$ and $\omega a_0/c \ll 1$ it is a good approximation to set

$$\frac{f(2,1,\omega)}{f(N,1,\omega)} \propto N^{3/2} \tag{7.17}$$

and independent of ω. Although (7.17) is likely to break down for $\omega a_0/c\sim 1$, one may hope that the range nonvalidity is small. Thus one finds from (6.13) and (6.14) that $\langle W(\mathbf{r})\rangle_{N,p}$ must be approximately given by the same function of ω_{E0} as $\langle W(\mathbf{r})\rangle_{2p}$, scaled by a factor proportional to N^{-3}, at least for E belonging to the discrete part of the hydrogen spectrum. Consequently the same argument leading to (7.16) should be valid also for $E=N$ and $l=1$ in general. Using for ω_{E0} the well-known expression of the hydrogen discrete spectrum we obtain that the outer part of the virtual cloud of photons is dominated by a series of concentric subshells of radii given by

$$r_{N,1}\sim 1.6\times 2\frac{\hbar a_0 c}{e^2}\frac{N^2}{N^2-1}\ . \tag{7.18}$$

More generally, we are led to the conclusion that the structure of the virtual cloud of photons in the ground state is in fact an "inside-out" mapping of the electronic structure of the hydrogen atom.

VIII. CONCLUSIONS

We summarize the results of the work reported in this paper as follows. We defined a CGED operator $W(r)$ in (5.9), which is related to the energy density of the electromagnetic field and which appears to be a suitable operator to describe the distribution of photons in space. In fact, as shown in Sec. V, $\int_{V_1} W(\mathbf{r})d^3r$ yields the energy carried by the photons within volume V_1 simply as the sum of the individual energies of each photon, while $W(r)$ satisfies an appropriate continuity equation. We have obtained the quantum-mechanical average $\langle W(\mathbf{r})\rangle$ in the ground state of a spinless hydrogen atom coupled to the electromagnetic field. This ground state has been obtained by perturbation theory, using the minimal coupling form of the radiation Hamiltonian. Within these limits, we have calculated $\langle W(\mathbf{r})\rangle$ exactly, and we have shown that it consists of an infinite number of contributions which add incoherently. Each contribution can be associated to the virtual transitions leading from the $1s$ ground state to all eigenstates of the bare hydrogen atom belonging to the same subshell. All bare atomic eigenstates, including the continuum, contribute to the total $\langle W(\mathbf{r})\rangle$, as implicit in (6.13) and (6.14). From a semiquantitative analysis of the latter expressions, we have found that each subshell belonging to the same shell of energy

$$E_{\text{shell}}-E_1 \equiv \hbar\omega_{E0}$$

yields a cloud of virtual photons extending out to a distance of the order of $r_E = c/\omega_{E0}$. This can be understood in modelistic terms by considering that the lifetime of a virtual photon of frequency ω, emitted in a virtual transition corresponding to an energy difference $\hbar\omega_{E0}$ between the atomic states involved, is $\sim(\omega+\omega_{E0})^{-1}$; hence this photon can reach out to a distance $\sim c(\omega+\omega_{E0})^{-1}$ from the atom. We see that $r_z = c/\omega_{E0}$ coincide with the limiting reach of the low-energy virtual photons. Thus the whole dressed hydrogen atom can be thought of as being contained within a sphere of radius $c/\omega_{2p,1s} \sim 1200$ Å, the other virtual Nl-$1s$ transitions yielding shells of virtual photons of progressively smaller dimensions. Within each shell, the contributions arising from the various subshells with different values of l are different from each other, since the $l=1$ subshell has been shown to dominate the others with $l \neq 1$ and since the r dependence of each subshell changes from r^{2l-2} to r^{2l-4} over a fairly broad region around r_E. In particular, exact calculations concerning the $2p$ subshell have been performed, which show that the r dependence of the CGED changes from r^{-4} to r^{-6} at $r_E \sim 1.6c/\omega_{2p,1s}$.

It may be of some interest to compare the rather complex shell-structured cloud of virtual phonons around a hydrogen atom discussed in this paper to the structure of the virtual clouds arising in other problems with different physical contexts. As it is well known, the ground state of a meson field linearly coupled to a static source entails a cloud of virtual mesons which decays exponentially over a distance of the order of the Compton wavelength of the meson.[29] Remarkably, the same qualitative agreement given above for the photon cloud can be used to explain this result, since the minimum energy cost to create a virtual meson is $\sim mc^2$, and consequently the lifetime of this meson is at most $\sim\hbar/mc^2$. In this time the virtual meson cannot travel over a distance larger than $\sim\hbar/mc$, which is its Compton wavelength. In the more complicated case, mentioned in the Introduction, of the cloudy bag model for the nucleon,[13] the charge density of the pion cloud is obtained as the sum of two contributions $\rho_{\pi N}$ and $\rho_{\pi\Delta}$ which arise, respectively, from the N and Δ degrees of freedom of the three quarks inside the bag. This is analogous to the shell structure in our approach to the photon cloud, although the situation in the cloudy bag model is more complicated because the leading contributions to the charge density arise from the quarks inside the bag rather than from the pion cloud. In the case of the optical polaron, also mentioned in the Introduction,[12] the polarization charge density does not display any evident shell structure, which is understandable since the source of the polarizing field is in this case structureless, contrary to the case of the hydrogen atom considered here or to the case of the MIT bag of the cloudy bag model for the nucleon. The familiar exponential behavior of the lattice polarization density at large disances from the electron[30] might seem rather surprising in the light of quantum field theory, since one would associate an exponential decay with a finite mass of the particles in the virtual cloud, and an inverse power law with massless particles such as photons or phonons. In the optical polaron case, however, this exponential behavior is likely to be associated with the optical character of the phonons contributing to the lattice polarization, which provides a finite excitation energy also for zero-momentum phonons.

Finally, we wish to spend a few words in relation to the work by Power and Thirunamachandran[14] mentioned in the Introduction. In our opinion, an approach of the sort they have developed is complementary to ours, in the sense that we work in the Schrödinger representation while their calculations are entirely developed in the Heisenberg picture. Consequently, we expect that a calculation of $W(r)$ using their technique should give the same results as in our approach. No comparison, however, is possible between our present results and those given in their paper, since most of the latter results have been obtained using the multipolar Hamiltonian, from which it is natural to calculate the transverse displacement field operator $\mathbf{d}(r)$. This operator, outside the source, is equal to the total electric field operator rather than to the transverse electric field which we use in this paper in the definition of $W(r)$. As a consequence, the r^{-7} behavior of the total energy density, which is proportional to the van der Waals interaction energy at large distances as obtained by Power and Thirunamachandran, cannot be compared to our r^{-6} behavior of $W(r)$ at the same distances, because the former is proportional to $d^2(\mathbf{r})$ and $W(r)$ is only a part contributing to it.

In conclusion, the shell structure that we propose in this paper for the cloud of virtual photons dressing the ground state of a hydrogen atom seems to be a well-founded and reasonable concept and perhaps a nontrivial contribution to the clarification of a the physical meaning of field-dressed particles.

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