Recoil contributions to the Lamb shift in the external-field approximation

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The calculation of recoil corrections to the Lamb shift in hydrogen is carried out by treating the Dirac electron in the presence of the nuclear Coulomb and convection potentials. The leading Dirac equation and its solutions were given by Grotch and Yennie many years ago. A systematic derivation of recoil corrections to the leading Lamb shift leads to results in agreement with previous work. New terms of order $\alpha(Z\alpha)^5m^2/M$ are derived by examining recoil corrections to higher-order radiative level shifts. These reduce the theoretical Lamb shift in the hydrogen $n=2$ state by about 2 ppm. It is expected, however, that additional contributions of the same order will also arise from corrections to the external-field approximation as well as from nonradiative recoil.

I. INTRODUCTION

The most demanding tests of quantum electrodynamics are to be found in the comparisons between theory and experiment of results which are sensitive to radiative corrections, since the theoretical analyses of these requires the full use of renormalization theory. Thus the anomalous moments of leptons and the Lamb shifts in hydrogenic atoms have provided fruitful testing grounds for many years.[1]

The Lamb shift has had a long history with many theoretical and experimental refinements, but there are still small discrepancies between theory and experiment which require additional work.[1] For the Lamb shift in hydrogen the current theoretical status is indicated by calculations which lead to the results[2] (in MHz)

$$S=\Delta E(2S_{1/2})-\Delta E(2P_{1/2})=1057.862$$

or 1057.884 ± 0.02, depending upon whether one uses the old rms proton electromagnetic radius[3] of $\langle r^2\rangle^{1/2}=0.805(11)$ fm or the more recent[4] value $\langle r^2\rangle^{1/2}=0.862(12)$ fm for the proton-size correction. The most recently published experimental results are[5]

$$S=1057.845(9)$$

(Lundeen and Pipkin) and

$$S=1057.8594(19)$$

(Sokolov and Yakovlev).

A comparison of theory and experiment suggests good agreement when the older rms proton radius is used. On the other hand, for the more recently published radius, the theoretical value is somewhat larger. The theoretical numbers arrived at by Mohr[2] through his numerical calculation have been partially confirmed by computations of Sapirstein.[6] The analytical results of Erickson,[7] however, are significantly larger than Mohr's results. This discrepancy occurs at the level of $\alpha(Z\alpha)^6$ terms.

The primary purpose of the present research is to study a part of the Lamb shift problem which has not, as yet, received the attention which now appears warranted. It is our intention to systematically analyze the nuclear recoil contributions to the Lamb shift. Before discussing the content of the present manuscript and related work, we will first examine some orders of magnitude appropriate to the Lamb shift.

Let us first assume that the nuclear mass M is infinite. Then the Lamb shift calculation entails an analysis of the radiative energy shifts of an electron bound by the Coulomb field of the nucleus.[8] The leading term in the expansion parameter $Z\alpha$ leads to energy shifts of order $m\alpha(Z\alpha)^4\ln(Z\alpha)$, $m\alpha(Z\alpha)^4$, and $m\alpha(Z\alpha)^4\ln(m/\Delta\epsilon_n)$. If one now relaxes the restriction of infinite nuclear mass and allows for a finite value, corrections of order $(m^2/M)\alpha(Z\alpha)^4\ln(Z\alpha)$, $(m^2/M)\alpha(Z\alpha)^4$, and

$$(m^2/M)\alpha(Z\alpha)^4\ln(m/\Delta\epsilon_n)$$

emerge. We believe these have already been evaluated correctly but in the present work we systematically rederive these results. There are also nonradiative recoil corrections of order $(m^2/M)(Z\alpha)^5$ which contribute to the Lamb shift. These have been worked out by a number of authors.[9–11]

Proceeding now to higher-order corrections to the Lamb shift resulting from lowest-order radiative corrections, we note that there are contributions of order $m\alpha(Z\alpha)^5$.[12] It was originally speculated that the modification of this result due to finite nuclear mass could be brought about exclusively through the modification of the square of the wave function at the origin. Although such a correction is surely present, it appears unlikely that this is the only correction. It is our opinion that the Lamb shift contributions of order $(m^2/M)\alpha(Z\alpha)^5$ are not presently known, and hence a major portion of our effort is to obtain these corrections. In addition to the above corrections, which we refer to as radiative recoil, there are also nonradiative recoil corrections of order $(m^2/M)(Z\alpha)^6$ as well as rather small corrections of order $(m^2/M)\alpha^2(Z\alpha)^4$.

In the present paper we present an analytic calculation of part of the $(m^2/M)\alpha(Z\alpha)^5$ radiative recoil correction. This part is based on the so-called external-field approximation. Another publication will deal with additional contributions of the same order as those presented here, except that these terms result from corrections to the external-field approximation. They can most readily be thought of as radiative corrections to the recoil corrections of order $(m^2/M)(Z\alpha)^5$ and hence are also of order $(m^2/M)\alpha(Z\alpha)^5$. Further work, not yet completed, will be devoted to the contributions of order $(m^2/M)(Z\alpha)^6$. These are nonradiative higher-order recoil corrections.

The plan of the present paper is as follows. In Sec. II A we review a modified Dirac equation and its solution. In Sec. II B we discuss the radiative level shifts in the external-field approximation. Section III contains our calculation of lowest-order results, that is mass corrections to the leading terms in the Lamb shift. Section IV provides a discussion of the mass corrections of order $(m^2/M)\alpha(Z\alpha)^5$ which result from the external-field approximation. Section V contains conclusions while an Appendix contains a brief discussion of the effective Dirac equation.

II. EXTERNAL-FIELD APPROXIMATION

A. Modified Dirac equation

In the absence of radiative corrections, the hydrogen atom is well described by the Hamiltonian[11]

$$H_D=\boldsymbol{\alpha}\cdot(\mathbf{p}-e\,\mathbf{A})+\beta m+V+\frac{p^2}{2M}\ . \tag{2.1}$$

A discussion of H_D is given in the Appendix. To arrive at this approximate Hamiltonian we assume that the dominant interaction between the electron and the proton (presumed to be spinless) is due to one photon exchange, evaluated in the Coulomb gauge. We assume that the proton can be placed on its positive energy mass shell in all ladder graphs contributing to the four point function. Corrections to this assumption must be examined.

In the above equation we note that

$$V=-\frac{Z\alpha}{r} \tag{2.2}$$

and

$$e\,\mathbf{A}=-\frac{1}{2M}V(\mathbf{p}+\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{p})=-\frac{1}{2M}(\mathbf{p}+\mathbf{p}\cdot\hat{\mathbf{r}}\hat{\mathbf{r}})V\ . \tag{2.3}$$

We refer to (2.1) as the external-field approximation since we have effectively reduced the Hamiltonian in the center-of-mass reference system to a one-particle Hamiltonian. Thus, the proton produces a four-potential A_μ which couples to the electron and also contributes a kinetic energy term $p^2/2M$.

The eigenvalue problem for H_D of Eq. (2.1) was studied by Grotch and Yennie, who found that to leading order in M^{-1} the eigenvalues and eigenvectors could readily be found.[11] The eigenvalues provide the correct reduced mass dependence to the Dirac energy levels including mass corrections to the fine-structure separation of order $m^2(Z\alpha)^4/M$. The eigenvectors are given by

$$|\,n\,\rangle=\left[1-\frac{1}{4M}[\mathbf{p}^2,W]\right]\left[\frac{1+\gamma_0\lambda}{1+\lambda}\right]|\,n\,) \tag{2.4}$$

with $W=-Z\alpha r$, $\lambda=m/2M$, and $|\,n\,)$ the solution of the Dirac Coulomb problem with m replaced by the reduced mass μ.

In this paper we will find it convenient to use the relation

$$\begin{aligned}[\mathbf{p}^2,W]&=-2V+2iZ\alpha\hat{\mathbf{r}}\cdot\mathbf{p}=2V+2i\mathbf{p}\cdot\hat{\mathbf{r}}Z\alpha\\&=-2V(1+i\mathbf{r}\cdot\mathbf{p})\\&=2(1-i\mathbf{p}\cdot\mathbf{r})V\ .\end{aligned} \tag{2.5}$$

Hence,

$$|\,n\,\rangle=\left[1+\frac{V}{2M}(1+i\mathbf{r}\cdot\mathbf{p})\right]\left[\frac{1+\gamma_0\lambda}{1+\lambda}\right]|\,n\,) \tag{2.6}$$

and

$$\begin{aligned}\langle\,n\,|&=(\,|\,n\,\rangle)^\dagger\gamma_0\\&=(\,n\,|\left[\frac{1+\gamma_0\lambda}{1+\lambda}\right]\left[1+(1-i\mathbf{p}\cdot\mathbf{r})\frac{V}{2M}\right]\ ,\end{aligned} \tag{2.7}$$

where we absorb γ_0 by assuming $(\,n\,|=[\,|\,n)]^\dagger\gamma_0$. In Eqs. (2.6) and (2.7) we now write

$$|\,n\,\rangle=|\,n_0\,\rangle+\delta\,|\,n\,\rangle\ ,$$

$$\langle\,n\,|=\langle\,n_0\,|+\delta\langle\,n\,|\ ,$$

where in both instances the correction term is the part proportional to $V/2M$. This decomposition will be used later.

It is useful to note that the upper and lower components of $|\,n_0\,\rangle$ are approximately given by

$$|\,n_0\,\rangle_u=|\,n_0)$$

and

$$|\,n_0\,\rangle_l=\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2m}\,|\,n_0)\ , \tag{2.8}$$

where $|\,n_0)$ is a two-component state vector with reduced mass. Thus $|\,n_0)$ is a nonrelativistic two-component reduced mass solution of the hydrogen atom.

In what follows we will find it convenient to write the Dirac equation in the form

$$(\not{\Pi}-m)\,|\,n\,\rangle=0 \tag{2.9}$$

with

$$\Pi^0=E_n-V-p^2/2M$$

and

$$\boldsymbol{\Pi}=\mathbf{p}-e\,\mathbf{A}\ . \tag{2.10}$$

Furthermore, the relations

$$ieE^i\equiv[\Pi^0,\Pi^i]=(-i\nabla^i V)\left[1+\frac{V}{M}\right]=\left[p^i,V+\frac{V^2}{2M}\right],\tag{2.11}$$

$$ie\epsilon^{ijk}B^k\equiv[\Pi^i,\Pi^j]=-\frac{i}{M}[(\nabla^i V)p^j-(\nabla^j V)p^i]\tag{2.12}$$

will also prove useful.

B. Radiative level shifts

Let us now briefly review the formalism for radiative level shifts in the external-field approximation. Our starting point is the formal expression for the radiative energy shift of an electron in a state $|n\rangle$, as given by Erickson and Yennie,[8]

$$\Delta E_n=\frac{\alpha}{4\pi^3}\int\frac{d^4k/i}{k^2}\langle n|\gamma_\mu\frac{1}{\not\Pi-\not k-m+i\epsilon}\gamma^\mu|n\rangle$$
$$-\delta m\langle n|n\rangle,\tag{2.13}$$

where now Π^μ is given in Eq. (2.10) and $|n\rangle$ is given by Eq. (2.6). This expression, which excludes vacuum polarization, is shown schematically in Fig. 1.

Before proceeding further let us discuss some of the diagrams which contribute to Fig. 1. In Fig. 2 one-photon exchange diagrams with radiative corrections on the electron side are illustrated. Such graphs represent approximate contributions to two-particle propagation and these graphs enter in a natural way in the Bethe-Salpeter construction of the complete four-point function.

On the proton side (the right side) of a higher-order graph, we decompose the proton propagators into double solid lines which give propagation on the positive energy mass shell while the remainder term denoted by the dot is the residual correction, as illustrated in Fig. 3. Then if the proton has four momentum $-p^\mu$ we find

$$S_F(-p^\mu)\gamma_0=-2\pi i\Lambda_+(-\mathbf{p})\delta[p_0+(\mathbf{p}^2+M^2)^{1/2}]$$
$$+\frac{1}{-p_0-i\epsilon-[\boldsymbol{\alpha}\cdot(-\mathbf{p})+\beta M]}.\tag{2.14}$$

The first term in Eq. (2.14) represents the double solid line while the second contains the remainder. This type of method has been discussed by various authors in a number of equivalent ways[13] and some discussion is provided in the Appendix.

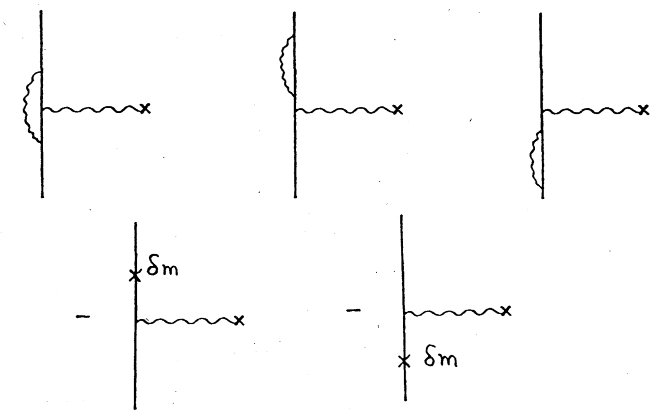

FIG. 2. One-photon exchange with radiative corrections to the electron.

In the external field approximation the proton propagator is consistently replaced by the first term of Eq. (2.14) and hence the graph shown in Fig. 4 is approximated as shown. Since the above procedure is approximate, at some stage we must study and evaluate all the terms which are missing as a result of the indicated replacement.

Extending the above discussion to multiphoton exchange entails the same basic ideas. Figure 1 contains all radiative corrections on the electron side to ladder diagrams in which the proton is consistently placed on the positive energy mass shell. The advantage of this approach is that the residual terms will be explicitly of order $1/M$ and will therefore involve radiative corrections to recoil diagrams. The disadvantage is that ultimately this separation could result in greater complexity in the actual evaluation of mass corrections.

The present paper consists primarily of the evaluation of Eq. (2.13). Following the standard discussion of Ref. 8 we convert Eq. (2.13) into

$$\Delta E_n=\frac{\alpha}{4\pi}\int_0^1 dz\int_0^\infty dK\int\frac{d^4k}{\pi^2 i}\langle n|I|n\rangle-\langle n|\delta m|n\rangle\tag{2.15}$$

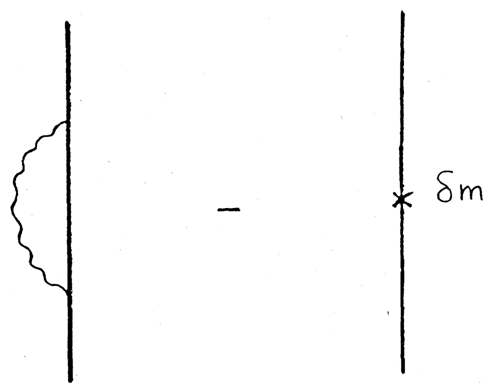

FIG. 1. Electron self-energy in the external field.

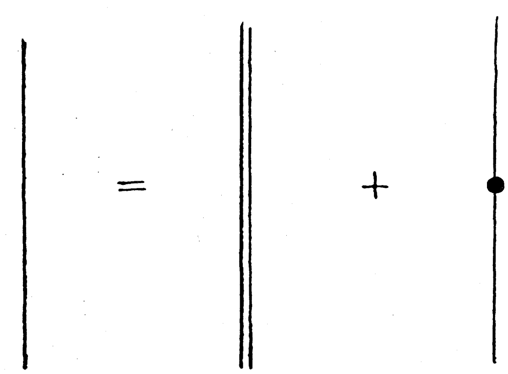

FIG. 3. Decomposition of proton propagator according to Eq. (2.14).

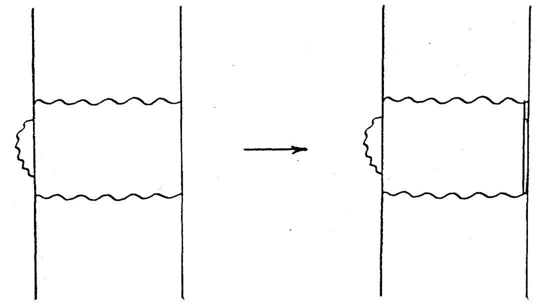

FIG. 4. An approximation to a diagram in which the proton propagates on shell.

with

$$I=2\gamma_\mu \frac{\not{\Pi}-\not{k}+m}{D^3}\gamma^\mu \tag{2.16}$$

and

$$D=K-k^2+2zk\cdot\Pi-z\not{\Pi}^2+zm^2 \;. \tag{2.17}$$

The calculation then proceeds very much like the usual Lamb shift calculation except now there is an M dependence contained in the states and in the operators. The reduction of (2.15) proceeds by separation into the sum of many terms,[14] namely, $I_L+I_{\mathscr{M}}+I_a+I_b+I_c+I_d+I_e+I_f$. In this paper we shall study each of these terms in order to extract its mass dependence.

III. LOWEST-ORDER RESULTS

In this section we shall present a systematic derivation of mass corrections to the lowest-order Lamb shift. The terms we are looking for are of order $\alpha(Z\alpha)^4m^2/M$. Since the leading contributions to the Lamb shift are contained in $I_{\mathscr{M}}$ and I_L we will examine only these. For the leading terms we find that the separation into various components $I_{\mathscr{M}}=I_{\mathscr{M}1}+I_{\mathscr{M}2}$ and $I_L=I_{L1}+I_{L2}+I_{L3}+I_{L4}$ is not needed since to lowest order the sums are given in the literature.

Thus,

$$\Delta E_n(\mathscr{M})=\frac{\alpha}{2\pi}\langle n \,|\left[-\frac{\mathscr{M}}{2m}\right]|\, n\rangle$$

$$=\frac{\alpha}{2\pi}\langle n \,|\left[-\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}+\frac{ie}{2m}\boldsymbol{\alpha}\cdot\mathbf{E}\right]|\, n\rangle \tag{3.1}$$

with $\mathbf{E}$ and $\mathbf{B}$ given in (2.11) and (2.12), respectively, and

$$\Delta E_n(L)=-\frac{\alpha}{\pi}\int_0^1 du\int_0^1 dz\, P(z,u)\langle n\,|p^i\frac{1}{\Delta}[p^i,V]\,|\,n\rangle \tag{3.2}$$

with

$$P(z,u)=-2(1-z^2)u(1-u)+1-z$$
$$+z(1-z)(1-u)+z^2(1-u)^2$$

and

$$\Delta=zm^2+u(1-z)2m\left[\frac{p^2}{2\mu}+V-\epsilon_n\right]$$

$$\equiv zm^2+u(1-z)H_{NR} \;.$$

We now turn to the calculation of $\Delta E_n(\mathscr{M})$. To order $1/M$ we can write the leading term as

$$\Delta E_n(\mathscr{M})=\frac{\alpha}{2\pi}\langle n_0 \,|\left[-\frac{e}{2m}\boldsymbol{\sigma}\cdot\frac{\mathbf{E}\times\mathbf{p}}{M}-\frac{i}{2m}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} V\right]|\, n_0\rangle$$

$$+\frac{\alpha}{2\pi}\langle \delta n_0 \,|\left[-\frac{i}{2m}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} V\right]|\, n_0\rangle+\frac{\alpha}{2\pi}\langle n_0 \,|\left[-\frac{i}{2m}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} V\right]|\, \delta n_0\rangle \;. \tag{3.3}$$

Note that in the expression for $\mathbf{E}$ (2.11) we have dropped the $V^2/2M$ term. Also note that the bra state contains a γ^0. The odd operators in the above expression connect the upper and lower components of the wave function and hence we obtain

$$\Delta E_n(\mathscr{M})=\frac{\alpha}{2\pi}(n_0\,|\,\frac{1}{2mM}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} V\times\mathbf{p}-\frac{i}{(2m)^2}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} V\boldsymbol{\sigma}\cdot\mathbf{p}-\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} V)\,|\,n_0) \tag{3.4}$$

for the first term of (3.3). The terms in (3.3) arising from the state vector correction will be quadratic in V and will also involve a momentum acting on the wave function. These terms are smaller than those considered in this paper [order $\alpha(Z\alpha)^5m^2/M$] and hence can be dropped.

We then find from (3.4) that we obtain

$$\Delta E_n(\mathscr{M})\cong\frac{\alpha}{2\pi}(n_0\,|\left[\frac{1}{2mM}+\frac{1}{2m^2}\right]\frac{dV}{dr}\boldsymbol{\sigma}\cdot\mathbf{L}$$

$$+\frac{1}{4m^2}\nabla^2V\,|\,n_0) \;. \tag{3.5}$$

This expression is easy to evaluate for hydrogen. It gives

$$\Delta E_n(\mathscr{M})=\frac{\alpha}{2\pi}\frac{(Z\alpha)^4 m}{n^3}\left[\frac{\mu}{m}\right]^3 \cong\frac{\alpha}{2\pi}\frac{(Z\alpha)^4 m}{n^3}\left[1-\frac{3m}{M}\right] \text{ for } l=0 \quad (3.6)$$

and

$$\Delta E_n(\mathscr{M})=\frac{\alpha}{2\pi}\frac{(Z\alpha)^4 m}{n^3}\left[\frac{\mu}{m}\right]^2 \cong\frac{\alpha}{2\pi}\frac{(Z\alpha)^4 m}{n^3}\left[1-\frac{2m}{M}\right]C_{1,j} \text{ for } l=1 \quad (3.7)$$

where $C_{1,1/2}=-\frac{1}{3}$ and $C_{1,3/2}=\frac{1}{6}$.

Thus for the $l=0$ state the nuclear mass correction enters through the square of the wave function at the origin since the $\nabla^2 V$ is localized at the origin. On the other hand, for $l\neq 0$ the mass correction contains two pieces.

(i) The first is the usual spin-orbit term which will have a mass correction contained in $1-3m/M$.

(ii) The second is a spin-other-orbit interaction arising from the magnetic field of (3.1). This term partially combines with the mass correction of (i), leading to a total of $-2m/M$.

We now turn to the terms arising from $\Delta E_n(L)$ of Eq. (3.2). The wave-function corrections will contribute important terms to be worked out in subsequent sections. They are not considered in this section. The mass corrections will now enter through the reduced mass factors in the upper component nonrelativistic state vector $|n_0)$ and also through the presence of reduced mass in H_{NR}.

We now repeat the calculation on p. 298 of Erickson and Yennie (EY).[8] Using their Eq. (3.20) and dropping the last term, which is a higher-order correction, we obtain

$$\begin{aligned}\Delta E_n(L)&=-\frac{\alpha}{\pi}\int_0^1 du(n_0|p^i\left[\frac{P(0,u)}{m^2}\ln\frac{m^2}{uH_{NR}}+\int_0^1 dz\frac{P(z,u)-P(0,u)}{zm^2}\right][p^i,V]|n_0)\\&=-\frac{\alpha}{\pi}\int_0^1 du(n_0|p^i\left[\frac{P(0,u)}{m^2}\left[\ln\frac{m/\mu}{u(Z\alpha)^2}+\ln\frac{\mu m(Z\alpha)^2}{H_{NR}}\right]+\int_0^1 dz\frac{P(z,u)-P(0,u)}{zm^2}\right][p^i,V]|n_0)\\&=-\frac{\alpha}{\pi m^2}(n_0|p^i[p^i,V]|n_0)\frac{2}{3}\left[\ln\frac{1}{(Z\alpha)^2}+\frac{11}{24}+\ln\left[1+\frac{m}{M}\right]\right]\\&\quad-\frac{\alpha}{\pi m^2}(n_0|p^i\ln\left[\frac{\mu m Z^2\alpha^2}{H_{NR}}\right][p^i,V]|n_0)\times\frac{2}{3}\,.\end{aligned} \quad (3.8)$$

In the first term we use

$$\frac{1}{m^2}(n_0|p^i[p^i,V]|n_0)=-\frac{2(Z\alpha)^4 m}{n^3}\left[\frac{\mu}{m}\right]^3\delta_{l0} \quad (3.9)$$

while in the second we use

$$\frac{1}{m^2}(n_0|p^i\ln\left[\frac{\mu m Z^2\alpha^2}{H_{NR}}\right][p^i,V]|n_0)\equiv-\frac{2(Z\alpha)^4 m}{n^3}\left[\frac{\mu}{m}\right]^3\ln\frac{\mu(Z\alpha)^2}{2\Delta\epsilon_n}\,, \quad (3.10)$$

where $\Delta\epsilon_n$ is the average excitation energy, including reduced mass factors. Using (3.9) and (3.10) we obtain

$$\Delta E_n(L)\cong\frac{4m\alpha(Z\alpha)^4}{3\pi n^3}\left[1-\frac{3m}{M}\right]\left\{\left[\ln\frac{1}{(Z\alpha)^2}+\frac{11}{24}+\ln\left[1+\frac{m}{M}\right]\right]\delta_{l0}+\ln\frac{\mu(Z\alpha)^2}{\Delta\epsilon_n}\right\} \quad (3.11)$$

thus verifying Eq. 4.1a of Ref. 8 except for the factor of $-\frac{1}{5}$ which arises from vacuum polarization.

IV. HIGHER-ORDER MASS CORRECTIONS

In the previous section we presented mass corrections to leading terms in the Lamb shift. In this section we discuss and present contributions of order $\alpha(Z\alpha)^5 m^2/M$.

The calculation proceeds by analysis of the EY reduction. Special care is needed to retain terms which were dropped by EY as a consequence of their choice of Π_μ. For example, for the infinite mass case, (2.3) and (2.10) above lead to (2.12), i.e., to noncommutativity of the components of Π_i. Such commutators can, in principle, lead to new contributions. However, if M tends to infinity these components commute, and hence such terms would not have been present in EY.

In the present work we closely follow the operator reduction of EY, which leads to terms I_L, $I_{\mathscr{M}}$, I_a, I_b, I_c, I_d, I_e, and I_f. From (2.73) of EY we note the possibility of an additional term arising from the noncommutativity of Π_0 and the field $F_{\mu\nu}$, but this turns out to vanish identically due to the Jacobi identity. The subsequent reduction leads to mass corrections of various origins. There are some contributions which may arise from noncommutativity of Π_0 and $F^{\mu\nu}$, while others originate when the denominators are reduced and corrections to the reduction procedure are no longer negligible.

Before proceeding to a detailed discussion of these new terms, we note that when the nucleus has finite mass the recoiling nuclear kinetic energy must be added to the electron kinetic energy. As a consequence of this, the operator

$$H \equiv m^2 - \not{\Pi}^2 \cong H^{NR} = p^2\left[1+\frac{m}{M}\right] + 2mV + \gamma^2 \quad (4.1)$$

is now modified, and hence $(Z\alpha)^5$ integrals on p. 464 of EY now have denominators which are altered. If $\lambda=0$, then

$$uz(1-z)p^2 \to uz(1-z)p^2\left[1+\frac{m}{M}\right], \quad (4.2)$$

thus leading to an extra factor $[1+(m/M)]^{-1/2}$. On the other hand, if $\lambda \neq 0$, then the denominator in the parametric integral is modified as

$$[1-z+\lambda^2(1-u)z]^{-1/2} \to \left[(1-z)\left[1+\frac{m}{M}\right]+\lambda^2(1-u)z\right]^{-1/2}. \quad (4.3)$$

In addition to this change, in adjustment (4) on p. 465 EY terms with $-p^2/2m$ instead of V will need an extra factor of $[1+(m/M)]^{-1}$ since the large momentum limit of V acting on the NR wave function is now the same as $-p^2/2\mu$. This second effect can be trivially included in the $(Z\alpha)^5$ results tabulated in Table VIII (EY, p. 493) by simply identifying those terms (only $L-p$ and $\mathscr{M}-p$) which are p^4V rather than Vp^2V and multiplying them by 4, rather than 3, reduced mass factors. The first effect, which is due to reduced-mass scaling for binding in the intermediate state, produces an extra reduced mass factor of $\frac{1}{2}$ for $L-H$ terms, but cannot be as readily scaled for the other terms. Nevertheless the integrals can be expanded in m/M and easily evaluated. The total contribution from the sources discussed above (not including the overall $1-3m/M$) contribution is[15]

$$\frac{\alpha(Z\alpha)^5 m^2}{n^3 M}\left(-8+\tfrac{19}{2}\ln 2+\tfrac{7}{32}\right). \quad (4.4)$$

We will now discuss in some detail the structure of the contributing terms, the extraction of those parts already contained in (4.4), and additional new corrections. To start, consider the terms I_L given by

$$I_{L1} = 8(1-z^2)z^3m^2\int_0^1 d\lambda^2 \frac{1}{D^\lambda}\Pi_\nu\frac{1}{D^\lambda}[\Pi^\nu,\not{\Pi}]\frac{1}{D_\lambda^2}\Bigg|\Bigg|\frac{1}{D_\lambda}, \quad (4.5)$$

$$I_{L2} = -4z(1-z)\Pi_\nu\frac{1}{D}[\Pi^\nu,\not{\Pi}]\frac{1}{D}\Bigg|\Bigg|\frac{1}{D^2}, \quad (4.6)$$

$$I_{L3} = -4z^2(1-z)\frac{1}{D}\Pi_\nu\frac{1}{D}[\Pi^\nu,\not{\Pi}]\frac{1}{D}\Bigg|\Bigg|\frac{1}{D}, \quad (4.7)$$

$$I_{L4} = -8z^3\frac{1}{D}\Pi_\nu\frac{1}{D}[\Pi^\nu,\not{\Pi}]\frac{1}{D^2}, \quad (4.8)$$

where

$$D = z^2m^2+K-(k-z\Pi)^2+z(1-z)H-z^2\mathscr{M}, \quad (4.9)$$

$$D_\lambda = z^2m^2+K-(k-\lambda z\Pi)^2+z(1-z)H. \quad (4.10)$$

Equation (4.9) is an alternate form of (2.17), where

$$H = m^2-\not{\Pi}^2 \quad (4.11)$$

$$\mathscr{M} = \not{\Pi}^2-\Pi^2 = e\boldsymbol{\sigma}\cdot\mathbf{B} - ie\boldsymbol{\alpha}\cdot\mathbf{E} \quad (4.12)$$

[see Eq. (3.1)].

For each $I_{L\eta}$ $(\eta=1,2,3,4)$ we write $I_{L\eta}=I_{L\eta}^0+I_{L\eta}^i$, where $I_{L\eta}^0$ denotes the part coming from the structure $\cdots\Pi^0\cdots[\Pi^0,\not{\Pi}]\cdots$ while $I_{L\eta}^i$ has the form $\cdots\Pi^i\cdots[\Pi^i,\not{\Pi}]\cdots$. The $I_{L\eta}^i$ terms contain the lowest-order Lamb shift terms as well as higher-order corrections. On the other hand, the $I_{L\eta}^0$ terms contain only higher-order terms.

A. I_{L1} contributions

To the desired accuracy, for $\eta=1$ we find

$$I_{L1}^i \cong -8(1-z^2)z^3m^2\int_0^1 d\lambda^2\frac{1}{D_\lambda}p^i\frac{1}{D_\lambda}[V,p^i]\frac{1}{D_\lambda^2}\Bigg|\Bigg|\frac{1}{D_\lambda}. \quad (4.13)$$

Our procedure for this term is to expand about $\lambda=0$ using the methods of EY. This gives

$$\begin{aligned} I_{L1}^i &\cong -8(1-z^2)z^3m^2\frac{1}{D_0}p^i\frac{1}{D_0}[V,p^i]\frac{1}{D_0^2}\Bigg|\Bigg|\frac{1}{D_0} \\ &\quad -8(1-z^2)z^3m^2\int_0^1\frac{d\lambda^2}{\lambda}(1-\lambda^2)\left[-\frac{1}{D_\lambda}p^i\frac{1}{D_\lambda}[V,p^i]\frac{1}{D_\lambda^2}\Bigg|\Bigg|\frac{1}{D_\lambda}\right]\Bigg|\Bigg|\frac{zk\cdot\Pi-\lambda z^2\Pi^2}{D_\lambda} \\ &\equiv I_{L1,0}^i + I_{L1,\mathrm{shift}}^i. \end{aligned} \quad (4.14)$$

Writing

$$zk\cdot\Pi-\lambda z^2\Pi^2 = zk_0\Pi_0-\lambda z^2\Pi_0^2 - z\mathbf{k}\cdot\mathbf{\Pi}+\lambda z^2\mathbf{\Pi}^2 \quad (4.15)$$

we separate the shift correction into two terms, $I^i_{L1-k_0}$ and I^i_{L1-p} where the first of these comes from the first two terms of (4.15) while the second comes from the last two terms. Thus, we have separated I_{L1} into contributions $I^i_{L1,0}$, $I^i_{L1-k_0}$, I^i_{L1-p}, and I^0_{L1}.

Using (2.15) it is relatively straightforward to show that to the desired accuracy

$$\Delta E(I^i_{L1,0})=\frac{\alpha}{\pi}\int_0^1 dz\int_0^1 du\, 2u(1-u)(1-z^2)\langle n\,|p^i\frac{1}{zm^2+u(1-z)H}[p^i,V]\,|\,n\rangle \tag{4.16}$$

which agrees with the first term of 3.27b of EY, except that H now includes mass corrections, as does the state vector $|\,n\rangle$. Specifically, we find

$$H\cong 2m\left[\frac{p^2}{2\mu}+V-\epsilon_n\right]+\delta H=H_{NR}+\delta H \tag{4.17}$$

with

$$\delta H=-V^2+2\epsilon_n V-\epsilon_n^2+\frac{\epsilon_n p^2}{2M}+\frac{1}{M}\mathbf{p}\cdot V\mathbf{p}+\frac{1}{4M}\{\mathbf{p}\cdot,[p^2,-i\nabla W]\}-\frac{e}{M}\sigma\cdot\mathbf{E}\times\mathbf{p}+ie\left[1+\frac{V}{m}\right]\alpha\cdot\nabla\left[\frac{e}{4\pi r}\right] \tag{4.18}$$

and from (2.4)

$$|\,n\rangle=\left[1+\frac{V}{2M}-\frac{iZ\alpha}{2M}\hat{\mathbf{r}}\cdot\mathbf{p}\right]\left[\frac{1+\gamma_0\lambda}{1+\lambda}\right]|\,n)=|\,n_0\rangle+\delta\,|\,n\rangle\ ,$$

$$\langle n\,|=(n\,|\left[\frac{1+\gamma_0\lambda}{1+\lambda}\right]\left[1+\frac{V}{2M}+\frac{iZ\alpha}{2M}\mathbf{p}\cdot\hat{\mathbf{r}}\right]=\langle n_0\,|+\delta\langle n\,|\ . \tag{4.19}$$

It is straightforward to show that terms from (4.16) arising from corrections to H_{NR}, namely, δH are too small. On the other hand, corrections can arise from the state vector corrections of (4.19).

We find that Eq. (4.16) becomes approximately

$$\Delta E(I^i_{L1,0})\cong\frac{\alpha}{\pi}\int_0^1 dz\int_0^1 du[2u(1-u)(1-z^2)](n\,|p^i\frac{1}{zm^2+u(1-z)H_{NR}}[p^i,V]\,|\,n)$$
$$+\frac{\alpha}{2\pi M}\int_0^1 dz\int_0^1 du[2u(1-u)(1-z^2)](n\,|\,Vp^i\frac{1}{zm^2+u(1-z)H_{NR}}[p^i,V]\,|\,n)\ , \tag{4.20}$$

where the second line comes only from the wave function correction $(n\,|\,V/2M$.

The first term of (4.20) gives a leading contribution to the Lamb shift as well as a correction $L1-H$ evaluated in EY. This correction is now modified by the presence of a reduced mass factor in H_{NR} as well as reduced mass in $|\,n)$. However, these corrections are already accounted for in (4.4) and in the reduced mass factor $1-3m/M$ which will also multiply all the $\alpha(Z\alpha)^5$ terms.

The additional term in (4.20) arising from the wave-function correction $(n\,|$, may readily be evaluated. For S states it yields

$$\Delta E^{\delta(n|}(I^i_{L1,0})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}\times\tfrac{2}{3}\ . \tag{4.21}$$

Next we calculate the shift correction

$$I^i_{L1-p}=16(1-z^2)z^3m^2\int_0^1 d\lambda(1-\lambda^2)\left[\frac{1}{D_\lambda}p^i\frac{1}{D_\lambda}[V,p^i]\frac{1}{D_\lambda^2}\right|\left|\frac{1}{D_\lambda}\right]\Bigg|\Bigg|\frac{-z\mathbf{k}\cdot\mathbf{p}+\lambda z^2\mathbf{p}^2}{D_\lambda} \tag{4.22}$$

noting that to the desired accuracy we replace $\mathbf{\Pi}$ by $\mathbf{p}$. Using the relation of EY

$$\Bigg|\Bigg|\frac{k_\nu}{D_\lambda}\approx\Bigg|\Bigg|\frac{\lambda z\Pi_\nu}{D_\lambda} \tag{4.23}$$

we can write the above as

$$8(1-z^2)z^5m^2\int_0^1 d\lambda^2(1-\lambda^2)\left[\frac{1}{D_\lambda}p^i\frac{1}{D_\lambda}[V,p^i]\frac{1}{D_\lambda^2}\right]\Bigg|\Bigg|\Bigg|\left[\frac{1}{D_\lambda}\frac{p^2}{D_\lambda}-\frac{p^j}{D_\lambda}\frac{p^j}{D_\lambda}\right]. \tag{4.24}$$

To work this out we expand D_λ in powers V keeping the full expression for H in D_λ. The calculation is long and tedious. It leads to the standard expression for the shift correction with mass corrections contained in (4.4) as well as some new terms. The leading term in the above-mentioned expansion will have a wave-function correction

$$\Delta E^{\delta|n)}(I^i_{L1-p})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(\tfrac{2}{3}-\ln 2) \tag{4.25}$$

while the nonleading terms in the expansion also yield a new term

$$\delta E(I^i_{L1-p})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(\tfrac{5}{2}\ln 2-\tfrac{11}{6}). \tag{4.26}$$

The other shift correction is

$$I^i_{L1-k_0}=-8(1-z^2)z^5m^2\int_0^1 d\lambda^2(1-\lambda^2)\left[\frac{1}{D_\lambda}p^i\frac{1}{D_\lambda}[V,p^i]\frac{1}{D_\lambda^2}\right]\Bigg|\Bigg|\Bigg|\left[\frac{\Pi_0}{D_\lambda}\frac{\Pi_0}{D_\lambda}-\frac{1}{D_\lambda}\frac{\Pi_0^2}{D_\lambda}\right]. \tag{4.27}$$

To the desired accuracy the insertion factor can be simplified to

$$\Bigg|\Bigg|\Bigg|\left[\frac{2V}{D_\lambda}\frac{p^2/2M}{D_\lambda}-\frac{1}{D_\lambda}\frac{\{V,p^2/2M\}}{D_\lambda}\right] \tag{4.28}$$

which contains at least one factor of V/M. By a judicious choice of ordering the insertions, a considerable simplification is achieved. After a long and laborious calculation we find a new contribution

$$\Delta E(I^i_{L1-k_0})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(\ln 2-\tfrac{3}{4}). \tag{4.29}$$

Finally, the last term of I_{L1} can be put in the form

$$I^0_{L1}=-8(1-z^2)z^3m^2\int_0^1 d\lambda^2\frac{1}{D_\lambda}\left[E_n-V-\frac{p^2}{2M}\right]\frac{1}{D_\lambda}[p^j,V]\gamma^j\frac{1}{D_\lambda^2}\Bigg|\Bigg|\frac{1}{D_\lambda} \tag{4.30}$$

after dropping some small numerator terms. The term with V on the left will not contribute since γ^j will either bring in additional momenta acting on the wave function or will bring in factors of $[\mathbf{p},V]$ through a V expansion of the denominators. In either case the contribution will be negligible. The E_n term can be shown to lead to zero by symmetrizing the expression. The $p^2/2M$ term will yield a contribution and will also require a V expansion of the denominators. After carrying out all expansions and integrals the result obtained is

$$\Delta E(I^0_{L1})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-\tfrac{9}{16}). \tag{4.31}$$

Adding up the contributions (4.21), (4.25), (4.26), (4.29), and (4.30) we obtain an additional term

$$\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(\tfrac{5}{2}\ln 2-\tfrac{29}{16}) \tag{4.32}$$

from I_{L1} beyond the mass correction contained in (4.4) or the standard $1-3m/M$ term multiplying the entire $\alpha(Z\alpha)^5$ term.

B. I_{L2} contributions

We break up I_{L2} into I^0_{L2} and I^i_{L2}, as for I_{L1}. To the desired accuracy we find that

$$I^i_{L2}\cong 4z(1-z)p^i\frac{1}{D}\gamma^0[V,p^i]\frac{1}{D}\Bigg|\Bigg|\frac{1}{D^2}. \tag{4.33}$$

To evaluate this we follow the standard EY reduction. If we expand up $\mathscr{M}$ terms from the denominator, these are found to be too small, and hence we may replace D of (4.33) with

$$D_1=z^2m^2+K-(k-z\Pi)^2+z(1-z)H. \tag{4.34}$$

We perform a shift to

$$D_0=z^2m^2+K-k^2+z(1-z)H \tag{4.35}$$

and then include shift corrections. Thus

$$I^i_{L2}\cong 4z(1-z)p^i\frac{1}{D_0}\gamma^0[V,p^i]\frac{1}{D_0}\bigg|\bigg|\frac{1}{D_0^2}-4z(1-z)\int_0^1 d\lambda\, p^i\frac{1}{D_\lambda}\gamma^0[p^i,V]\frac{1}{D_\lambda}\bigg|\bigg|\frac{1}{D_\lambda^2}\frac{2\lambda z^2\Pi^2-2zk\cdot\Pi}{D_\lambda}\ . \tag{4.36}$$

In the reduction of the shift correction we find that we obtain standard terms such as $L2-p$ along with mass corrections contained in (4.4). However, the shift correction in this case will lead to an operator structure of the form p^4V, and hence the state vector correction $\delta(n\mid=(n\mid V/2M$ is significant. We find

$$\Delta E^{\delta(n\mid}(I^i_{L2-p})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}[4(1-\ln 2)-1]\ . \tag{4.37}$$

In addition to this correction, which is a wave function correction to the shift correction, the leading term of (4.36) will also have a wave-function correction, which we denote by $\Delta E^{\delta(n\mid}(I^i_{L2,0})$. We obtain

$$\Delta E^{\delta(n\mid}(I^i_{L2,0})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-2)\ . \tag{4.38}$$

Combining (4.37) and (4.38) we find the additional correction from I^i_{L2}, which is entirely a wave function correction, to be

$$\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(1-4\ln 2)\ . \tag{4.39}$$

Next we need to examine I^0_{L2}. After a straightforward examination of this term we find

$$I^0_{L2}=-4z(1-z)\frac{p^i}{2m}\frac{p^2}{2M}\frac{1}{D}p^iV\frac{1}{D}\bigg|\bigg|\frac{1}{D^2}\ . \tag{4.40}$$

The calculation of this correction leads to

$$\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(4\ln 2-1) \tag{4.41}$$

which, remarkably, just cancels the contribution of (4.39).

Thus it appears that there are no additional corrections to I_{L2} beyond those mentioned earlier in Eq. (4.4).

C. I_{L3} contributions

I_{L3} is broken up into I^i_{L3} and I^0_{L3} and an $\mathcal{M}$ expansion of the expression is made writing $D=D_1-z^2\mathcal{M}$. We find that for I^i_{L3} the factor $z^2\mathcal{M}$ can be dropped to the desired accuracy, and that

$$I^i_{L3}\cong 4z^2(1-z)\frac{1}{D_1}p^i\frac{1}{D_1}\gamma^0[V,p^i]\frac{1}{D_1}\bigg|\bigg|\frac{1}{D_1}\ . \tag{4.42}$$

We now carry out a shift correction, thus writing

$$I^i_{L3}=I^i_{L3,0}+I^i_{L3-p}+I^i_{L3-k_0}\ , \tag{4.43}$$

where

$$I^i_{L3,0}=4z^2(1-z)\frac{1}{D_0}p^i\frac{1}{D_0}[V,p^i]\frac{1}{D_0}\bigg|\bigg|\frac{1}{D_0}\ , \tag{4.44}$$

$$I^i_{L3-p}=-8z^2(1-z)\int_0^1 d\lambda\left[\frac{1}{D_\lambda}p^i\frac{1}{D_\lambda}[V,p^i]\frac{1}{D_\lambda}\bigg|\bigg|\frac{1}{D_\lambda}\right]\bigg|\bigg|\frac{-z\mathbf{k}\cdot\mathbf{p}+\lambda z^2p^2}{D_\lambda}\ , \tag{4.45}$$

$$I^i_{L3-k_0}=-8z^2(1-z)\int_0^1 d\lambda\left[\frac{1}{D_\lambda}p^i\frac{1}{D_\lambda}[V,p^i]\frac{1}{D_\lambda}\bigg|\bigg|\frac{1}{D_\lambda}\right]\bigg|\bigg|\frac{zk_0\Pi_0-\lambda z^2\Pi_0^2}{D_\lambda}\ . \tag{4.46}$$

The analysis of $I^i_{L3,0}$ is quite straightforward. It leads to a lowest-order Lamb shift correction as well as a term $L3-H$, and these have appropriate mass corrections already accounted for earlier. We find, however, that there is also a wave function correction of the type $(n\mid V/2M$, which leads to

$$\Delta E^{\delta(n\mid}(I^i_{L3,0})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-\tfrac{1}{3})\ . \tag{4.47}$$

Next we turn to I_{L3-p} and carry out a V expansion of the denominators. Only the first denominator (at the far left) will lead to a correction term with an extra V. After doing the V expansion we carry out the insertion process where necessary. Structures of the type $f(p^2)p^4V$ and $Vg(p^2)p^4V$ are encountered, the first of which requires a wave function correction. This correction turns out to give

$$\Delta E^{\delta(n\mid}(I_{L3-p})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-2\ln 2+\tfrac{17}{12})\ . \tag{4.48}$$

What remains we call $\delta E(I^i_{L3-p})$. It is an involved expression, consisting of many terms, which when evaluated gives

$$\delta E(I^i_{L3-p})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(8\ln 2-\tfrac{65}{12})\ . \tag{4.49}$$

The shift correction $L3-k_0$ is first simplified by rewriting the insertion factors in terms of $Vp^2/2M$. Once this is done the simplification of denominators is more transparent. No special difficulties are encountered and the result is

$$\Delta E(I^i_{L3-k_0})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-4\ln 2+\tfrac{65}{24})\ . \tag{4.50}$$

Finally, we also need to evaluate I^0_{L3}. To do this we find it convenient to use symmetric averaging. Unfortunately, there are many terms which result from this procedure, but each one of them is not especially difficult. The end result gives terms that are included in Eq. (4.4) along with some new corrections, which give

$$\Delta E(I^0_{L3})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-3\ln 2+2+\tfrac{3}{8}+\tfrac{1}{192}) . \quad (4.51)$$

Collecting all corrections from I_{L3} we obtain

$$\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-\ln 2+\tfrac{3}{4}+\tfrac{1}{192}) . \quad (4.52)$$

D. I_{L4} corrections

The separation of I_{L4} into I^0_{L4} and I^i_{L4} is similar to that done for the other terms. We expand I^i_{L4} in a series in $\mathscr{M}$ but only the zeroth-order term is needed. Then I^i_{L4} is handled by shifting the k integration and picking up shift corrections. We find that

$$I^i_{L4,0}=8z^3\frac{1}{D_0}p^i\frac{1}{D_0}[V,p^i]\frac{1}{D_0^2} , \quad (4.53)$$

$$I^i_{L4-p}=-16z^3\int_0^1 d\lambda\frac{1}{D_\lambda}p^i\frac{1}{D_\lambda}[V,p^i]\frac{1}{D_\lambda^2}\left|\left|\frac{-z\mathbf{k}\cdot\mathbf{p}+\lambda z^2p^2}{D_\lambda}\right.\right. , \quad (4.54)$$

$$I^i_{L4-k_0}=-16z^3\int_0^1 d\lambda\frac{1}{D_\lambda}p^i\frac{1}{D_\lambda}[V,p^i]\frac{1}{D_\lambda^2}\left|\left|\frac{zk_0\Pi_0-\lambda z^2\Pi_0^2}{D_\lambda}\right.\right. . \quad (4.55)$$

The leading term (4.53) is quite straightforward and gives corrections already obtained as well as a new term from the wave-function correction. This new term is

$$\Delta E^{\delta(n|}(I^i_{L4,0})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-\tfrac{4}{5}) . \quad (4.56)$$

For I^i_{L4-p} we carry out the insertion procedure and find a term $L4-p$ obtained earlier except that now there is a wave-function correction to this of

$$\Delta E^{\delta(n|}(I^i_{L4-p})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(3\ln 2-\tfrac{37}{20}) . \quad (4.57)$$

I^i_{L4} has a term from the V expansion of the shift correction. We obtain this term, already contained in (4.4) but now an additional piece is also present

$$\delta E(I^i_{L4-p-V})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-\tfrac{9}{4}\ln 2+\tfrac{73}{40}) . \quad (4.58)$$

The other shift correction, $I^i_{L4-k_0}$, provides another long but straightforward calculation, leading to a result

$$\Delta E(I^i_{L4-k_0})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(3\ln 2-\tfrac{99}{40}) . \quad (4.59)$$

Finally we must also evaluate I^0_{L4}. After some simplification we find that this takes the form

$$I^0_{L4}\cong-\frac{4z^3}{M}\frac{1}{D}p^2\frac{1}{D}\gamma^j[p^j,V]\frac{1}{D^2} . \quad (4.60)$$

We now have an odd Dirac operator and hence contributions of the correct order can come from the lower components of the wave function or from the odd Dirac operator in D. Taking both of these into account we obtain

$$I^0_{L4}\cong-\frac{2z^3}{mM}\frac{1}{D_1^2}p^4V\frac{1}{D_1^2}+\frac{4z^4}{M}\frac{1}{D_1}V\frac{p^4}{D_1^2}V\frac{1}{D_1^2} . \quad (4.61)$$

The subsequent calculation leads to

$$\Delta E(I^0_{L4})=\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-6\ln 2+\tfrac{9}{2}) . \quad (4.62)$$

Combining all additional corrections from I_{L4} we obtain

$$\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-\tfrac{9}{4}\ln 2+\tfrac{6}{5}) . \quad (4.63)$$

Thus the total additional I_L contribution is given by the sum of Eqs. (4.32), (4.52), and (4.63) and is

$$\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(-\tfrac{3}{4}\ln 2-\tfrac{11}{192}+\tfrac{1}{5}) . \quad (4.64)$$

We also analyzed all other terms, such as $I_{\mathscr{M}}$, I_a, I_b, I_c, I_d, I_d, I_e, I_f for additional contributions not already contained in (4.4). Our conclusion is that all such additional terms are smaller than those we have retained, and thus the recoil correction from radiative diagrams within the external-field approximation appears to be given by Eqs. (4.4) and (4.64).

V. CONCLUSIONS

If we combine Eqs. (4.4) and (4.64), we obtain the total $\alpha(Z\alpha)^5m^2/M$ external field correction beyond the customary $3m/M$ factor. Thus we find a new term

$$\frac{\alpha(Z\alpha)^5}{n^3}\frac{m^2}{M}(\tfrac{35}{4}\ln 2-8+\tfrac{1}{5}+\tfrac{31}{192}) . \quad (5.1)$$

For hydrogen in the $n=2$ state this gives a correction of -0.0019 MHz or about -2 ppm for the theoretical $2S$ correction to the Lamb shift. To the desired accuracy the $2P$ state correction appears to be negligible.

The above correction is not large enough to resolve the discrepancy between the theoretical and experimental values of the Lamb shift. It should be kept in mind, however, that there are additional radiative corrections to those given above which result from corrections to the external field approximation. We are now in the process of evaluating these numerically. At the present time we do not yet know how these will affect the result. Additionally, it is known that $(Z\alpha)^6 m^2/M$ can also add significant corrections from nonradiative processes and an analysis of these contributions is also in progress.

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APPENDIX: EFFECTIVE DIRAC EQUATION

The Bethe-Salpeter (BS) equation for the four-point function G of a two-particle system formally reads

$$G=G_0+G_0KG\ , \tag{A1}$$

where certain momentum (or coordinate) integrals are implicit. G_0 represents free propagation and K is the sum of all irreducible interaction kernels for the system. For the H atom, K is the sum indicated in Fig. 5. The iterative solution of Eq. (A1) then reproduces the perturbation expansion of G consisting of all graphs, reducible and irreducible.

It is well known that for precise and systematic calculation of bound-state energies it is often difficult to work with the homogeneous four-dimensional BS integral equation corresponding to the four point function (A1). An alternative is to carry out a reduction of this equation by truncating K so that a solvable three-dimensional equation results, which contains the dominant binding of the two particles. Then the remainder of K can be included by generating the pertubation expansion about this three-dimensional equation.

Our Eq. (2.1) is just such an equation. Detailed use of this equation is made in GY and is derived qualitatively there. A more rigorous and systematic derivation appears in the work of Gorelick and Grotch,[13] Lepage[13] and, more recently Bodwin *et al.*[13] have discussed similar equations without the assumption of a very massive proton. Equation (2.1) is arrived at by retaining in (A1) only the first graph of Fig. 5 and also by constraining the proton on its positive energy mass shell everywhere. Therefore, the ladder graphs for the electron-proton interaction are included in this leading order approximation.

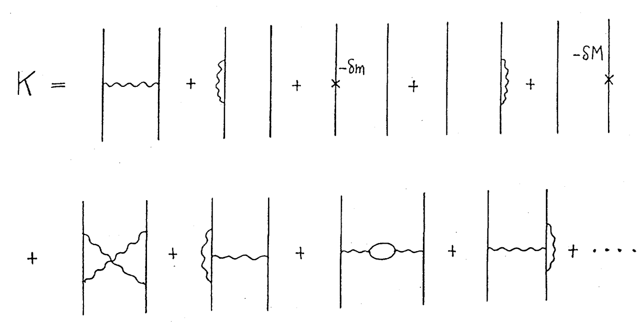

FIG. 5. Irreducible contributions to the kernel K.

In the c.m. system, the equation for the wave function takes the form

$$\left[\boldsymbol{\alpha}\cdot\mathbf{p}+\frac{1}{2M}\{\boldsymbol{\alpha}\cdot\mathbf{p},V\}+\frac{1}{4m}[\boldsymbol{\alpha}\cdot\mathbf{p},[\mathbf{p}^2,\mathbf{W}]] +V+\frac{\mathbf{p}^2}{2M}+\beta m\right]\psi(\mathbf{r})=E\psi(\mathbf{r})\ , \tag{A2}$$

where

$$V=-\frac{Z\alpha}{r}\quad\text{and}\quad W=-Z\alpha r\ .$$

This is an effective Dirac equation for the electron in the potential V and

$$e\,\mathbf{A}\equiv-\frac{1}{2M}\{\mathbf{p},V\}-\frac{1}{4M}[\mathbf{p},[\mathbf{p}^2,W]]\ .$$

Also the proton in this procedure brings in a contribution of $\mathbf{p}^2/2M$ to the Hamiltonian. The eigenvalue problem for the above equation is discussed in GY. Here, let us note only that in the nonrelativistic regime, Eq. (A2) reduces to the Schrödinger equation with reduced mass, and that if $M\to\infty$ it reproduces the Dirac-Coulomb equation.

Energy level shifts

In the derivation of (A2) all but the leading irreducible kernel were neglected. In principle, therefore, energy shifts will occur due to the additional irreducible kernels as well as due to the off-shell proton terms in the ladders obtained by repetition of the leading kernel (Fig. 5). If all these perturbation kernels are called ΔK, then the shifts are given to first order in ΔK, by

$$\Delta E_n=i\langle n\mid\Delta K\mid n\rangle\ ,$$

where $\mid n\rangle$ satisfies (A2), i.e.,

$$(\not{H}-m)\mid n\rangle=0\ .$$

The present paper considers the energy shifts (see Fig. 1) of relative order $Z\alpha(m/M)$ that come from the quantized radiation field of the effective electron of Eq. (A2). In Fig. 1 the dark internal solid line represents the exact electron propagator for an electron in the external field satisfying (A2). Graphically, this line can be decomposed as shown in Fig. 6. Such a separation is analogous to the familiar decomposition into one-potential and many-potential parts in the usual Lamb shift ($M\to\infty$) calculation. This indicates what perturbation kernels Fig. 1 in-

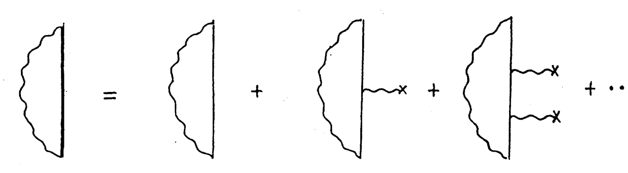

FIG. 6. Expansion of electron propagator in the external field.

corporates. The perturbation kernel for this can be written as

$$i\Delta K^{(1)}=\frac{\alpha}{4\pi^3}\int\frac{d^4k/i}{k^2+i\epsilon}\gamma_\mu\frac{1}{\not{\Pi}-\not{k}-m+i\epsilon}\gamma^\mu-\delta m \ .$$

Rather than expanding the propagator of the effective electron in a series, as suggested by Fig. 6, we have adopted, for the extraction of terms of the order of interest, the procedure of EY.

We only note here that the energy shift can be written in terms of matrix elements between eigenfunctions of Eq. (A2) in the following manner. We have $\Delta E_n = i\langle n \mid \Delta K \mid n\rangle$. We insert complete sets of two-particle momentum eigenstates in the c.m. system to get

$$\Delta E_n=\int d^3p_3\int d^3p_1\langle n\mid \mathbf{p}_3,-\mathbf{p}_3\rangle\langle \mathbf{p}_3,-\mathbf{p}_3\mid i\Delta K\mid \mathbf{p}_1,-\mathbf{p}_1\rangle\langle \mathbf{p}_1,-\mathbf{p}_1\mid n\rangle$$
$$=\int d^3p_3\int d^3p_1\psi_n^\dagger(\mathbf{p}_3)\langle \mathbf{p}_3,-\mathbf{p}_3\mid i\Delta K\mid \mathbf{p}_1,-\mathbf{p}_1\rangle\psi_n(\mathbf{p}_1) \ ,$$

where ψ is a four-component spinor in the electron space and a two-component spinor in the space of proton. The amplitude ΔK is written following the standard Feynman rules.

The important perturbation kernels that result from having the proton in the ladder graphs off-shell as well as the irreducible kernels that give rise to contributions of the order $\alpha(Z\alpha)^5(m^2/M)$ now arise from two-photon exchange diagrams.

[1]T. Kinoshita and J. Sapirstein, in Proceedings of the Ninth International Conference on Atomic Physics, University of Washington, Seattle, Washington, July (1984) (in press).

[2]P. J. Mohr, Phys. Rev. Lett. **34**, 1050 (1975).

[3]D. J. Drickey and L. N. Hand, Phys. Rev. Lett. **9**, 521 (1962); L. N. Hand, D. J. Miller, and Richard Wilson, Rev. Mod. Phys. **35**, 335 (1963).

[4]G. G. Simon, Ch. Schmitt, F. Borkowski, and V. H. Walther, Nucl. Phys. **A333**, 381 (1980).

[5]S. R. Lundeen and F. M. Pipkin, Phys. Rev. Lett. **46**, 232 (1981); Yu. L. Sokolov and V. P. Yakovlev, Zh. Eksp. Teor. Fiz. **83**, 15 (1982) [Sov. Phys.—JETP **56**, 7 (1982)].

[6]J. Sapirstein, Phys. Rev. Lett. **47**, 1723 (1981).

[7]G. W. Erickson, Phys. Rev. Lett. **27**, 780 (1971).

[8]G. W. Erickson and D. R. Yennie, Ann. Phys. (N.Y.) **35**, 271 (1965); **35**, 47 (1965). In the present paper these are often referred to as EY.

[9]E. E. Salpeter, Phys. Rev. **87**, 838 (1952).

[10]T. Fulton and P. C. Martin, Phys. Rev. **95**, 811 (1954).

[11]H. Grotch and D. R. Yennie, Rev. Mod. Phys. **41**, 350 (1969). Hereafter referred to as GY.

[12]R. Karplus, A. Klein, and J. Schwinger, Phys. Rev. **86**, 288 (1952); M. Baranger, H. A. Bethe, and R. P. Feynman, *ibid.* **92**, 482 (1953); A. J. Layzer, Phys. Rev. Lett. **4**, 580 (1960); H. M. Fried and D. R. Yennie, *ibid.* **4**, 583 (1960); A. J. Layzer, J. Math. Phys. **2**, 292, 308 (1961); R. Novick, E. Lipworth, and P. Yergin, Phys. Rev. **100**, 1153 (1955).

[13]See Ref. 11 and F. Gross, Phys. Rev. **186**, 1448 (1969); G. P. Lepage, Phys. Rev. A **16**, 863 (1977); A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1983); R. N. Faustov, Sov. J. Part. Nucl. **3**, 153 (1972); J. L. Gorelick and H. Grotch, J. Phys. G **3**, 751 (1977); R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966); G. T. Bodwin, D. R. Yennie, and M. A. Gregorio, Cornell University Report No. CLNS-84/618 (unpublished), for an incomplete but representative sampling of relevant papers.

[14]See also S. J. Brodsky and G. W. Erickson, Phys. Rev. **148**, 26 (1966).

[15]This result has been independently worked out by G. W. Erickson, and is in complete agreement with ours.