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Chaos and nonlinear modes in a perturbed Toda chain

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The role of spatially coherent modes in a chaotic state of the damped and driven Toda chain is investigated with the help of the scattering transform.

In many chaotic (or turbulent) systems spatially coherent nonlinear modes seem to play an important role. However, these modes are difficult to define analytically. Frequently one only knows about the presence and nature of a coherent mode when a "lump" appears in one's data! Until they are more precisely defined, possible uses for these coherent modes, e.g., to provide a description of low-dimensional attractors, remain speculative.

When the dynamical system is a perturbation of a completely integrable nonlinear wave equation the situation is clearer: solitons become natural candidates for the coherent states. Even in these cases, however, since the integrable system can be strongly perturbed, the practicality of representing a field in a soliton basis must be carefully investigated. Are the number and properties of the solitons which comprise the field sufficiently stable that the representation is useful?

Such questions can be answered with a numerical procedure which transforms the field to its soliton representation.[1–3] This numerical transform is just being developed and has only been tested for mild perturbations with regular data and responses. Here, we take the transform in the simplest and most developed framework (the periodic Toda lattice) and extend its use to both irregular data and strong perturbations. Specifically, we use the transform to study two examples: (i) the appearance of a coherent spatial state from a chaotic one, and (ii) the dynamics of the lattice system under the presence of an external inhomogeneous force and dissipation.

Self-organization of a turbulent (or random) state into a state of spatial coherence (order) is an important phenomenon shared by many nonlinear dissipative systems. A possible theoretical description of this process[4–8] is as follows. Let $X(t)$ denote the state of the system at time t. In the absence of dissipation, X satisfies a nonlinear, conservative evolution equation that possesses several constants of motion $\{(I_j(X)\}$. In the presence of dissipation, these invariants $I_j(X)$ will decay at different rates. We assume I_0 decays slower than I_1, I_1 slower than I_2, and so on. Thus, as time increases, the nonlinear dissipative system dynamically evolves into a state that satisfies the following minimization problem: minimize $I_1(X)$ subject to the constraint that $I_0(X)$ is invariant. The organized state $X_{\rm org}$ is a solution of this minimization problem. If, in addition, the nonlinear dissipative system possesses a cascade from high spatial wave numbers toward lower wave numbers, one expects the organized state to exhibit spatial coherence. Of course, on the longest time scale governing $I_0(X)$ the organized state will eventually disappear.

This scenario for self-organization has been successively tested in magnetohydrodynamics[4,5] and two-dimensional Navier-Stokes flow.[6,7] Hasegawa, Kodama, and Watanabe[8] observed that the Korteweg–de Vries (KdV) equation, perturbed by dissipation, provides a simpler testing ground for these ideas. There the authors use beautiful, albeit heuristic, reasoning to argue that a random state of the periodic KdV equation will, in the presence of small dissipation, self-organize into a soliton wave train with the largest wavelength that the system can support. The numerical checks of this conjecture are somewhat indirect (as given in Ref. 8); here, we want to give direct evidence in a discretized Korteweg–de Vries system.

The specific model we discuss is the Toda lattice with periodic boundary conditions under the influence of an external driving force and damping. It has several advantages for the questions we want to ask about the importance of coherent structures: first, the discrete nature of the model removes any doubts about the effects of discretization of a partial differential equation; second, the system without perturbation (driving and damping) is completely integrable with the help, e.g., of the scattering transform.[9] Furthermore, the scattering transform provides us with a natural set of coordinates which can give direct information about the existence and nature of coherent spatial structures. The formalism for using the transform as a numerical device for measuring the soliton content in any given state has been developed elsewhere.[1,2] The purpose of this paper is to implement this device and use it to confirm directly the scenario of Ref. 8 and test the behavior of the system in the presence of perturbations.

The Toda equations with weak dissipation take the (normalized) form

$$\ddot{u}_n = e^{(u_{n-1}-u_n)} - e^{(u_n-u_{n+1})} + v_n + f_n \ , \qquad (1)$$

where v_n denotes the dissipation and f_n the external forcing. $\{n\}$ labels the Toda particles and a "dot" signifies a time derivative. We have used two types of nonlocal dissipation, linear and quadratic, with corresponding coefficients η_1 and η_2:

$$v_n = (\dot{u}_{n+1} - 2\dot{u}_n + \dot{u}_{n-1})(\eta_1 + \eta_2|\dot{u}_{n+1} - \dot{u}_{n-1}|) \ . \quad (2)$$

The external force was chosen to be sinusoidal with amplitude γ and frequency ω, and spatially inhomogeneous:

$$f_n = \gamma \sin(\omega t)\delta_{n,1} \ . \quad (3)$$

Using the numerical methods described in Ref. 2 we integrated the Toda equations and, at successive instants in time, measured the configuration $[u_n(t), \dot{u}_n(t)]$ with the discriminant measuring device defined and developed in Ref. 1. In addition, we implemented a procedure to calculate the zeros and areas between the zeros of the discriminant which are related to the velocities and actions of the excited modes (see Ref. 1). This procedure determines precisely the soliton and radiation ("phonon") content in the configuration at each time t.

The self-organization process should apply to many soliton systems that are perturbed by dissipation, for example, to the Toda lattice. We studied a 15 particle lattice and used a high-temperature heat bath to generate the random initial conditions. Precisely, we chose initial conditions of the displacement $u_n = 0$ at lattice sites with the initial velocities $\dot{u}_n$ each chosen independently from a Maxwell distribution with temperature $T=1$. Total linear momentum was zero.

For the run shown in Fig. 1 we set $\eta_2 = \gamma = 0$. Changing the form of dissipation did not qualitatively alter the results. The run that we have depicted is typical. Figure 1(a) shows the discriminant of the initial data. Following the rules of Ref. 1 we notice that all modes are appreciably excited —substantial high-frequency radiation and a few long-wavelength, coherent soliton modes. Because of the dissipation, the discriminant changes with t. Figures 1(b) and 1(c) show the temporal evolution: notice that as time increases the high-frequency radiation modes (for which zeros of the discriminant are close to zero; see Ref. 1) decay; the higher their frequency the faster their decay. By time 15, only two long-wavelength solitons running to the right and two long-wavelength solitons running to the left remain in the configuration; at later times only two solitons remain and continue to decay slowly. Thus, self-organization does indeed occur, and its evolution is quantitatively measured by the discriminant.

Obviously, since only the difference of two adjacent coordinates enters the nonlinear Toda potential, a *homogeneous* external driving force will not produce chaos but merely move the center of mass. This is in contrast to local non-

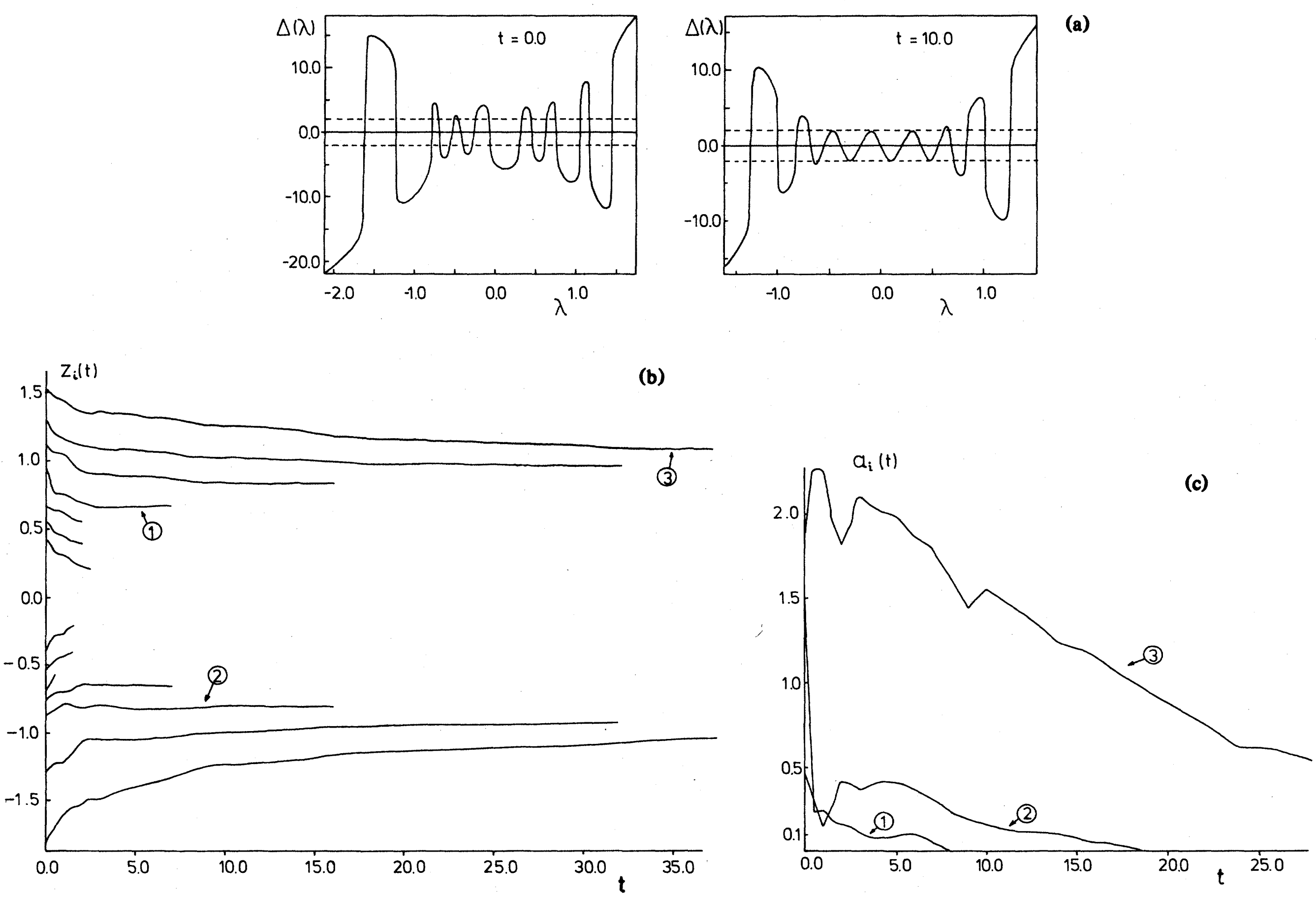

FIG. 1. (a) Toda discriminants of the configuration at time $t=0.0$ and $t=10.0$ ($\eta_1=0.4$, $\eta_2=0.0$, $\gamma=0.0$). (b) Zeros $Z_i(t)$ of the discriminant as function of time with corresponding action greater than 0.05. (c) Actions of the three marked modes in (b) as function of time.

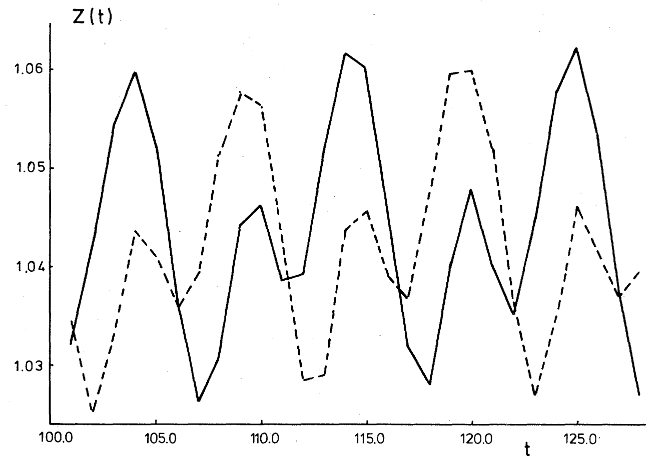

FIG. 2. Zeros $Z_l(t)$ of the Toda discriminant as function of time with corresponding action greater than 0.05 ($\eta_1=0.4$, $\eta_2=0.0$, $\gamma=0.5$). Solid line: $-Z(t)$ of the soliton moving to the right. Broken line: $Z(t)$ of the soliton moving to the left.

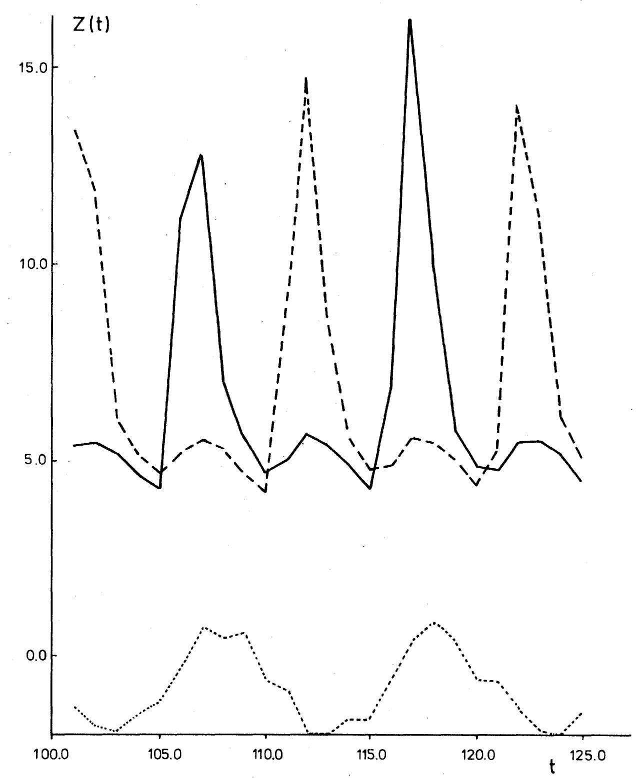

FIG. 3. Selected zeros $Z_l(t)$ of the discriminant as function of time ($\eta_1=0.4$, $\eta_2=0.0$, $\gamma=50.0$). Solid line: $-Z(t)$ of the fastest soliton moving to the right. Broken line: $Z(t)$ of the fastest soliton moving to the left. Dotted line: a typical phonon mode.

linearities as, e.g., in the case of the sine-Gordon potential.[10,11] In addition, because of the presence of dissipation, any initial spatial structures will smooth out and the system is then no longer influenced by the nonlinearity of the potential. Therefore, in order to see interesting spatial behavior, we have to use an inhomogeneous driver. This can be done in many different ways; here, we have chosen to drive one particle only [see Eq. (3)]. The change of dynamical response as the inhomogeneity of the driver is varied has been studied elsewhere.[12]

The shortest chain one can study, namely, with just two particles,[13] reduces, due to periodic boundary conditions, to the motion of a *single* particle in a Toda potential with damping and driving forces. Note that in contrast to many other nonlinear potentials we do not have a metastable state (local maximum or sepratrix) here, except perhaps at infinity. We have found that the system goes chaotic via a period-doubling bifurcation for moderate driving amplitudes, whereas for high amplitudes we typically find intermittent behavior.[14,15] There is a clear analogy between a ball bouncing on a vibrating table and this single-particle Toda problem. The former has been studied in detail.[16]

To investigate many-particle effects, we have chosen a chain consisting of 15 particles and kept the damping ($\eta_1=0.4, \eta_2=0$) and driving frequency fixed ($\omega=0.6$). The only parameter we varied was the driving amplitude γ. We started with flat initial conditions; i.e., $u_n=\dot{u}_n=0$. After integrating for approximately 10 periods of the driver we examined the soliton content of the profile with the help of the discriminant technique (see Ref. 1). The results are shown in Figs. 2 ($\gamma=0.5$) and 3 ($\gamma=50$). For both values of γ we find *periodic* behavior of the chain. This can be seen in the time evolution of the scattering data (see Figs. 2 and 3) and from phase plane plots and Poincaré sections for the behavior of any single particle on the chain as well (not shown here). We find that for low driving amplitudes only a few modes are excited, these modes being spatially coherent modes as one would expect from the results of the first case above. Figure 2 shows this periodic motion of the soliton modes; the actions of the few excited phonon modes are smaller than 0.05. This picture changes drastically as we go to higher driving amplitudes. Not only do the velocities of the solitons become larger, but also all degrees of freedom are excited (including phonon modes). In Fig. 3 the time dependence of only selected modes is depicted. As in the case of small driving amplitudes we can clearly see the periodic nature of the response. [Owing to a low-order interpolation scheme in evaluating a figure such as Fig. 1(a), not all maxima have precisely the same height.]

So far our studies have not revealed chaotic many-particle response. It seems that large driving amplitudes will be necessary to generate chaos in a Toda chain. At such large amplitudes the chaotic configuration, or even a locked periodic configuration (as in the case above), cannot be described by a few *Toda* modes, since the dynamical system is too large a perturbation from the Toda lattice. One situation in which the extended Toda system *can* be described by a few Toda modes is where damping dominates potential sources of chaos. This is already the case for the self-organization process above. The results presented here for the Toda chain are in contrast for those found for driven, damped nonlinear Klein-Gordon equations.[10,11,17] In those cases a few solitary wave modes can accurately describe even chaotic states. Further studies with more than the 15 particles used here are in progress to clarify this striking difference.

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