Subordination of fast-relaxing degrees of freedom to order parameters under Ginzburg-Landau regimes

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The role of fluctuations is studied in the locally attractive and locally invariant center manifold for a system which is a truncation of Hopf's model for hydrodynamic turbulence. The anlaysis is carried out in a regime where the system sustains a hard-mode instability and the real part of the Floquet coefficients does not change sign. The Gaussian width of the time-independent factor of the probability density is shown to measure the subordination of the fast-relaxing degree of freedom. A physically meaningful equation is derived relating the Gaussian width, the intensity of the additive noise, and the external control parameter. The characteristic curves of the reduced Fokker-Planck equation are the limit cycles derived from bifurcation theory.

I. INTRODUCTION

The center-manifold approach to analyze fluctuations in systems displaying dissipative structures was implemented by a number of people: Knobloch and Weisenfeld classified the instabilities according to the normal form of the system;[1] Fernández and Sinanoğlu have identified the dissipative structures occurring in open reactive systems operating far from equilibrium with center manifolds.[2–5]

In a recent paper, these authors have considered a dynamical system capable of sustaining a hard-mode instability under a Ginzburg-Landau regime but such that when the real part of the Floquet exponents changes from negative to positive, the Torus-type bifurcation can no longer be described by the Ginzburg-Landau equations.[5] This model is a truncation of Hopf's system for fluid dynamic turbulence.[6] Since the system is already in the Poincaré normal form, the separation between the fast-relaxing degree of freedom from the Haken's order parameters is straightforward and becomes evident if one considers that the Jacobian matrix is in Jordan normal form.[5,7]

The center-manifold equation is obtained by adiabatic elimination and the time-independent factor of the probability density functional is a Gaussian peaked at this center manifold. The inverse of the distance from its maximum to its inflection point gives a measure of the local attractivity of the center manifold. (The more spread the probability is, the less attractive.) It will be shown in this paper that *this Gaussian width is directly proportional to the root square of the intensity of the additive noise and inversely proportional to the root square of the bifurcation control parameter.* A result which agrees with the intuitive picture that the local attractivity of the center manifold should decrease with the intensity of the noise.

II. CENTER MANIFOLD

The following dynamical system is already in the Poincaré normal form:

$$\dot{X}_1=(\nu-1)X_1-X_2+X_1X_3;\quad \dot{X}_2=X_1+(\nu-1)X_2+X_2X_3\ ;$$
$$\dot{X}_3=\nu X_3-(X_1^2+X_2^2+X_3^2)\ . \tag{1}$$

The Jacobian matrix being in the Jordan normal form at the nontrivial stationary state $\mathbf{X}=(0,0,\nu)$:

$$\underline{J}=\begin{bmatrix}2\nu-1 & -1 & 0\\ 1 & 2\nu-1 & 0\\ 0 & 0 & -\nu\end{bmatrix}\ . \tag{2}$$

One can identify the order parameters as X_1 and X_2 and the fast-relaxing degree of freedom as X_3 (cf., for example, in Haken[8]). After a relaxation time of the order of the reciprocal of the damping constant (ν^{-1}), we have, as a consequence of the adiabatic elimination approximation

$$\dot{X}_3\approx 0\ . \tag{3}$$

This relation gives as the center-manifold equation

$$X_3=F(X_1,X_2)=\tfrac{1}{2}\nu+[\tfrac{1}{4}\nu^2-(X_1^2+X_2^2)]^{1/2}\ ;$$
$$\nu\in(\tfrac{1}{2}-\delta,\ \tfrac{1}{2}+\delta)\ . \tag{4}$$

The real part of the Floquet exponents of the limit cycle do not change sign (it remains negative) in the open interval $\frac{1}{2}\leq\nu\leq\frac{2}{3}$. It was proven in an early paper that the bifurcation can, in that interval, be described by the Ginzburg-Landau equations.[5] This description does not hold for a more extended interval since, beyond $\nu=\frac{2}{3}$ a Torus bifurcates and the real part of the Floquet coefficients becomes positive.

To analyze the role of fluctuations for the interval where Eq. (4) is valid, we consider the coupling of Eq. (1) with additive white Gaussian noise δ correlated on the fast-relaxing degree of freedom X_3. This perturbation obeys the relation

$$\langle u(t)\rangle=0;\quad \langle u(t)u(t')\rangle=2d\delta(t-t')\ . \tag{5}$$

The quantity d is the intensity of the noise. The center manifold is tangent at the steady state to the eigenspace of $\underline{J}$ with associated eigenvalues $\pm i$. It contains the recurrent solutions of the system. In the deterministic limit $d\rightarrow 0$, the time-independent part of the probability density is the Dirac delta peaked at $X_3=F(X_1,X_2)$. The relation being valid after a relaxation time of the order ν^{-1}.

III. FOKKER-PLANCK EQUATION AT THE CENTER MANIFOLD

The general Fokker-Planck equation for the probability density functional $P(X_1,X_2,X_3,\nu,t)$ reads

$$\partial_t P = -\partial_{X_1}(\dot{X}_1 P) - \partial_{X_2}(\dot{X}_2 P) - \partial_{X_3}\{[\dot{X}_3 - u(t)]P\} + d\partial_{X_3}\partial_{X_3}P \quad . \tag{6}$$

The time-dependent factor obeys a reduced Fokker-Planck equation in the order parameter space.[1,8] Therefore, X_3 is given by Eq. (4):

$$\partial_t T = X_2\partial_{X_1}T - X_1\partial_{X_2}T - \partial_{X_1}[X_1F(X_1,X_2)T] - \partial_{X_2}[X_2F(X_1,X_2)T] \quad . \tag{7}$$

The time-independent factor $L(X_3)$ is a Gaussian whose width should be parametrically dependent on the position in the center manifold given by the coordinates X_1 and X_2. This has to be so to allow for the continuous flow of probability about the center manifold [cf. (1)]:

$$L(X_3) = (g/\pi)^{1/2}\exp\{-g(X_1,X_2)[X_3 - F(X_1,X_2)]^2\} \quad . \tag{8}$$

Since the width g becomes infinitely large for $d=0$ (cf. below), it is a valid assumption that, in the sense of the theory of distributions,

$$\int \Psi(\mathbf{X})L(\mathbf{X})\,dX_3 \cong \Psi(X_1,X_2,F(X_1,X_2)) \cong \int \Psi(\mathbf{X})\delta(X_3 - F(X_1,X_2))\,dX_3 \tag{9}$$

for any $\Psi(\mathbf{X})$ of square integrable.

Equation (9) is valid, of course, only for small noise intensities. $g(X_1,X_2) = g(0,0) + O(1)$, and we denote the average inverse width $g(0,0)$ by g_0.

The integration of Eq. (6) with respect to X_3 (fast degree of freedom) will be performed using the relation (9):

$$\begin{aligned}\partial_t T = {} & [-(\tfrac{1}{2}g)\partial_{X_1}g][-\tfrac{1}{2}X_1 - X_2 + X_1F(X_1,X_2)]T - \partial_{X_1}\{[-\tfrac{1}{2}X_1 - X_2 + X_1F(X_1,X_2)]T\} \\ & -(\tfrac{1}{2}g\partial_{X_2}g)[X_1 - \tfrac{1}{2}X_2 + X_2F(X_1,X_2)]T - \partial_{X_2}\{[X_1 - \tfrac{1}{2}X_2 + X_2F(X_1,X_2)]T\} \\ & - T\partial_{X_3}[\tfrac{1}{2}X_3 - (X_1^2 + X_2^2 + F^2)] + d(\partial_{X_3}\partial_{X_3}T - 2gT) \quad .\end{aligned} \tag{10}$$

We have considered the integrated form at the bifurcation value of the parameter $\nu = \frac{1}{2}$.

A comparison between relations (7) and (10) gives for the average inverse width

$$g_0 = \tfrac{1}{4}d^{-1} \quad \text{(at the bifurcation point)} \; ,$$
$$g_0 = \nu/2d \quad (\text{for } \tfrac{1}{2} \le \nu \le \tfrac{2}{3}) \quad . \tag{11}$$

Relation (11) justifies the assumption given in Eq. (9). The distance from the maximum of $L(X_3)$ to its inflection point is $k = (2g)^{-1/2}$, a quantity parametrically dependent on the slowly relaxing degrees of freedom.

The average quantity $\bar{k}^{-1}$ provides, as we have stated in Sec. I, a measure of the local attractivity of the center manifold. We arrive at the desired formula:

$$\bar{k}^{-1} = (2g_0)^{1/2} = (\nu/d)^{1/2} \quad . \tag{12}$$

(Roughly speaking, the local attractivity increases as we depart from the bifurcation value $\frac{1}{2}$ and decreases, as expected, when the noise intensity increases.)

The characteristic curves of the reduced Fokker-Planck equation are precisely the limit cycles analytically found in previous work by Fernández and Sinanoğlu:

$$(\nu - \tfrac{1}{2}) - (X_1^2 + X_2^2) = 0 \quad . \tag{13}$$

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