Dynamics of SU(1,1) coherent states

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We derive the most general Hamiltonian which preserves the Perelomov SU(1,1) coherent states under time evolution. It is shown that the Hamiltonian of the degenerate parametric oscillator from nonlinear optics, which does not preserve ordinary coherent states, does preserve the SU(1,1) coherent states under time evolution.

I. INTRODUCTION

Over the years there have been many studies on the time evolution of the coherent states of the harmonic oscillator.[1–4] These states, also known as the ordinary coherent states, are associated with the algebra consisting of the operators a, $a^\dagger$, and I such that $[a,a^\dagger]=I$. Of particular interest has been the determination of the Hamiltonian operator for which an initial coherent state remains coherent under time evolution. In Refs. 1–4 using a variety of methods, it is determined that this Hamiltonian has the form

$$H(t)=\omega(t)a^\dagger a+f(t)a^\dagger+f^*(t)a+\beta(t) \ , \qquad (1.1)$$

where $\omega(t)$ and $\beta(t)$ are real and $f(t)$ is a complex function of time, otherwise arbitrary. That Eq. (1.1) is determined by several different methods is a reflection of the fact that the ordinary coherent states may be generated in several ways: (1) as minimum uncertainty states, (2) as eigenstates of the annihilation operator, and (3) as states displaced from the ground state via the operator $D(z)=\exp(za^\dagger-z^*a)$, where z is a complex number. The states generated by these three methods are equivalent and have the property that for the Hamiltonian of Eq. (1.1) they follow the classical motion in the usual two-dimensional phase space.

The notion of generalized coherent states arises in the attempt to find quasiclassical states for systems whose Hamiltonian cannot be cast into the form of Eq. (1.1).[5] Of particular interest here are those states associated with the Lie group SU(1,1) whose generators and unitary irreducible representations (UIR) have long been known to produce the spectrum of a number of quantum systems.[6] Previously,[7–9] we have studied path integrals represented in terms of the SU(1,1) coherent states (CS) for systems where the Hamiltonian could be expressed as a polynomial in the generators of the group. We have also obtained a useful phase integral approximation based on the notion of large N (number of dimensions) as a semiclassical limit.[10] These calculations, however, were performed without regard to the most general form of the Hamiltonian which leaves an initial SU(1,1) CS coherent under time evolution. It is this problem which is to be addressed in this paper. Furthermore, we show that our results are applicable to a problem in nonlinear quantum optics, namely, that of a degenerate parametric oscillator.[11]

The SU(1,1) CS to be employed in this paper are those of Perelomov,[12] which are generated via a displacement-type operator, rather than those of Barut and Girardello,[13] which are eigenstates of the lowering operator. (The different definitions in this case lead to nonequivalent sets of states.) The reason for our preference is discussed in Ref. 7.

In Sec. II some necessary properties of the states are given, and the most general coherence preserving Hamiltonian is derived. Comparision with the classical equations of motion on the relevant (curved) phase space is made. In Sec. III an example is given: the degenerate parametric oscillator as mentioned above.

II. COHERENCE PRESERVING HAMILTONIAN

The Lie algebra of SU(1,1) consists of the generators K_0, K_+, and K_- satisfying the commutation relations

$$[K_0,K_\pm]=\pm K_\pm, \quad [K_-,K_+]=2K_0 \ . \qquad (2.1)$$

The Casimir invariant is given by

$$C=K_0^2-\tfrac{1}{2}(K_+K_-+K_-K_+) \ , \qquad (2.2)$$

which for a UIR has the eigenvalue $k(k-1)$, where k is the so-called Bargmann index.[14] Our interest will be confined to the representation known as the positive discrete series, $\mathscr{D}^+(k)$, where the states $|n,k\rangle$ diagonalize the compact generator K_0 as $K_0|n,k\rangle=(n+k)|n,k\rangle$, where $n=0,1,2,\ldots$ and $k>0$. The operators K_+ and K_- act as raising and lowering operators, respectively, within $\mathscr{D}^+(k)$, i.e., $K_-|n,k\rangle\propto|n-1,k\rangle$ and $K_+|n,k\rangle\propto|n+1,k\rangle$.

Following Perelomov,[12] the SU(1,1) CS are defined as

$$|\xi,k\rangle=D(\alpha)|0,k\rangle \ , \qquad (2.3)$$

where

$$D(\alpha)=\exp(\alpha K_+-\alpha^*K_-)=e^{\xi K_+}e^{\gamma K_0}e^{-\xi^*K_-} \ , \qquad (2.4)$$

and where $\alpha=-\frac{1}{2}\tau e^{-i\phi}$, $\xi=-\tanh(\frac{1}{2}\tau)e^{-i\phi}$, and $\gamma=\ln(1-|\xi|^2)$. These states have most of the properties of the ordinary coherent states. In particular, they are nonorthogonal and over complete. The overlap of two states $|\xi',k\rangle$ and $|\xi'',k\rangle$ gives the reproducing kernel as

$$K_k(\xi'',\xi')=\langle\xi'',k|\xi',k\rangle=(1-|\xi''|^2)^k(1-|\xi'|^2)^k(1-\xi''^*\xi')^{-2k} \ , \qquad (2.5)$$

such that

$$K_k(\xi'',\xi')=\int d\mu_k(\xi)K_k(\xi'',\xi)K_k(\xi,\xi') , \quad (2.6)$$

where

$$d\mu_k(\xi)=\frac{2k-1}{\pi}\frac{d^2\xi}{1-|\xi|^2} . \quad (2.7)$$

Also unity is resolved as

$$I=\int d\mu_k(\xi)|\xi,k\rangle\langle\xi,k| . \quad (2.8)$$

To obtain the Hamiltonian which preserves coherence, we adopt an argument used by Stoler[3] for ordinary coherent states. This makes use of the displacement operator definition. For a particular k sector, $|0,k\rangle$ is the ground state. We then find the most general Hamiltonian H_g which preserves the ground state under time evolution. The coherence preserving Hamiltonian H_{coh} is then obtained by transformation with the displacement operator $D(\alpha(t))$ of Eq. (2.4). One obtains

$$H_{\text{coh}}=D(\alpha(t))H_gD^\dagger(\alpha(t))-iD(\alpha(t))\frac{\partial}{\partial t}D^\dagger(\alpha(t)) . \quad (2.9)$$

The ground state $|0,k\rangle$ evolves with H_g into $|0,k\rangle_t$ which, to also be a ground state, must satisfy the requirement $K_-|0,k\rangle_t=0$ implying that $|0,k\rangle_t\propto|0,k\rangle$. And since $|0,k\rangle_t$ must satisfy the Schrödinger equation, one must have $K_-H_g|0,k\rangle_t=0$ so that $|0,k\rangle_t$ is an eigenstate of H_g. Thus we choose H_g as $H_g=a(t)K_0+\beta(t)$, where $a(t)$ and $\beta(t)$ are arbitrary functions of time. Other Hermitian operators such as K_0^2 of K_+K_- might be added to H_g as they would also preserve the ground state as an eigenstate under evolution. However, the transformed Hamiltonian from Eq. (2.9) will not preserve arbitrary coherent states.

Now with $D(\alpha)$ given in Eq. (2.4), so that

$$D^\dagger(\alpha)=e^{\xi^*K_-}e^{-\gamma K_0}e^{-\xi K_+} , \quad (2.10)$$

with the Baker-Hausdorff-Campbell formula

$$e^ABe^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots , \quad (2.11)$$

one obtains

$$D(\alpha)K_0D^\dagger(\alpha)=(\cosh\tau)K_0+\tfrac{1}{2}e^{-i\phi}(\sinh\tau)K_+ +\tfrac{1}{2}e^{i\phi}(\sinh\tau)K_- , \quad (2.12a)$$

$$D(\alpha)K_-D^\dagger(\alpha)=(K_--2\xi K_0+\xi^2K_+)/(1-|\xi|^2) . \quad (2.12b)$$

Then, using these results, from Eq. (2.9) one obtains

$$H_{\text{coh}}=A(t)K_0+f(t)K_++f^*(t)K_-+\beta(t) , \quad (2.13)$$

where

$$A(t)=a(t)\cosh\tau+2\,\text{Im}(\dot\xi\xi^*)/(1-|\xi|^2) , \quad (2.14a)$$

$$f(t)=\tfrac{1}{2}a(t)e^{-i\phi}\sinh\tau+i\dot\xi/(1-|\xi|^2) . \quad (2.14b)$$

From Eq. (2.14b) we obtain the equations for $\dot\phi$ and $\dot\tau$ as

$$\dot\phi=a(t)-2\,\text{Re}[f(t)e^{i\phi}]\text{csch}\tau , \quad (2.15a)$$

$$\dot\tau=-2\,\text{Im}[f(t)e^{i\phi}] . \quad (2.15b)$$

We now show that these equations are consistent with the corresponding classical equations of motion. These equations may be found in the continuous limit of the propagator when expressed as a path integral over SU(1,1) CS. For the Hamiltonian of Eq. (2.13) one has the propagator as

$$G_k(\xi'',t'',\xi',t')=\langle\xi'',k|T\exp\left[-i\int_{t'}^{t''}H_{\text{coh}}(t)dt\right]|\xi',k\rangle , \quad (2.16)$$

where T is the time ordering operator. As discussed in Ref. 7, this may be expressed as the path integral

$$G_k(\xi'',t'',\xi',t')=\int_{\xi'}^{\xi''}\mathscr{D}\mu_k(\xi)\exp\left\{i\int_{t'}^{t''}\mathscr{L}(\xi,\xi^*,\dot\xi,\dot\xi^*)dt\right\} , \quad (2.17)$$

where

$$\mathscr{L}=\frac{ik}{(1-|\xi|^2)}(\xi^*\dot\xi-\xi\dot\xi^*)-\mathscr{H}_k(\xi,\xi^*,t) , \quad (2.18)$$

with

$$\mathscr{H}_k(\xi,\xi^*,t)=\langle\xi,k|H_{\text{coh}}(t)|\xi,k\rangle . \quad (2.19)$$

The Euler-Lagrange equations lead to the equation of motion (Hamilton's equation)

$$\dot\xi=\{\xi,\mathscr{H}_k\} , \quad (2.20)$$

where

$$\{A,B\}=\frac{(1-|\xi|^2)^2}{2ik}\left[\frac{\partial A}{\partial\xi}\frac{\partial B}{\partial\xi^*}-\frac{\partial A}{\partial\xi^*}\frac{\partial B}{\partial\xi}\right] \quad (2.21)$$

$$=\frac{1}{k\sinh\tau}\left[\frac{\partial A}{\partial\phi}\frac{\partial B}{\partial\tau}-\frac{\partial A}{\partial\tau}\frac{\partial B}{\partial\phi}\right] . \quad (2.22)$$

The phase space here is a curved one—the Lobachevskii plane. (We have corrected a minor error in Ref. 7.)

With the matrix elements of the SU(1,1) generators given by

$$\langle\xi,k|K_0|\xi,k\rangle=k(1+|\xi|^2)/(1-|\xi|^2)=k\cosh\tau , \quad (2.23a)$$

$$\langle\xi,k|K_+|\xi,k\rangle=2k\xi^*/(1-|\xi|^2)=-ke^{i\phi}\sinh\tau , \quad (2.23b)$$

$$\langle\xi,k|K_-|\xi,k\rangle=2k\xi/(1-|\xi|^2)=-ke^{-i\phi}\sinh\tau , \quad (2.23c)$$

one obtains

$$\mathscr{H}_k=\frac{k}{(1-|\xi|^2)}[A(t)(1+|\xi|^2)-2f(t)\xi^*-2f^*(t)\xi]+\beta(t) \quad (2.24a)$$

$$=k[A(t)\cosh\tau-f(t)e^{i\phi}\sinh\tau-f^*(t)e^{-i\phi}\sinh\tau]+\beta(t) . \quad (2.24b)$$

The equations of motion yield

$$\dot\xi=-iA(t)\xi-\tfrac{1}{2}if^*(t)\xi^2-\tfrac{1}{2}if(t) , \quad (2.25)$$

or in terms of ϕ and τ we have

$$\dot{\phi}=A(t)-2\,\mathrm{Re}[f(t)e^{i\phi}]\coth\tau \ , \qquad (2.26a)$$

$$\dot{\tau}=-2\,\mathrm{Im}[f(t)e^{i\phi}] \ . \qquad (2.26b)$$

Note that Eq. (2.26b) is the same as Eq. (2.15b). Equation (2.26a) may be converted to Eq. (2.15a) by the substitution of $A(t)$ from Eq. (2.14a), thus demonstrating the consistency between Eq. (2.15) and (2.26).

III. EXAMPLE OF THE DEGENERATE PARAMETRIC OSCILLATOR

In this section we present an example of a system which possesses an SU(1,1) CS preserving Hamiltonian. This example comes from nonlinear quantum optics: the degenerate parametric oscillator. The Hamiltonian for this system is

$$H(t)=\omega a^{\dagger}a+\kappa[e^{-2i\omega t}(a^{\dagger})^2+e^{2i\omega t}a^2] \ , \qquad (3.1)$$

where the pump mode is treated classically (the factors $(e^{\pm i2\omega t})$ and κ is a coupling constant. The significance of this Hamiltonian is that it is predicted to produce photon antibunching and to generate squeezed states. Indeed, the SU(1,1) CS seem to be a special class of squeezed states.

This Hamiltonian has previously been treated with the ordinary coherent states in Ref. 11; the propagator was evaluated by path integration over these states. Note, however, Eq. (3.1) is not of the form (1.1) and thus does not preserve ordinary coherent states. But the generators of SU(1,1) may be realized as[6]

$$K_{+}=\tfrac{1}{2}(a^{\dagger})^2 \ , \qquad (3.2a)$$

$$K_{-}=\tfrac{1}{2}a^2 \ , \qquad (3.2b)$$

$$K_0=\tfrac{1}{4}(a^{\dagger}a+aa^{\dagger}) \ , \qquad (3.2c)$$

so that we may write

$$H(t)=2\omega K_0+2\kappa(K_{+}e^{-2i\omega t}+K_{-}e^{2i\omega t}) \qquad (3.3)$$

(where an additive constant has been dropped), which is of the form of Eq. (2.13) with $A=2\omega$, $f(t)=2e^{-2i\omega t}$, and $\beta(t)=0$. The classical equation of motion is then

$$\dot{\xi}=-2i\omega\xi-i\kappa e^{2i\omega t}\xi^2-i\kappa e^{-2i\omega t} \ . \qquad (3.4)$$

This nonlinear differential equation may be solved to give

$$\xi=e^{-2i\omega t-i\pi/2}\tanh[\kappa(t-t_i)] \ , \qquad (3.5)$$

where t_i is some initial arbitrary time. The differential Eq. (3.4) is very similar to one that occurs in the path integration with the ordinary coherent states of Ref. 11. Note that in the limit $\kappa\rightarrow 0$, Eq. (3.3) reduces to the Hamiltonian of the harmonic oscillator. The solution of Eq. (3.4) is then $\xi=\xi_0 e^{-2i\omega t}$, where ξ_0 is a constant, which agrees with Ref. 7. Note that the solution in Eq. (3.5) reduces, as $\kappa\rightarrow 0$, to the particular case when $\xi_0=0$.

In another work we shall discuss the application of the SU(1,1) CS to the degenerate parametric oscillator more fully, including the path integration of its propagator in these states.

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