Nonlinear Hamiltonians with applications to quantum hydrodynamics

P. Goldberg and W. M. Schreiber
The College of Staten Island and Graduate Center of The City University of New York, Staten Island, New York 10301

L. F. Landovitz
Graduate Center of The City University of New York, New York, New York 10036

G. Sandri
Boston University, Boston, Massachusetts 02215
(Received 12 April 1984)

The quantum mechanics of nonlinear Hamiltonians whose classical limit is of the form $H=Kq^2p^2$ is studied in several dimensions. Such Hamiltonians arise in the canonical formulation of the hydrodynamics of an ideal, incompressible fluid, where the canonical variables are Clebsch potentials. They also appear in the study of the propagation of electromagnetic radiation through an optically active medium. Despite the nonlinearity, exact solutions are obtained in both the classical and quantum-mechanical formulations. The relevant Heisenberg operators for the dynamical behavior of the system are found and compared to their corresponding classical counterparts.

I. INTRODUCTION

Symmetrized dynamical equations for many physical theories may be interpreted as referring to either classical functions or to quantum-mechanical operators. In this paper we formulate the hydrodynamics of an ideal, incompressible fluid so that the equations refer to quantum-mechanical operators as well as to classical fields. Although a fundamental view would consider the appropriate hydrodynamic quantities as statistical averages over microscopic variables, interesting results are obtained if the averaging process is assumed completed and "macroscopic" variables are used.

We consider the velocity and pressure fields for an inviscid, incompressible fluid, $\mathbf{u}(\mathbf{x},t)$ and $P(\mathbf{x},t)$ to be quantum-mechanical operators satisfying the equations

$$\frac{\partial \mathbf{u}}{\partial t}+\tfrac{1}{2}\{\mathbf{u}\cdot\nabla\mathbf{u}\}+\nabla P=0\;, \tag{1.1}$$

$$\nabla\cdot\mathbf{u}=0\;, \tag{1.2}$$

where the curly brackets denote symmetrization so that

$$\{\mathbf{u}\cdot\nabla u_i\}=u_j\frac{\partial u_i}{\partial x_j}+\frac{\partial u_i}{\partial x_j}u_j\;. \tag{1.3}$$

We assume the classical velocity and pressure, $\mathbf{v},\bar{P}$, are given by the mean of the corresponding quantum-mechanical operators,

$$\mathbf{v}(\mathbf{x},t)=\langle\mathbf{u}(\mathbf{x},t)\rangle\;, \tag{1.4}$$

$$\bar{P}(\mathbf{x},t)=\langle P(\mathbf{x},t)\rangle\;. \tag{1.5}$$

Letting

$$\mathbf{w}(\mathbf{x},t)=\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)\;, \tag{1.6}$$

we obtain from Eqs. (1.1) and (1.2)

$$\frac{\partial \mathbf{v}}{\partial t}+\langle\tfrac{1}{2}\{(\mathbf{w}+\mathbf{v})\cdot\nabla(\mathbf{w}+\mathbf{v})\}+\nabla P\rangle=0\;, \tag{1.7}$$

$$\nabla\cdot\mathbf{v}=0\;. \tag{1.8}$$

Thus the evolution of the fluid is given by

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\langle\tfrac{1}{2}\{\mathbf{w}\cdot\nabla\mathbf{w}\}+\nabla P\rangle\;. \tag{1.9}$$

The quantum-mechanical fluctuations, therefore, give rise to an internal stress which does not appear in the inviscid Navier-Stokes equation. Using the incompressibility condition, we obtain

$$\tfrac{1}{2}\langle\{\mathbf{w}\cdot\nabla\mathbf{w}\}\rangle=\frac{\partial}{\partial x_j}(\tfrac{1}{2}\langle\mathbf{w}_i\mathbf{w}_j+\mathbf{w}_j\mathbf{w}_i\rangle) \tag{1.10}$$

so that the total stress is given by

$$\frac{\partial}{\partial x_j}(T_{ij}+\bar{P}\delta_{ij}) \tag{1.11}$$

where

$$T_{ij}=\tfrac{1}{2}\langle\mathbf{w}_i\mathbf{w}_j+\mathbf{w}_j\mathbf{w}_i\rangle\;. \tag{1.12}$$

Note that the above result does not depend upon the details of the commutation relations.

For a two-dimensional incompressible fluid, $\mathbf{u}\cdot\boldsymbol{\omega}=0$, the vorticity field $\boldsymbol{\omega}(\mathbf{x},t)=\nabla\times\mathbf{u}=\omega\hat{\mathbf{k}}$, and the dynamical equation reduces to vorticity convection

$$\frac{\partial\omega}{\partial t}+(\mathbf{u}\cdot\nabla)\omega=0\;. \tag{1.13}$$

For such a system it has been shown[1] that a canonical formulation exists using variables of the type first employed by Clebsch.[2–4] In this formulation, the velocity field is given by

$$\mathbf{u}=\mathsf{P}\cdot(\alpha\nabla\beta)\;, \tag{1.14}$$

where P is a suitable projection operator and α,β are canonical variables for the fluid. The canonical equations of motion obtained by variation of the Hamiltonian[5]

$$H=\tfrac{1}{2}\int u^2 d\mathbf{x} \tag{1.15}$$

are given by

$$\frac{\partial\alpha}{\partial t}+\mathbf{u}\cdot\nabla\alpha=0\ , \tag{1.16}$$

$$\frac{\partial\beta}{\partial t}+\mathbf{u}\cdot\nabla\beta=0\ . \tag{1.17}$$

The fluid equation (1.13) is obtained from the above equations using the relationship $\boldsymbol{\omega}=\nabla\alpha\times\nabla\beta$. Equal-time canonical commutation relations are assumed for α and β.

We may introduce a complete set of functions $f_m(x)$ and define

$$\alpha=\sum_m q_m f_m(x)\ , \tag{1.18}$$

$$\beta=\sum_m p_m f_m(x)\ . \tag{1.19}$$

Since the α,β are canonically conjugate and the f_m are complete, we have the Poisson bracket

$$\{q_m,p_{m'}\}_{\rm PB}=\delta_{mm'}\ . \tag{1.20}$$

The functions f_m may be chosen such that

$$\int d\mathbf{x}\,\mathsf{P}\cdot(f_m\nabla f_{m'})\cdot\mathsf{P}\cdot(f_n\nabla f_{n'}) \tag{1.21}$$

$$=\int d\mathbf{x}(f_m\nabla f_{m'})\cdot\mathsf{P}\cdot(f_n\nabla f_{n'}) \tag{1.22}$$

$$=K_{mm'}\delta_{mn}\delta_{m'n'} \tag{1.23}$$

so that the Hamiltonian is given by

$$H=\sum_{m,m'}K_{mm'}q_m^2p_{m'}^2\ . \tag{1.24}$$

For our model we choose a small number of modes and assume $K_{mm'}\approx K\delta_{mm'}$ for these modes. We are, therefore, led to the study of a model Hamiltonian for quantum hydrodynamics composed of qp pairs with the classical limit

$$H_{\rm cl}=\tfrac{1}{2}Kq^2p^2\ . \tag{1.25}$$

The corresponding Lagrangian is

$$L=\frac{1}{2}\frac{1}{Kq^2}\dot{q}^2\ . \tag{1.26}$$

Thus, the Hamiltonian (1.25) may be viewed as describing a system with a position-dependent mass. The transition to quantum mechanics proceeds with the standard assumption that

$$\{q_m,p_{m'}\}_{\rm PB}\to\frac{1}{i\hbar}[q_m,p_{m'}]\ .$$

Another system of current interest to which this analysis may be applied is the propagation of electromagnetic radiation through an optically active medium. For each mode of the field, $q\sim E$ and $p\sim B$;[6] thus terms of the form q^2p^2 will correspond to a medium in which there is a Faraday effect. In addition, Hamiltonians of this form also arise in the study of plasma transport perpendicular to a uniform magnetic field.

II. GENERAL STRUCTURE

For a dynamical system with $2N$ degrees of freedom, one can construct N^2 linearly independent Hermitian operators O_i, $i=1,2,\ldots,N^2$, from the N^2 forms q_ip_j. The canonical operators q_i and p_j satisfy the commutation relation

$$[q_i,p_j]=i\hbar\delta_{ij},\ \ i,j=1,2,\ldots,N\ . \tag{2.1}$$

The N^2 operators O_i satisfy the closed algebraic system

$$[O_i,O_j]=i\hbar c_{ijk}O_k \tag{2.2}$$

where the structure constants c_{ijk} are real.

A quantum Hamiltonian is constructed which has the classical limit

$$H_{\rm cl}=\tfrac{1}{2}Kq^2p^2 \tag{2.3}$$

which is obtained assuming q_i and p_i commute. The operators O_i are chosen such that the quantum Hamiltonian contains only terms of the form O_i^2.

In Sec. III we present both the classical and quantum-mechanical solutions for the one-dimensional case. For this case, results are given for the more general Hamiltonian

$$H=Kq^np^n \tag{2.4}$$

and its corresponding quantum analog, as well as specific results for $n=2$. We note that for this model, energy surfaces in phase space are hyperbolic.

Solutions for a two-dimensional system, of particular interest for the fluid model described in Sec. I, are presented in Sec. IV. Solutions for the three-dimensional case are given in Sec. V.

The solutions given for the two- and three-dimensional models exhibit conservation of angular momentum. Using Eqs. (1.18) and (1.19) for the canonical variables α and β, the vorticity is given by

$$\boldsymbol{\omega}=\nabla\alpha\times\nabla\beta=\sum_{m,m'}q_mp_{m'}\nabla f_m\times\nabla f_{m'}\ . \tag{2.5}$$

Therefore, for our model

$$\boldsymbol{\omega}=\sum_i L_i\mathbf{F}_i\ , \tag{2.6}$$

where

$$L_i=\epsilon_{ijk}q_jp_k\ , \tag{2.7}$$

and

$$\mathbf{F}_i=\tfrac{1}{2}\epsilon_{ijk}\nabla f_i\times\nabla f_k\ . \tag{2.8}$$

Note that the enstrophy

$$N=\int\omega^2 d\mathbf{x} \tag{2.9}$$

is a constant of the motion.

In terms of the q,p variables, we see that the vorticity is related to components of the angular momentum in the multidimensional space they define. Hence the quantization of angular momentum leads to the quantization of vorticity.

III. ONE-DIMENSIONAL CLASSICAL AND QUANTUM SOLUTIONS

We consider the classical Hamiltonian

$$H = Ku^n \tag{3.1}$$

where

$$u = qp \ . \tag{3.2}$$

Clearly, u is a conserved quantity. The resulting Hamilton equations are

$$\dot{q} = Knu^{n-1}q \ , \tag{3.3}$$

$$\dot{p} = -Knu^{n-1}p \ , \tag{3.4}$$

with solutions given by

$$q(t) = q(0)e^{Knu^{n-1}t} \ , \tag{3.5}$$

$$p(t) = p(0)e^{-Knu^{n-1}t} \ . \tag{3.6}$$

For the interesting case of $n=2$ we have the "hyperbolic" oscillator

$$H = \tfrac{1}{2}Ku^2 \tag{3.7}$$

for which we obtain

$$q(t) = q(0)e^{Kut} \ , \tag{3.8}$$

$$p(t) = p(0)e^{-Kut} \ . \tag{3.9}$$

The corresponding quantum-mechanical Hamiltonians are obtained by setting

$$u = \tfrac{1}{2}(qp + pq) \tag{3.10}$$

which makes these Hamiltonians Hermitian.

Defining the Heisenberg operator[7]

$$O_+ = U^+ O U \tag{3.11}$$

where

$$U = e^{-(i/\hbar)Ht} \ , \tag{3.12}$$

one obtains

$$\dot{q}_+ = \frac{1}{i\hbar}[q_+, H] \ , \tag{3.13}$$

$$\dot{p}_+ = \frac{1}{i\hbar}[p_+, H] \ . \tag{3.14}$$

Note that Eq. (3.7) implies

$$u_+ = u \tag{3.15}$$

so that u is a conserved operator.

From Eq. (3.11) we have

$$q_+ = e^{(i/2\hbar)Ku^2t} q e^{-(i/2\hbar)Ku^2t} \ , \tag{3.16}$$

$$p_+ = e^{(i/2\hbar)Ku^2t} p e^{-(i/2\hbar)Ku^2t} \ . \tag{3.17}$$

Using the canonical commutation relation, Eq. (2.1), we obtain

$$q_+ = e^{(K/2)ut} q e^{(K/2)ut} \ , \tag{3.18}$$

$$p_+ = e^{-(K/2)ut} p e^{-(K/2)ut} \ . \tag{3.19}$$

Equations (3.18) and (3.19) are in close correlation with the corresponding classical expressions of Eqs. (3.8) and (3.9). Higher moments of q and p are obtainable from q_+ and p_+ since Eq. (3.11) implies

$$\left[\prod_{i=1}^{s} O_i\right]_+ = \prod_{i=1}^{s} (O_i)_+ \ . \tag{3.20}$$

One can find eigenfunctions $\psi_u(q)$ of the Schrödinger operator

$$u = \frac{\hbar}{2i}\left[q\frac{\partial}{\partial q} + \frac{\partial}{\partial q}q\right] \tag{3.21}$$

corresponding to the eigenvalue u. The resulting expression is

$$\psi_u(q) = \frac{1}{2\sqrt{\pi\hbar|q|}} e^{(i/\hbar)u\ln|q|} \tag{3.22}$$

where

$$\int_{-\infty}^{\infty} dq\, \psi_{u'}^* \psi_u = \delta(u-u') \ . \tag{3.23}$$

Thus the spectrum of H is continuous.

The resulting Green's function for the $n=2$ case is

$$\begin{aligned} G(q_2,q_1,t) &= \langle q_2 | U(t) | q_1 \rangle \\ &= \int_{-\infty}^{\infty} \langle q_2 | U(t) | u \rangle \langle u | q_1 \rangle du \ . \end{aligned} \tag{3.24}$$

Thus

$$G(q_2,q_1,t) = \int_{-\infty}^{\infty} du\, e^{-(i/2\hbar)Ku^2t} \psi_u(q_2)\psi_u^*(q_1) \ . \tag{3.25}$$

Upon performing the above integration, one obtains

$$G(q_2,q_1,t) = e^{-(i\pi/4)} \frac{1}{2\sqrt{2\pi\hbar Kt\,|q_1|\,|q_2|}} \times e^{-(i/2\hbar Kt)}\left[\ln\frac{|q_1|}{|q_2|}\right]^2 \ . \tag{3.26}$$

We now study the more general Hamiltonian for arbitrary n

$$H = Ku^n \tag{3.27}$$

for which

$$q_+ = e^{(i/\hbar)Ku^nt} q e^{-(i/\hbar)Ku^nt} \ , \tag{3.28}$$

$$p_+ = e^{(i/\hbar)Ku^nt} p e^{-(i/\hbar)Ku^nt} \ . \tag{3.29}$$

Equations (3.13) and (3.14) yield

$$\frac{\partial q_+}{\partial t} = -\frac{i}{\hbar} K U^\dagger [q, u^n] U \ , \tag{3.30}$$

$$\frac{\partial p_+}{\partial t} = -\frac{i}{\hbar} K U^\dagger [p, u^n] U \ . \tag{3.31}$$

Using the canonical commutation relations, one finds by induction

$$[q,u^n]=[(u+i\hbar)^n-u^n]q \ , \tag{3.32}$$

$$[p,u^n]=[(u-i\hbar)^n-u^n]p \ . \tag{3.33}$$

Equations (3.30) and (3.31) become, therefore,

$$\frac{\partial q_+}{\partial t}=-\frac{i}{\hbar}K[(u+i\hbar)^n-u^n]q_+ \ , \tag{3.34}$$

$$\frac{\partial p_+}{\partial t}=-\frac{i}{\hbar}K[(u-i\hbar)^n-u^n]p_+ \ , \tag{3.35}$$

so that

$$q_+=e^{-(i/\hbar)K[(u+i\hbar)^n-u^n]t}q =qe^{(i/\hbar)K[(u-i\hbar)^n-u^n]t} \ , \tag{3.36}$$

$$p_+=e^{-(i/\hbar)K[(u-i\hbar)^n-u^n]t}p =pe^{(i/\hbar)K[(u+i\hbar)^n-u^n]t} \ . \tag{3.37}$$

The latter expressions in Eqs. (3.36) and (3.37) follow from the fact that q_+ and p_+ are Hermitian operators. Equations (3.36) and (3.37) imply that

$$(qp)_+=qp \ , \tag{3.38}$$

$$(pq)_+=pq \ . \tag{3.39}$$

Therefore, not only is u a conserved operator but also the operators qp and pq are separately conserved, in close proximity to the classical result.

IV. TWO-DIMENSIONAL MODEL

The four linearly independent Hermitian operators chosen are

$$u_1=\tfrac{1}{2}(q_1p_2+q_2p_1) \ , \tag{4.1}$$

$$u_2=\tfrac{1}{2}(q_1,p_1-q_2p_2) \ , \tag{4.2}$$

$$\Omega=\tfrac{1}{2}(q_1p_2-q_2p_1) \ , \tag{4.3}$$

$$C=\tfrac{1}{4}[(q_1p_1+p_1q_1)+(q_2p_2+p_2q_2)] \ . \tag{4.4}$$

The three operators u_1,u_2,Ω are closed under commutation

$$[u_1,u_2]=i\hbar\Omega \ , \tag{4.5}$$

$$[u_1,\Omega]=i\hbar u_2 \ , \tag{4.6}$$

$$[u_2,\Omega]=-i\hbar u_1 \ . \tag{4.7}$$

The operator C commutes with u_1,u_2,Ω. Note that Ω is one-half the angular momentum operator L_3. It is also of interest to note that the Lie algebra described by Eqs. (4.5)–(4.7) is isomorphic to the two-dimensional Lorentz algebra.[8]

The Hamiltonian

$$H=\tfrac{1}{2}K(u_1^2+u_2^2) \tag{4.8}$$

has the classical limit of Eq. (2.3). Since

$$[\Omega,H]=0 \tag{4.9}$$

the operator Ω is conserved. As H,Ω, and C commute with each other, one can find simultaneous eigenfunctions of these operators.

In cylindrical coordinates

$$H=-\frac{K\hbar^2}{8}\left[r^2\frac{\partial^2}{\partial r^2}+3r\frac{\partial}{\partial r}+\frac{\partial^2}{\partial\theta^2}\right] . \tag{4.10}$$

We search for eigenstates of the Hamiltonian such that

$$H\psi(r,\theta)=E\psi(r,\theta) \ . \tag{4.11}$$

Setting

$$\psi(r,\theta)=F(r)G(\theta) \ , \tag{4.12}$$

the resulting equation is

$$\frac{1}{F}\left[-r^2\frac{d^2F}{dr^2}-3r\frac{dF}{dr}\right]-\frac{8E}{K\hbar^2}=\frac{1}{G}\frac{d^2G}{d\theta^2} \ . \tag{4.13}$$

Setting each side of Eq. (4.13) equal to $-m^2$ and requiring periodicity in θ yields

$$G(\theta)=e^{im\theta}, \quad m=0,\pm1,\pm2.\ldots \ . \tag{4.14}$$

Setting

$$F(r)=r^{\gamma}f(r) \tag{4.15}$$

the equation which $f(r)$ must satisfy is

$$r^2f''(r)+(2\gamma+3)rf'(r) +\left[\frac{8E}{K\hbar^2}-m^2+\gamma(\gamma+2)\right]f=0 \ . \tag{4.16}$$

Choosing $\gamma=-\frac{3}{2}$ eliminates the second term of Eq. (4.16). The equation then has the solution

$$f(r)=r^{(ik+1/2)} \tag{4.17}$$

for

$$E=\frac{K\hbar^2}{8}(k^2+m^2+1) \ . \tag{4.18}$$

In Eqs. (4.17) and (4.18) k is continuous.

The eigenfunction $\psi_{km}(r,\theta)$ of H is

$$\psi_{km}(r,\theta)=\frac{1}{2\pi r}r^{ik}e^{im\theta}, \quad -\infty<k<\infty, \quad m=0,\pm1,\ldots \ . \tag{4.19}$$

In Eq. (4.19) we have normalized $\psi_{km}(r,\theta)$ such that

$$\int_0^\infty dr\,r\int_0^{2\pi}d\theta\,\psi^*_{k'm'}\psi_{km}=\delta(k'-k)\delta_{m'm} \ . \tag{4.20}$$

In cylindrical coordinates, the expressions for the operators Ω and C are

$$\Omega=-\frac{i\hbar}{2}\frac{\partial}{\partial\theta} \ , \tag{4.21}$$

$$C=-\frac{i\hbar}{2}\left[r\frac{\partial}{\partial r}+1\right] . \tag{4.22}$$

Thus $\psi_{km}(r,\theta)$ is a simultaneous eigenfunction of the mutually commuting operators H, Ω and C, with associated eigenvalues

$$E_{km}=\frac{K\hbar^2}{8}(k^2+m^2+1)\ , \tag{4.23}$$

$$\Omega_m=\frac{m\hbar}{2}\ , \tag{4.24}$$

$$C_k=\frac{\hbar k}{2}\ . \tag{4.25}$$

The Green's function is given by

$$G(r,\theta,r',\theta',t)=\langle r,\theta\mid U(t)\mid r'\theta'\rangle\ . \tag{4.26}$$

Thus

$$G(r,\theta,r',\theta',t)=\langle r,\theta\mid e^{-(i/\hbar)Ht}\mid r',\theta'\rangle$$
$$=\int_{-\infty}^{\infty}dk\sum_{m=-\infty}^{\infty}e^{-(i/\hbar)E_{km}t}\langle r,\theta\mid k,m\rangle\langle k,m\mid r',\theta'\rangle\ . \tag{4.27}$$

Evaluating the above expression we obtain

$$G(r,\theta,r',\theta',t)=e^{-i(\pi/4)}\frac{1}{2\pi^2}\left[\frac{2\pi}{\hbar Kt}\right]^{1/2}\frac{1}{rr'}\exp\left\{i\left[\frac{2}{\hbar Kt}\right]\left[\ln^2\left[\frac{r'}{r}\right]+(\theta'-\theta)^2\right]\right\}$$
$$\times\exp\left[-\frac{i\hbar Kt}{8}\right]\sum_{m=-\infty}^{\infty}\exp\left[-\frac{i\hbar Kt}{8}\left[m+\frac{4}{\hbar Kt}(\theta'-\theta)\right]^2\right]\ . \tag{4.28}$$

Defining the expressions

$$u^{(+)}=u_1+iu_2\ , \tag{4.29}$$

$$u^{(-)}=u_1-iu_2\ , \tag{4.30}$$

the Hamiltonian (4.8) can be written as the sum of two commuting Hermitian parts. Thus

$$H=H^{(+-)}+H^{(-+)}\ , \tag{4.31}$$

where

$$H^{(+-)}=\frac{K}{4}u^{(+)}u^{(-)}\ , \tag{4.32}$$

$$H^{(-+)}=\frac{K}{4}u^{(-)}u^{(+)}\ . \tag{4.33}$$

The states $\psi_{km}(r,\theta)$ are eigenstates of the operators $H^{(+-)}$ and $H^{(-+)}$ with respective eigenvalues

$$E_{km}^{(+-)}=\frac{K\hbar^2}{16}[k^2+(m+1)^2]\ , \tag{4.34}$$

$$E_{km}^{(-+)}=\frac{K\hbar^2}{16}[k^2+(m-1)^2]\ . \tag{4.35}$$

Since

$$\frac{\partial(u_1)_+}{\partial t}=\frac{1}{i\hbar}U^\dagger[u_1,H]U$$
$$=\frac{K}{2}[\Omega(u_2)_++(u_2)_+\Omega]\ , \tag{4.36}$$

$$\frac{\partial(u_2)_+}{\partial t}=\frac{1}{i\hbar}U^\dagger[u_2,H]U$$
$$=-\frac{K}{2}[\Omega(u_1)_++(u_1)_+\Omega]\ , \tag{4.37}$$

one obtains

$$\frac{\partial u_+^{(G)}}{\partial t}=-i\frac{K}{2}(\Omega u_+^{(+)}+u_+^{(+)}\Omega)\ , \tag{4.38}$$

$$\frac{\partial u_+^{(-)}}{\partial t}=i\frac{K}{2}(\Omega u_+^{(-)}+u_+^{(-)}\Omega)\ . \tag{4.39}$$

These equations imply

$$u_+^{(+)}=e^{-i(K/2)\Omega t}u^{(+)}e^{-i(K/2)\Omega t}\ , \tag{4.40}$$

$$u_+^{(-)}=e^{i(K/2)\Omega t}u^{(-)}e^{i(K/2)\Omega t}\ . \tag{4.41}$$

Using the definitions of Eqs. (4.29) and (4.30) we obtain the result

$$(u_1)_+=\tfrac{1}{2}(e^{-i(K/2)\Omega t}u^{(+)}e^{-i(K/2)\Omega t}$$
$$+e^{i(K/2)\Omega t}u^{(-)}e^{i(K/2)\Omega t})\ , \tag{4.42}$$

$$(u_2)_+=-\frac{i}{2}(e^{-i(K/2)\Omega t}u^{(+)}e^{-i(K/2)\Omega t}$$
$$-e^{i(K/2)\Omega t}u^{(-)}e^{i(K/2)\Omega t})\ . \tag{4.43}$$

V. THREE-DIMENSIONAL CASE

We choose the following nine linearly independent Hermitian operators of the form q_ip_j:

$$u_1=q_1p_2+q_2p_1\ , \tag{5.1}$$

$$\mathbf{w}_1=\tfrac{1}{2}(q_1p_1+p_1q_1)\ , \tag{5.2}$$

$$\Omega_1=q_1p_2-q_2p_1\ , \tag{5.3}$$

$$u_2=q_1p_3+q_3p_1\ , \tag{5.4}$$

$$\mathbf{w}_2=\tfrac{1}{2}(q_2p_2+p_2q_2)\ , \tag{5.5}$$

$$\Omega_2=q_1p_3-q_3p_1\ , \tag{5.6}$$

$$u_3=q_2p_3+q_3p_2\ , \tag{5.7}$$

$$\mathbf{w}_3=\tfrac{1}{2}(q_3p_3+p_3q_3)\ , \tag{5.8}$$

$$\Omega_3=q_2p_3-q_3p_2\ . \tag{5.9}$$

Alternatively, instead of the three $\mathbf{w}$ operators, we can construct the operators

$$C_1=\tfrac{1}{2}(q_1p_1+p_1q_1+q_2p_2+p_2q_2)\ , \tag{5.10}$$

$$C_2=\tfrac{1}{2}(q_1p_1+p_1q_1+q_3p_3+p_3q_3)\ , \tag{5.11}$$

$$C_3=\tfrac{1}{2}(q_2p_2+p_2q_2+q_3p_3+p_3q_3)\ , \tag{5.12}$$

so that

$$C_1=\mathbf{w}_1+\mathbf{w}_2\ , \tag{5.13}$$

$$C_2=\mathbf{w}_1+\mathbf{w}_3\ , \tag{5.14}$$

$$C_3=\mathbf{w}_2+\mathbf{w}_3\ . \tag{5.15}$$

The operators C_i, $i=1,2,3$, commute with the operators $u_i,\mathbf{w}_i,\Omega_i$. Furthermore, their sum

$$C=C_1+C_2+C_3 \tag{5.16}$$

commutes with all of the operators of Eqs. (5.1)—(5.9). These nine operators form an algebraic system that is closed under commutation. Denoting the nine operators of Eqs. (5.1)—(5.9) by O_i, $i=1,2,\ldots,9$, one obtains

$$[O_i,O_j]=i\hbar\sum_{k=1}^{q} c_{ijk}O_k\ . \tag{5.17}$$

The nonzero structure constants c_{ijk} have the values ± 1 and ± 2. Specifically

$$c_{123}=c_{347}=c_{369}=c_{491}=c_{693}=c_{897}=1\ , \tag{5.18}$$

$$\begin{aligned}c_{149}&=c_{153}=c_{167}=c_{176}=c_{194}=c_{231}=c_{246}=c_{264}=c_{351}\\&=c_{374}=c_{396}=c_{473}=c_{486}=c_{579}=c_{597}=c_{671}=c_{684}\\&=c_{789}=-1\ ,\end{aligned} \tag{5.19}$$

$$c_{132}=c_{462}=c_{795}=2\ , \tag{5.20}$$

$$c_{135}=c_{468}=c_{798}=-2\ . \tag{5.21}$$

From Eq. (5.17)

$$c_{ijk}=-c_{jik} \tag{5.22}$$

so these quantities need not be listed explicitly.

The Hamiltonian

$$H=\frac{K}{2}\sum_{i=1}^{3}(u_i^2+v_i^2-\mathbf{w}_i^2) \tag{5.23}$$

satisfies Eq. (2.3) in the classical limit with the assumption that q_i,p_i commute. In Eq. (5.23) we have used

$$v_1=q_1p_1-q_2p_2\ , \tag{5.24}$$

$$v_2=q_1p_1-q_3p_3\ , \tag{5.25}$$

$$v_3=q_2p_2-q_3p_3\ . \tag{5.26}$$

Note that the v operators are linearly dependent. They may be expressed in terms of the $\mathbf{w}$ operators by

$$v_1=\mathbf{w}_1-\mathbf{w}_2\ , \tag{5.27}$$

$$v_2=\mathbf{w}_1-\mathbf{w}_3\ , \tag{5.28}$$

$$v_3=\mathbf{w}_2-\mathbf{w}_3\ . \tag{5.29}$$

Substituting Eqs. (5.27)—(5.29) into Eq. (5.23) yields the alternative expression for the Hamiltonian

$$H=\frac{K}{2}\left[\sum_{i=1}^{3}(u_i^2+\mathbf{w}_i^2)-2\sum_{\substack{i=1\\i<j}}^{2}\sum_{j=2}^{3}\mathbf{w}_i\mathbf{w}_j\right]. \tag{5.30}$$

With the structure constants given in Eqs. (5.18)—(5.22), we find

$$[L_i,H]=0,\quad i=1,2,3 \tag{5.31}$$

where the L_i are the components of angular momentum. Thus, angular momentum is conserved and one can find simultaneous eigenfunctions of L_3 and H. In spherical coordinates, the Hamiltonian is

$$H=-\frac{K}{2}\hbar^2\left[r^2\frac{\partial^2}{\partial r^2}+4r\frac{\partial}{\partial r}-\frac{L^2}{\hbar^2}-\frac{3}{4}\right] \tag{5.32}$$

where

$$L^2=\sum_{i=1}^{3}L_i^2\ .$$

Setting

$$H\psi=E\psi \tag{5.33}$$

one finds the normalized eigenfunctions

$$\psi_{klm}(r,\theta,\phi)=\frac{1}{\sqrt{2\pi}}\frac{1}{r^{3/2}}r^{ik}Y_l^m(\theta,\phi),\quad -\infty<k<\infty,\quad l=0,1,2,\ldots,\quad m=-l,-l+1,\ldots,l\ . \tag{5.34}$$

The normalization of these eigenfunctions is given by

$$\int \psi^*_{klm}\psi_{k'l'm'}d\mathbf{x}=\delta(k'-k)\delta_{ll'}\delta_{mm'}\ . \tag{5.35}$$

The corresponding eigenvalues of H are

$$E_{kl}=\frac{\hbar^2K}{2}[k^2+l(l+1)+3]\ . \tag{5.36}$$

The Green's function is given by

$$G(r,\theta,\phi,t,r',\theta'\phi')=\langle r,\theta,\phi\,|\,e^{-(i/\hbar)Ht}\,|\,r',\theta',\phi'\rangle$$
$$=\int_{-\infty}^{\infty}dk\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\langle r,\theta,\phi\,|\,e^{-(i/\hbar)Ht}\,|\,k,l,m\rangle\langle k,l,m\,|\,r',\theta',\phi'\rangle$$
$$=\int_{-\infty}^{\infty}dk\sum_{l=0}^{\infty}\sum_{m=-l}^{l}e^{-(i\hbar K/2)[k^2+l(l+1)+3]t}\psi_{klm}(r,\theta,\phi)\psi^*_{klm}(r',\theta',\phi')\ . \tag{5.37}$$

Performing the k integration and using the addition theorem, one obtains

$$G(r,\theta,\phi,t,r',\theta',\phi')=\frac{1}{4\pi}e^{-i\pi/4}\left[\frac{1}{4\pi\hbar Kt}\right]^{1/2}e^{-i(3/2)\hbar Kt}\frac{1}{r^{3/2}r'^{3/2}}$$
$$\times\exp\left\{\frac{i}{2\hbar Kt}\left[\ln^2\left[\frac{r'}{r}\right]\right]\right\}\sum_{l=0}^{\infty}(2l+1)P_L(\cos\gamma)e^{-(i\hbar K/2)l(l+1)t}\ , \tag{5.38}$$

where P_L is a Legendre polynomial and γ is the angle between (θ',ϕ') and (θ,ϕ).

VI. CONCLUSION

We have shown that there exists a Hamiltonian formulation of phenomenological fluid mechanics. We have then proceeded to quantize the model and obtained exact solutions of a truncated version of the full nonlinear Hamiltonian, which has the classical limit, $H_{\rm cl}=\frac{1}{2}Kq^2p^2$. These solutions exhibit conservation of angular momentum which we have shown is related to the quantization of vorticity. Although the motivation for this work arose from fluid mechanics, similar Hamiltonians are found in electrodynamics.

We are presently developing the field theoretical extension of this work. The model Hamiltonian of interest for quantum hydrodynamics is of the form

$$H=\sum_{\alpha,\beta,\gamma,\delta}K_{\alpha\beta\gamma\delta}q_\alpha q_\beta p_\gamma p_\delta\ , \tag{6.1}$$

where the summation is over field modes, so that mode-mixing phenomena must now be included. We plan to apply the results obtained from the field theory to fluid flow and to quantum-optical systems.

ACKNOWLEDGMENT

This research was supported in part by Grant No. 13711, PSC-CUNY (Professional Staff Congress) Research Award Program of the City University of New York.

APPENDIX: DERIVATION OF THE CANONICAL EQUATIONS

The velocity field $\mathbf{u}$ satisfying Eq. (1.1) and the incompressibility condition Eq. (1.2) may be written as

$$\mathbf{u}=\mathsf{P}\cdot(\alpha\nabla\beta)\ , \tag{A1}$$

where α and β are nonlocal potentials and the projection operator P has components

$$\mathsf{P}_{ij}=\mathsf{I}_{ij}-\frac{\partial}{\partial x_i}\mathsf{G}_{\rm op}\frac{\partial}{\partial x_j}\ . \tag{A2}$$

The Green's-function operator satisfies

$$\nabla^2\mathsf{G}_{\rm op}=1 \tag{A3}$$

corresponding to the Green's function

$$\nabla^2\mathsf{G}(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}')\ . \tag{A4}$$

The projection operator ensures that the constraint condition Eq. (1.2) is satisfied. The basic properties of P are given by

$$\sum_j\mathsf{P}_{ij}\mathsf{P}_{jk}=\mathsf{P}_{ik}\ , \tag{A5}$$

$$\sum_i\frac{\partial}{\partial x_i}\mathsf{P}_{ij}=0\ , \tag{A6}$$

$$\sum_j\mathsf{P}_{ij}\frac{\partial}{\partial x_j}=0\ , \tag{A7}$$

so that

$$\mathsf{P}\cdot\mathbf{u}=\mathbf{u} \tag{A8}$$

and

$$\nabla\cdot\mathbf{u}=0\ . \tag{A9}$$

We examine the variational derivatives of

$$H=\tfrac{1}{2}\int u^2d\mathbf{x}$$
$$=\tfrac{1}{2}\int(\alpha\nabla\beta)\cdot\mathsf{P}\cdot(\alpha\nabla\beta)d\mathbf{x}$$
$$=\tfrac{1}{2}(\alpha\nabla\beta,\ \mathsf{P}\cdot(\alpha\nabla\beta))\ , \tag{A10}$$

where we have employed Eq. (A8) and the Hermiticity of P to simplify the integral. The variation of H gives

$$\delta H_{\delta\alpha}=\tfrac{1}{2}((\delta\alpha)\nabla\beta,\ \mathsf{P}\cdot(\alpha\nabla\beta))$$
$$+\tfrac{1}{2}(\alpha\nabla\beta,\ \mathsf{P}\cdot(\delta\alpha)\nabla\beta) \tag{A11}$$
$$=(\delta\alpha,\nabla\beta\cdot\mathsf{P}\cdot(\alpha\nabla\beta)) \tag{A12}$$
$$=(\delta\alpha,\mathbf{u}\cdot\nabla\beta)\ . \tag{A13}$$

Therefore, we obtain from Hamilton's equation

$$\frac{\partial\beta}{\partial t} \equiv -\frac{\delta H}{\delta\alpha} = -\mathbf{u}\cdot\nabla\beta \,. \tag{A14}$$

Similarly, variation of H with respect to β gives

$$\frac{\partial\alpha}{\partial t} \equiv \frac{\delta H}{\delta\beta} = -\mathbf{u}\cdot\nabla\alpha \,. \tag{A15}$$

The canonical variables may be viewed as Lagrangian "particle" labels that follow the flow of the fluid.

[1]P. Goldberg and G. Sandri, Bull. Am. Phys. Soc. **26**, 1012 (1981).

[2]H. Lamb, *Hydrodynamics* (Dover, New York, 1932), Article 167.

[3]P. Morrison and J. Greene, Phys. Rev. Lett. **45**, 790 (1980).

[4]O. Buneman, Phys. Fluids **23**, 1716 (1980).

[5]See Appendix.

[6]See, e.g., M. Sargent III, M. O. Scully and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, Mass., 1974), Chap. 14.

[7]L. F. Landovitz, Phys. Rev. A **11**, 67 (1975).

[8]See, e.g., M. Hamermesh, *Group Theory and its Applications to Physical Problems* (Addison-Wesley, Reading, Mass., 1962).