# Pump-field fluctuations in resonance fluorescence with two-photon resonant excitation

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The effects of amplitude and phase fluctuations of the pump laser on the resonance fluorescence under two-photon resonant excitation are studied. The averaging of the stochastic Bloch equations is carried out within the approximation scheme that has recently been applied to the case of one-photon resonant excitation. In the case of two-photon resonant excitation the stochastic pump field gives rise to multiplicative nonlinear Gaussian fluctuation effects. The phase fluctuations are treated in the phase-diffusion model, and the averaging over the amplitude fluctuations is performed by means of generating-function methods. Both the intensity and the intensity correlation of the fluorescent light are studied. When the amplitude fluctuations are weak, the cases of one- and two-photon resonant excitations are similar. The difference between both cases consists of a modification of the Rabi frequency and a faster damping of the Rabi oscillations in the two-photon case. For sufficiently fast correlation decay of the amplitude fluctuations their effect can simply be described by a rate. In the general case of $n$-photon resonant excitation this rate is larger by a factor of $n^2$ than that in the case of one-photon resonant excitation. When the amplitude fluctuations are sufficiently strong, the Rabi oscillations are "washed out." In the case of one-photon resonant excitation both the intensity and the intensity correlation function of the scattered light develop into step functions over time. In the two-photon case the dynamics of both quantities is determined by a new time constant which acts as both the oscillation time and the damping time. Consequently an overshoot peak occurs in both the intensity and the intensity correlation instead of the tendency to become step functions.

## I. INTRODUCTION

Since 1930, when Weisskopf and Wigner[1] developed the first theory of the resonance fluorescence as a typical example of resonant interaction between atomic systems and light fields, much effort has been spent on the investigation of the properties of the usual resonance fluorescence with one-photon resonant excitation. In the last years considerable activity has focused on the influence of different statistical properties of the exciting radiation on the scattered light.[2-14]

The model widely used in the theory of the resonance fluorescence with one-photon resonant excitation is the two-level atom driven by an external source of light, the dynamics of the two-level atom under the influence of the driving field being described by means of the Bloch equations. When the atom is excited by a fluctuating light field the Bloch equations become Langevin-type equations with multiplicative linear complex noise.

This system of stochastic differential equations for the density-matrix elements of the two-level atom is very difficult to solve. Instead of pursuing a direct integration a theorem can be used which states that in the case of Markovian field fluctuations the averaged density-matrix elements can be found from the solution of a system of partial differential equations.[15] However, it should be noted that this system of partial differential equations is equally hard to solve. In general, it leads to an infinite set of coupled differential equations, which can be solved, after truncation, numerically. In this way exact results for the intensity, the spectrum, and the intensity correlation function have been obtained for the cases when the exciting field fluctuations have been described by the extended phase-diffusion model with a non-Lorentzian line shape,[14] the chaotic field,[6,9,14] and the real Gaussian field.[9]

On the other hand there have been attempts to solve the stochastic Bloch equations directly in order to find (approximative) analytical solutions. Only in the limit of Gaussian white-noise fluctuations of the external driving field have general methods of solution been given. In particular, they have been applied to the case when the exciting field undergoes phase fluctuations, which can be described by a Wiener-Levy process (phase-diffusion model).[2] The situation becomes difficult when the external driving noise cannot be treated as a standard Gaussian white noise because its power spectrum has a finite bandwidth. Closed solutions have been presented for some limiting cases. First, calculations have been performed for the case when the intensity of the pump field is weak.[5,7] In particular, chaotic fields[5] and single-mode laser fields with phase and amplitude fluctuations[7] have been studied. Second, solutions have been given for the cases of chaotic fields and real Gaussian amplitude fields when the atomic damping rates are negligibly small.[8] Third, the problem of resonant interaction between a two-level atom and a fluctuating single-mode laser field has been treated by utilizing a high-driving-field approximation.[11] Fourth, the situation for a fluctuating single-mode laser field has been studied for the case when the relative mean-square deviation of the Rabi frequency is small compared with the ratio of the characteristic atomic relaxation time to the correlation time of the amplitude fluctuations.[13]

When $n$ photons ($n>1$) of the incident stochastic light field are needed in order to fulfill the resonance condition ($n$-photon resonant excitation) the corresponding Bloch equations that describe the atomic dynamics become Langevin-type equations with multiplicative nonlinear complex noise. If the atom is driven with laser light, the amplitude of which remains stable while its phase fluctuates according to the phase-diffusion model, the problem of averaging the Bloch equations simply consists of substituting $\Gamma_2+n^2\Gamma_L$ for the atomic dephasing rate $\Gamma_2$,[16] $2\Gamma_L$ being the full width at half maximum of the laser line. This is the natural generalization of the result well known for the case of one-photon resonant excitation ($n=1$). With the exception of light fields undergoing only phase fluctuations the nonlinear noise in the Bloch equations prevents its treatment as a Gaussian noise and hence it can be expected to lead to effects of field fluctuations on the scattered light, which are quite different from those known for the case of one-photon resonant excitation.

In the present paper we study the simplest example of multiphoton excitation in resonance fluorescence: the two-photon case. Assuming the external stochastic light source is a laser field undergoing phase and amplitude fluctuations, in addition to the multiplicative linear noise, a quadratic noise occurs in the Bloch equations. Describing the phase fluctuations within the usual phase-diffusion model and applying the method used by us in the theory of one-photon resonance fluorescence,[13] we present in Sec. II an approximative solution of the Bloch equations. This enables us to calculate the relevant quantum-mechanical correlation functions (averaged over the phase fluctuations of the exciting field), which determine the intensity, the intensity correlation function, and the spectrum of the scattered light. In order to perform the averaging over the amplitude fluctuations we describe them by an Ornstein-Uhlenbeck process and make use of the methods of generating functions[15,17] recently applied in the theory of photoelectron counting.[18] In Sec. III we present some results for the intensity and the intensity correlation function of the resonance fluorescence light. In particular, we show that when the amplitude fluctuations are sufficiently strong the quadratic noise leads, in comparison to the linear one, to drastically different features in the temporal evolution of the intensity and the intensity correlation function of the scattered light.

## II. THEORY

Let us consider a two-level atom, which is excited by a fluctuating laser field of the following form:

$$E_L(t)=E_L^{(+)}(t)+E_L^{(-)}(t)\ , \tag{2.1}$$

$$E_L^{(+)}(t)=\tfrac{1}{2}[\hat{E}_L+\delta\hat{E}_L(t)]\times\exp\{-i[\omega_L t+\varphi_L(t)]\}\ , \tag{2.2}$$

$$E_L^{(-)}(t)=[E_L^{(+)}(t)]^*\ , \tag{2.3}$$

where $\delta E_L(t)$ and $\varphi_L(t)$, respectively, are real Gaussian random variables for the amplitude and the phase fluctuations. The dynamics of the two-level atom with ground state $|1\rangle$ and excited state $|2\rangle$ that are separated by an energy $\hbar\omega_{21}$ and are coupled by a two-photon transition is described by the Bloch equations. It should be noted that a two-level atom does not allow for two-photon transitions and the two-level model used in this paper has to be considered as an effective one. The general multilevel density-matrix equation needed for studying multiphoton processes, however, can be reduced to a multiphoton Bloch equation via the adiabatic elimination of the virtual levels. Details of the derivation can be found in the report of Allen and Stroud.[19] The two-photon Bloch equations derived in this way are completely analogous to the ordinary Bloch equations for one-photon transitions in a two-level system:

$$\frac{d}{dt}|\Phi(t))=M(t)|\Phi(t))\ , \tag{2.4}$$

where the components of the four-dimensional vector $|\Phi(t))$ are defined by

$$(1|\Phi(t))=\langle A_{22}(t)\rangle\ , \tag{2.5}$$

$$(2|\Phi(t))=\langle A_{11}(t)\rangle\ , \tag{2.6}$$

$$(3|\Phi(t))=\langle \tilde{A}_{12}(t)\rangle\ , \tag{2.7}$$

$$(4|\Phi(t))=\langle \tilde{A}_{21}(t)\rangle\ , \tag{2.8}$$

and the $4\times4$ matrix $M(t)$ is given by

$$M(t)=\begin{bmatrix} -\Gamma_1 & 0 & -\frac{1}{2}i\omega_R(t) & \frac{1}{2}i\omega_R(t) \\ \Gamma_1 & 0 & \frac{1}{2}i\omega_R(t) & -\frac{1}{2}i\omega_R(t) \\ -\frac{1}{2}i\omega_R(t) & \frac{1}{2}i\omega_R(t) & -i[\delta\omega-2\dot{\varphi}_L(t)]-\Gamma_2 & 0 \\ \frac{1}{2}i\omega_R(t) & -\frac{1}{2}i\omega_R(t) & 0 & i[\delta\omega-2\dot{\varphi}_L(t)]-\Gamma_2 \end{bmatrix}\ . \tag{2.9}$$

In Eqs. (2.5)–(2.8) $A_{nm}(t)$ ($n,m=1,2$) denote atomic flip operators $[A_{nm}(t)|_{t=0}=|n\rangle\langle m|]$, $\tilde{A}_{12}(t)$ and $\tilde{A}_{21}(t)$ being the corresponding slowly varying operators

$$\tilde{A}_{12}(t)=A_{12}(t)\exp\{i2[\omega_L t+\varphi_L(t)]\}\ , \tag{2.10}$$

$$\tilde{A}_{21}(t)=\tilde{A}_{12}^{\dagger}(t)\ . \tag{2.11}$$

In Eq. (2.9) $\Gamma_1$ and $\Gamma_2$, respectively, are the rates of energy and phase relaxation, $\delta\omega=\omega_{21}-2\omega_L$. The stochastic two-photon Rabi frequency $\omega_R(t)$ is defined as follows:

$$\omega_R(t)=\frac{1}{4\hbar}\alpha_{12}[\hat{E}_L+\delta\hat{E}_L(t)]^2\ , \tag{2.12}$$

where $\alpha_{12}$ denotes the polarizibility. The nonfluctuating part of the two-photon Rabi frequency is given by

$$\Omega_R=\frac{1}{4\hbar}\alpha_{12}\hat{E}_L^2 . \qquad (2.13)$$

Using the abbreviation

$$\delta\Omega_R=\frac{1}{4\hbar}\alpha_{12}[\delta\hat{E}_L(t)]^2 \qquad (2.14)$$

we can then write Eq. (2.12) as

$$\omega_R(t)=\Omega_R+2(\Omega_R)^{1/2}[\delta\Omega_R(t)]^{1/2}+\delta\Omega_R(t) . \qquad (2.15)$$

Note that $[\delta\Omega_R(t)]^{1/2}$ is a Gaussian stochastic variable describing the amplitude fluctuations of the exciting laser field.

When the vector $|\Phi(t))$ is known at time $t_1$ we find it at time $t_2$ $(t_2\geq t_1)$ by formal integration of Eq. (2.4). The result is

$$|\Phi(t_2))=S(t_1,t_2)\,|\Phi(t_1)) , \qquad (2.16)$$

where $S(t_1,t_2)$ is the time-ordered exponential matrix

$$S(t_1,t_2)=T\exp\left[\int_{t_1}^{t_2}d\tau M(\tau)\right] . \qquad (2.17)$$

We now assume that at time $t=0$ the atom is in the ground state and hence following from the definitions in Eqs. (2.5)—(2.8) the initial condition is

$$|\Phi(0))=|2) . \qquad (2.18)$$

The (stochastic) excited-state population at time $t$, which is proportional to the intensity of resonance fluorescence, is given by $\sigma_{22}(t)=\langle A_{22}(t)\rangle=(1\,|\,\Phi(t))$. Making use of Eqs. (2.16) and (2.18) we obtain

$$\sigma_{22}(t)=(1\,|\,S(0,t)\,|\,2) . \qquad (2.19)$$

For a complete description of the resonance fluorescence light, knowledge of correlation functions of the field to all orders is desired. The spectrum is obtained by Fourier transformation of the correlation function $\langle A_{21}(t)A_{12}(t+\tau)\rangle$, which is proportional to the correlation function of second order in the field strength of the scattered light. In analogy to the case of one-photon excitation the correlation function $\langle A_{21}(t)A_{12}(t+\tau)\rangle$ can be expressed in terms of the time-ordered exponential matrix $S(t_1,t_2)$ as follows:[13]

$$\begin{aligned}\langle A_{21}(t)A_{12}(t+\tau)\rangle &\\ =\exp[&-i2\omega_L\tau-i2\varphi_L(t+\tau)+i2\varphi_L(t)]\\ &\times[(3\,|\,S(t,t+\tau)\,|\,2)(4\,|\,S(0,t)\,|\,2)\\ &+(3\,|\,S(t,t+\tau)\,|\,3)(1\,|\,S(0,t)\,|\,2)] . \end{aligned}\qquad (2.20)$$

Note that in the limit $\tau\to 0$ Eq. (2.20) reduces to Eq. (2.19) for the excited-state population: $\lim_{\tau\to 0}\langle A_{21}(t)A_{12}(t+\tau)\rangle=\langle A_{22}(t)\rangle$. Another correlation function, which has been a subject of increasing interest, is the intensity correlation function. This correlation function of fourth order in the field strength is proportional to the atomic correlation function $G_{22}(t,t+\tau)=\langle A_{21}(t)A_{22}(t+\tau)A_{12}(t)\rangle$, which can be expressed in terms of the time-ordered exponential matrix $S(t_1,t_2)$ as well:

$$G_{22}(t,t+\tau)=(1\,|\,S(t,t+\tau)\,|\,2)(1\,|\,S(0,t)\,|\,2) . \qquad (2.21)$$

In the case of a stochastic driving field we have to average Eqs. (2.19)—(2.21) over the field fluctuations; for example,

$$\langle\sigma_{22}(t)\rangle_{st}=\langle(1\,|\,S(0,t)\,|\,2)\rangle_{st} , \qquad (2.22)$$

$$\langle G_{22}(t,t+\tau)\rangle_{st}=\langle(1\,|\,S(t,t+\tau)\,|\,2)(1\,|\,S(0,t)\,|\,2)\rangle_{st} . \qquad (2.23)$$

In what follows we assume the phase of the exciting laser field is a Wiener-Levy process. The averaging over the phase fluctuations in Eqs. (2.22) and (2.23) can simply be performed by substituting $-4\Gamma_L$ for the $2i\dot{\varphi}_L$ and $-2i\dot{\varphi}_L$ in Eq. (2.9) for the $M$ matrix, $2\Gamma_L$ being the full width at half maximum of the laser line.[16] Note that this substitution leads to a modification of the atomic dephasing rate $\Gamma_2$, namely, $\Gamma_2\to\Gamma_2+4\Gamma_L$.

The averaging over the amplitude fluctuations is very difficult since they cannot, in general, be assumed to be δ-correlated and the $M$ matrices at different times do not commute. In order to find an approximative solution we follow the procedure that has recently been applied to the problem of the usual resonance fluorescence in a fluctuating laser field.[13] Briefly, we subdivide the matrix $M(t)$ as follows (after substituting $-4\Gamma_L$ for the $2i\dot{\varphi}_L$ and $-2i\dot{\varphi}_L$):

$$\begin{aligned}M(t)=M_0&\left[1+2\left[\frac{\delta\Omega_R(t)}{\Omega_R}\right]^{1/2}+\frac{\delta\Omega_R(t)}{\Omega_R}\right]\\ &-M_1\left[2\left[\frac{\delta\Omega_R(t)}{\Omega_R}\right]^{1/2}+\frac{\delta\Omega_R(t)}{\Omega_R}\right] ,\end{aligned}\qquad (2.24)$$

where the matrices $M_0$ and $M_1$ are defined by the relations

$$M_0=M(t)\,|_{\omega_R(t)=\Omega_R} , \qquad (2.25)$$

$$M_1=M(t)\,|_{\omega_R(t)=0} . \qquad (2.26)$$

At this point we note that the problem of performing the averaging over the amplitude fluctuations of the exciting laser field in Eqs. (2.22) and (2.23) can be simplified drastically when the $M_1$ term in Eq. (2.24) can be omitted. Confining ourselves to the case when the resonance condition is exactly fulfilled from Eqs. (2.26) and (2.9) the matrix $M_1$ is easily seen to vanish when the relaxation rates $\Gamma_1$ and $\Gamma_2+4\Gamma_L$ are negligibly small, strictly speaking, when the exciting field is strong and in the time scale under consideration the effect of relaxation is weak. In the case when the relaxation must be taken into account, the neglect of the $M_1$ term in Eq. (2.24) requires the following condition to be fulfilled:

$$\frac{\langle\delta\Omega_R(t)\rangle_{st}}{\Omega_R}\max(1,4\Gamma\tau_A)\min(1,\Gamma t)\ll 1 , \qquad (2.27)$$

$$\Gamma=\max(\Gamma_1,\Gamma_2+4\Gamma_L) , \qquad (2.28)$$

where $\tau_A$ denotes the correlation time of the amplitude fluctuations. This condition can be derived following the scheme for the case of one-photon resonant excitation,[13] however, the nonlinearity of the noise must be taken into account.

Disregarding the $M_1$ term in Eq. (2.24) we approximate $S(t_1,t_2)$ defined by Eq. (2.17) as follows:

$$S(t_1,t_2)\approx\exp\left\{M_0(t_2-t_1)+M_0\int_{t_1}^{t_2}d\tau\left[2\left[\frac{d\Omega_R(\tau)}{\Omega_R}\right]^{1/2}+\frac{\delta\Omega_R(\tau)}{\Omega_R}\right]\right\}. \quad (2.29)$$

Inserting Eq. (2.29) into Eqs. (2.22) and (2.23) and making use of the $M_0$ representation,

$$M_0\,|\,\lambda_k)=\lambda_k\,|\,\lambda_k)\ , \quad (2.30)$$

we rewrite Eqs. (2.22) and (2.23) as

$$\langle\sigma_{22}(t)\rangle_{\rm st}=\sum_l C_l^{1,2}e^{\lambda_l t}\langle\varphi_l(t)\rangle_{\rm st}\ , \quad (2.31)$$

$$\langle G_{22}(t,t+\tau)\rangle_{\rm st}=\sum_{l,k}C_l^{1,2}C_k^{1,2}e^{\lambda_l t+\lambda_k\tau}\times\langle\varphi_{lk}(t,t+\tau)\rangle_{\rm st}\ , \quad (2.32)$$

where

$$\langle\varphi_{lk}(t,t+\tau)\rangle_{\rm st}=\left\langle\exp\left[\frac{\lambda_l}{\Omega_R}\int_0^t d\tau'\{2\Omega_R^{1/2}[\delta\Omega_R(\tau')]^{1/2}+\delta\Omega_R(\tau')\}+\frac{\lambda_k}{\Omega_R}\int_t^{t+\tau}d\tau'\{2\Omega_R^{1/2}[\delta\Omega_R(\tau')]^{1/2}+\delta\Omega_R(\tau')\}\right]\right\rangle_{\rm st}, \quad (2.33)$$

$$\langle\varphi_l(t)\rangle_{\rm st}=\langle\varphi_{lk}(t,t)\rangle_{\rm st}\ . \quad (2.34)$$

In Eqs. (2.31) and (2.32) the coefficients $C_l^{i,j}$ defined by the relation

$$C_l^{i,j}=(i\,|\,\lambda_l)(\lambda_l\,|\,j) \quad (2.35)$$

are derived to be

$$C_1^{1,2}=\frac{1}{2}\left[1-\frac{\Gamma_1(\Gamma_2+4\Gamma_L)}{\lambda_3\lambda_4}\right],$$
$$C_2^{1,2}=0\ ,$$
$$C_3^{1,2}=-\frac{1}{2}\frac{(\Gamma_1+\lambda_4)(\Gamma_1+\lambda_3)}{\lambda_3(\lambda_4-\lambda_3)}\ , \quad (2.36)$$
$$C_4^{1,2}=-\frac{1}{2}\frac{(\Gamma_1+\lambda_3)(\Gamma_1+\lambda_4)}{\lambda_4(\lambda_3-\lambda_4)}\ ,$$

where

$$\lambda_1=0\ ,$$
$$\lambda_2=-(\Gamma_2+4\Gamma_L)\ , \quad (2.37)$$
$$\lambda_3=-\tfrac{1}{2}(\Gamma_1+\Gamma_2+4\Gamma_L)+[\tfrac{1}{4}(\Gamma_1-\Gamma_2-4\Gamma_L)^2-\Omega_R^2]^{1/2}\ ,$$
$$\lambda_4=-\tfrac{1}{2}(\Gamma_1+\Gamma_2+4\Gamma_L)-[\tfrac{1}{4}(\Gamma_1-\Gamma_2-4\Gamma_L)^2-\Omega_R^2]^{1/2}\ .$$

We now assume that the amplitude of the exciting laser field fluctuates according to an Ornstein-Uhlenbeck process. In order to perform the averaging in Eq. (2.33) we apply the method outlined in the Appendix. Making the identifications

$$x(t)\to[\delta\Omega_R(t)]^{1/2},\quad \gamma\to\Gamma_A\equiv\tau_A^{-1}\ ,$$
$$a\to\lambda_l/\Omega_R,\quad b\to\lambda_k/\Omega_R,\quad c\to2(\Omega_R)^{1/2}\ ,$$
$$t'\to t+\tau,\quad \tilde{\varphi}(t,t')\to\varphi_{lk}(t,t+\tau)$$

and taking into account that Eqs. (A8) and (A9) yield $q=\Gamma_A\langle\delta\Omega_R\rangle_{\rm st}$ we obtain from Eq. (A50) together with Eqs. (A36), (A37), (A42), and (A44)–(A49) the result

$$\langle\varphi_{lk}(t,t+\tau)\rangle_{\rm st}=\exp[g_l(t)+g_{lk}(t,t+\tau)]\ , \quad (2.38)$$

where

$$g_l(t)=\frac{1}{2}\left[\Gamma_A-2\lambda_l\left[1-\frac{\Gamma_A^2}{w_l^2}\right]\right]t-\tfrac{1}{2}\ln\left[\cosh(w_lt)+\frac{u_l}{\Gamma_A}\sinh(w_lt)\right]+\frac{\lambda_l\Gamma_A^2}{w_l^3}\left[\frac{w_l^2}{u_l\Gamma_A}-\frac{\left[\dfrac{w_l^2}{u_l}-2(w_l-u_l)\right]\cosh(w_lt)+\Gamma_A\sinh(w_lt)+2(w_l-u_l)}{\Gamma_A\cosh(w_lt)+u_l\sinh(w_lt)}\right], \quad (2.39)$$

$$g_{lk}(t,t+\tau)=\frac{1}{2}\left[\Gamma_A-2\lambda_l\left[1-\frac{\Gamma_A^2}{w_k^2}\right]\right]\tau-\tfrac{1}{2}\ln\left[\cosh(w_k\tau)+\frac{u_{lk}(t)}{\Gamma_A}\sinh(w_k\tau)\right]+\frac{\lambda_k\Gamma_A^2}{w_k}\left[\frac{w_{lk}^2(t)}{u_{lk}(t)\Gamma_A}-\frac{\left[\dfrac{w_{lk}^2(t)}{u_{lk}(t)}-2[w_{lk}(t)-u_{lk}(t)]\right]\cosh(w_k\tau)+\Gamma_A\sinh(w_k\tau)+2[w_{lk}(t)-u_{lk}(t)]}{\Gamma_A\cosh(w_k\tau)+u_{lk}(t)\sinh(w_k\tau)}\right], \quad (2.40)$$

$$w_l=\Gamma_A\left[1-4\frac{\lambda_l}{\Omega_R}\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Gamma_A}\right]^{1/2}, \tag{2.41}$$

$$u_l=\frac{\Gamma_A^2}{w_l}\left[1-2\frac{\lambda_l}{\Omega_R}\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Gamma_A}\right], \tag{2.42}$$

$$w_{lk}(t)=w_k\frac{\Gamma_A}{w_l}\frac{w_l-u_l+u_l\cosh(w_lt)+\Gamma_A\sinh(w_lt)}{\Gamma_A\cosh(w_lt)+u_l\sinh(w_lt)}, \tag{2.43}$$

$$u_{lk}(t)=\frac{\Gamma_A^2}{w_k}\left[1-\frac{\lambda_k}{\lambda_l}\frac{2(\lambda_l/\Omega_R)\langle\delta\Omega_R\rangle_{\rm st}\cosh(w_lt)+(u_l-w_l)\sinh(w_lt)}{\Gamma_A\cosh(w_lt)+u_l\sinh(w_lt)}\right]. \tag{2.44}$$

Since $g_{lk}(t,t)=0$ from Eqs. (2.34) and (2.38) the average $\langle\varphi_l(t)\rangle_{\rm st}$ is easily found to be

$$\langle\varphi_l(t)\rangle_{\rm st}=\exp[g_l(t)]. \tag{2.45}$$

Equations (2.31) and (2.32) then read as follows:

$$\langle\sigma_{22}(t)\rangle_{\rm st}=\sum_l C_l^{1,2}e^{\lambda_lt}\exp[g_l(t)], \tag{2.46}$$

$$\langle G_{22}(t,t+\tau)\rangle_{\rm st}=\sum_{l,k}C_l^{1,2}C_k^{1,2}e^{\lambda_lt+\lambda_k\tau}\times\exp[g_l(t)+g_{lk}(t,t+\tau)]. \tag{2.47}$$

Finally, we note that the averaging of Eq. (2.20) can be performed in an analogous way. Provided that the linewidth of the exciting laser field can be assumed to be small, that is, the strength of the phase fluctuations is small, the result of averaging over the phase fluctuations consists in substituting into Eq. (2.20) for $-i2\varphi_L(t+\tau)+i2\varphi_L(t)$ the quantity $-4\Gamma_L\tau$ and in substituting into the matrix $M(t)$ defined in Eq. (2.9) for both $2i\dot{\varphi}_L$ and $-2i\dot{\varphi}_L$ the quantity $-4\Gamma_L$. Performing the averaging over the amplitude fluctuations in the approximation given by Eq. (2.29) we obtain

$$\langle\langle A_{21}(t)A_{12}(t+\tau)\rangle\rangle_{\rm st}=e^{-i\omega_L\tau-4\Gamma_L\tau}\sum_{l,k}(C_l^{4,2}C_k^{3,2}+C_l^{1,2}C_k^{3,3})e^{\lambda_lt+\lambda_k\tau}\times\exp[g_l(t)+g_{lk}(t,t+\tau)]. \tag{2.48}$$

The explicit form of the coefficients $C_l^{i,j}$ is given (replacing $\Gamma_L\to4\Gamma_L$), for example, in Ref. 13.

## III. DISCUSSION

In this section we discuss the effects of the driving-field fluctuations on the intensity and the intensity correlation of the scattered light in more detail. From an inspection of Eqs. (2.46) and (2.47) the amplitude fluctuations of the exciting laser field are seen to prevent, in general, the factorization of the intensity correlation function $\langle G_{22}(t,t+\tau)\rangle_{\rm st}$ into the product of intensities $\langle\sigma_{22}(t)\rangle_{\rm st}\langle\sigma_{22}(\tau)\rangle_{\rm st}$. Only in the steady-state case is such a decomposition found. This is due to the fact that as the time $t$ goes to infinity the sum over $l$ in Eq. (2.47) reduces to the term with $l=1$ and the functions $g_{1k}(t,t+\tau)$ tend to $g_k(\tau)$,

$$\lim_{t\to\infty}g_{1k}(t,t+\tau)=g_k(\tau). \tag{3.1}$$

Combining Eqs. (3.1), (2.47), and (2.46) we indeed find

$$\langle G_{22}(\tau)\rangle_{\rm st}\equiv\lim_{t\to\infty}\langle G_{22}(t,t+\tau)\rangle_{\rm st}=\langle\sigma_{22}(\infty)\rangle_{\rm st}\langle\sigma_{22}(\tau)\rangle_{\rm st}, \tag{3.2}$$

where

$$\langle\sigma_{22}(\infty)\rangle_{\rm st}\equiv\lim_{t\to\infty}\langle\sigma_{22}(t)\rangle_{\rm st}=C_1^{1,2} \tag{3.3}$$

[note that $\lambda_1=0$ implies $g_1(t)=0$]. This result is well known from the case of usual one-photon resonant excitation.[13]

When the amplitude fluctuations of the exciting laser field are sufficiently weak [$\langle\delta\Omega_R\rangle_{\rm st}/\Gamma_A\ll1$] the results of two-photon resonant excitation and one-photon resonant excitation are very similar. In this case Eqs. (2.41) and (2.42) can be simplified as follows:

$$w_l\approx\Gamma_A\left[1-2\frac{\lambda_l}{\Omega_R}\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Gamma_A}\right], \tag{3.4}$$

$$u_l\approx\Gamma_A, \tag{3.5}$$

and hence from Eqs. (2.43) and (2.44) we obtain

$$w_{lk}(t)\approx w_k\left[1+2\frac{\lambda_l}{\Omega_R}\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Gamma_A}(1-e^{-\Gamma_At})\right], \tag{3.6}$$

$$u_{lk}(t)\approx u_k. \tag{3.7}$$

Combining Eqs. (3.4)–(3.7), (2.39), and (2.40) the functions $g_l(t)$ and $g_{lk}(t,t+\tau)$ can approximately be expressed as

$$g_l(t)\approx\frac{\lambda_l}{\Omega_R}\langle\delta\Omega_R\rangle_{\rm st}t+\frac{\lambda_l^2}{\Omega_R^2}g^{(1)}(t), \tag{3.8}$$

$$g_{lk}(t,t+\tau)\approx g_k(\tau)+\frac{\lambda_l\lambda_k}{\Omega_R^2}g^{(2)}(t,\tau), \tag{3.9}$$

where

$$g^{(1)}(t)=4\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Gamma_A}\Omega_R\left[t+\frac{1}{\Gamma_A}(e^{-\Gamma_At}-1)\right], \tag{3.10}$$

$$g^{(2)}(t,t')=4\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Gamma_A}\frac{\Omega_R}{\Gamma_A}(e^{-\Gamma_A t}-1)(e^{-\Gamma_A t'}-1)\ . \quad (3.11)$$

Inserting Eqs. (3.8) and (3.9) into Eqs. (2.46) and (2.47) we find the following expressions for the averaged intensity and the averaged intensity correlation function of the scattered light:

$$\langle\sigma_{22}(t)\rangle_{\rm st}\approx\sum_l C_l^{1,2}\exp\left[\lambda_l\left[1+\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Omega_R}\right]t\right]\exp\left[\frac{\lambda_l^2}{\Omega_R^2}g^{(1)}(t)\right]\ , \quad (3.12)$$

$$\langle G_{22}(t,t+\tau)\rangle_{\rm st}\approx\sum_{l,k} C_l^{1,2}C_k^{1,2}\exp\left[\lambda_l\left[1+\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Omega_R}\right]t+\lambda_k\left[1+\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Omega_R}\right]\tau\right]$$
$$\times\exp\left[\frac{1}{\Omega_R^2}[\lambda_l^2 g^{(1)}(t)+\lambda_k^2 g^{(1)}(\tau)+\lambda_l\lambda_k g^{(2)}(t,\tau)]\right]\ . \quad (3.13)$$

Indeed, this result is very similar to that derived for the case when one photon is needed for resonant excitation of the two-level atom considered.[13] It corresponds to a cumulant expansion up to the second order in $\delta\hat{E}_L(t)$, which is exact in the case of one-photon resonant excitation when the amplitude fluctuations of the exciting laser field give rise to a $\delta\Omega_R(t)$ that is linear in $\delta\hat{E}_L(t)$. It is obvious that in this case the amplitude fluctuations do not modify the Rabi frequency. From Eqs. (3.12) and (3.13) it is seen that in the case of two-photon resonant excitation the amplitude fluctuations change the Rabi frequency. This effect results from the term proportional to $\delta\Omega_R(t)\sim[\delta\hat{E}_L(t)]^2$ in Eq. (2.29), the stochastic expectation value of which does not vanish.

In both the case of one-photon resonant excitation and, if the amplitude fluctuations of the exciting laser field are weak, the case of two-photon resonant excitation the effect of the amplitude fluctuations on the damping of the Rabi oscillations can be described by a decay rate $\tilde{\Gamma}$ provided that the bandwidth of the amplitude fluctuations is sufficiently large. From Eqs. (3.12) and (3.13) together with Eqs. (3.10) and (3.11) this rate, which results from the term proportional to $[\delta\Omega_R(t)]^{1/2}\sim\delta\hat{E}_L(t)$ in Eq. (2.29), is seen to be

$$\tilde{\Gamma}^{(2)}=4\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Gamma_A}\Omega_R=4\frac{\langle(\delta\hat{E}_L)^2\rangle_{\rm st}}{\hat{E}_L^2}\frac{\Omega_R^2}{\Gamma_A}\ . \quad (3.14)$$

Comparing this result with the corresponding one-photon result[13] we find, under the conditions of equal Rabi frequencies $\Omega_R$, equal correlation times $\tau_A=\Gamma_A^{-1}$, and equal relative amplitude fluctuations $\langle(\delta\hat{E}_L)^2\rangle_{\rm st}/\hat{E}_L^2$,

$$\frac{\tilde{\Gamma}^{(2)}}{\tilde{\Gamma}^{(1)}}=4\ , \quad (3.15)$$

where the indices (1) and (2), respectively, denote the cases of one-photon resonant excitation and two-photon resonant excitation. In the latter case the amplitude fluctuations lead to a decay of the Rabi oscillations, which is 4 times faster than in the former case. At this point we note that the generalization of Eq. (3.14) to the case when $n$ photons of the exciting light field are needed in order to guarantee resonance can be easily found by cumulant expansion. We derive

$$\tilde{\Gamma}^{(n)}=n^2\frac{\langle(\delta\hat{E}_L)^2\rangle_{\rm st}}{\hat{E}_L^2}\frac{\Omega_R^2}{\Gamma_A}\ , \quad (3.16)$$

and hence the generalization of Eq. (3.15) is

$$\frac{\tilde{\Gamma}^{(n)}}{\tilde{\Gamma}^{(1)}}=n^2\ . \quad (3.17)$$

We now turn to the study of the case when the amplitude fluctuations of the exciting laser field are strong: $\langle\delta\Omega_R\rangle_{\rm st}/\Gamma_A\gg1$. It is clear that the main effect then results from the quadratic noise $\delta\Omega_R(t)\sim[\delta\hat{E}_L(t)]^2$ in Eq. (2.29). Since its correct description requires the summation of high-order terms in the cumulant expansion we expect features in the averaged intensity and averaged intensity correlation function of the scattered light which are quite different from those known for the case of one-photon resonant excitation. To demonstrate them and to avoid lengthy formulas let us consider the case when the relaxation parameters $\Gamma_1$ and $\Gamma_2+4\Gamma_L$ can be disregarded in the time scale of interest. We confine ourselves to the study of the intensity and the steady-state intensity correlation function.

Omitting the relaxation rates $\Gamma_1$ and $\Gamma_2+4\Gamma_L$ in Eqs. (2.36) and (2.37) we rewrite Eq. (2.46) as follows:

$$\langle\sigma_{22}(t)\rangle_{\rm st}=\tfrac{1}{2}(1-\tfrac{1}{2}\{\exp[\lambda_3 t+g_3(t)]+{\rm c.c.}\}) \quad (3.18)$$

(note that $\lambda_3=i\Omega_R$). In order to calculate $g_3(t)$ we remember the condition $\langle\delta\Omega_R\rangle_{\rm st}/\Gamma_A\gg1$ and expand Eqs. (2.41) and (2.42). This yields

$$w_3\approx\Gamma_A\left[2\frac{\langle\delta\Omega_R\rangle_{\rm st}}{\Gamma_A}\right]^{1/2}(1-i)\ , \quad (3.19)$$

$$u_3\approx\tfrac{1}{2}w_3\ . \quad (3.20)$$

Combining Eqs. (2.39), (3.19), and (3.20) we find that the third term in Eq. (2.39) can be neglected when the amplitude fluctuations are sufficiently strong. We therefore obtain

$$g_3(t) \approx -i\Omega_R t + \tfrac{1}{2}\Gamma_A t + \ln\left[\frac{u_3}{\Gamma_A}\sinh(w_3 t) + \cosh(w_3 t)\right]^{-1/2} . \tag{3.21}$$

Inserting Eq. (3.21) into Eq. (3.18) we derive the asymptotic strong-fluctuation behavior of $\langle \sigma_{22}(t)\rangle_{st}$ to be

$$\langle \sigma_{22}(t)\rangle_{st} \approx \frac{1}{2}\left[1 - \frac{e^{-\omega t}}{[f^{(1)}(t)f^{(2)}(t)]^{1/2}}(\{[f^{(2)}(t)]^{1/2}+1\}^{1/2}\cos(\omega t) - \{[f^{(2)}(t)]^{1/2}-1\}^{1/2}\sin(\omega t))\right] , \tag{3.22}$$

where

$$\omega = (\tfrac{1}{2}\Gamma_A \langle \delta\Omega_R\rangle_{st})^{1/2} , \tag{3.23}$$

$$f^{(1)}(t) = 1 + \frac{\omega}{\Gamma_A} + e^{-4\omega t}\left[\left[1 - \frac{\omega}{\Gamma_A}\right]\cos(4\omega t) - \frac{\omega}{\Gamma_A}\sin(4\omega t)\right] , \tag{3.24}$$

$$f^{(2)}(t) = 1 + \left[\frac{\dfrac{\omega}{\Gamma_A} - e^{-4\omega t}\left[\left[1 - \dfrac{\omega}{\Gamma_A}\right]\sin(4\omega t) + \dfrac{\omega}{\Gamma_A}\cos(4\omega t)\right]}{1 + \dfrac{\omega}{\Gamma_A} + e^{-4\omega t}\left[\left[1 - \dfrac{\omega}{\Gamma_A}\right]\cos(4\omega t) - \dfrac{\omega}{\Gamma_A}\sin(4\omega t)\right]}\right]^2 . \tag{3.25}$$

Making use of Eqs. (3.2) and (3.3) the steady-state intensity correlation function is simply given by $\langle G_{22}(\tau)\rangle_{st} = \frac{1}{2}\langle \sigma_{22}(\tau)\rangle_{st}$ with $\langle \sigma_{22}(\tau)\rangle_{st}$ from Eq. (3.22).

Comparing Eq. (3.22) with Eq. (3.12) we see that in the case of two-photon resonant excitation of the two-level atom the cases of weak and strong amplitude fluctuations of the driving field lead to qualitatively different features in the dynamical behavior of the scattered light. Whereas in the former case the amplitude fluctuations give rise to a damping of the usual Rabi oscillations in the intensity and the intensity correlation of the scattered light, the time scales of oscillation and damping, respectively, being $\Omega_R^{-1}$ and $\tilde{\Gamma}^{-1}$, in the latter case the Rabi oscillations are completely suppressed. Instead of them only a single

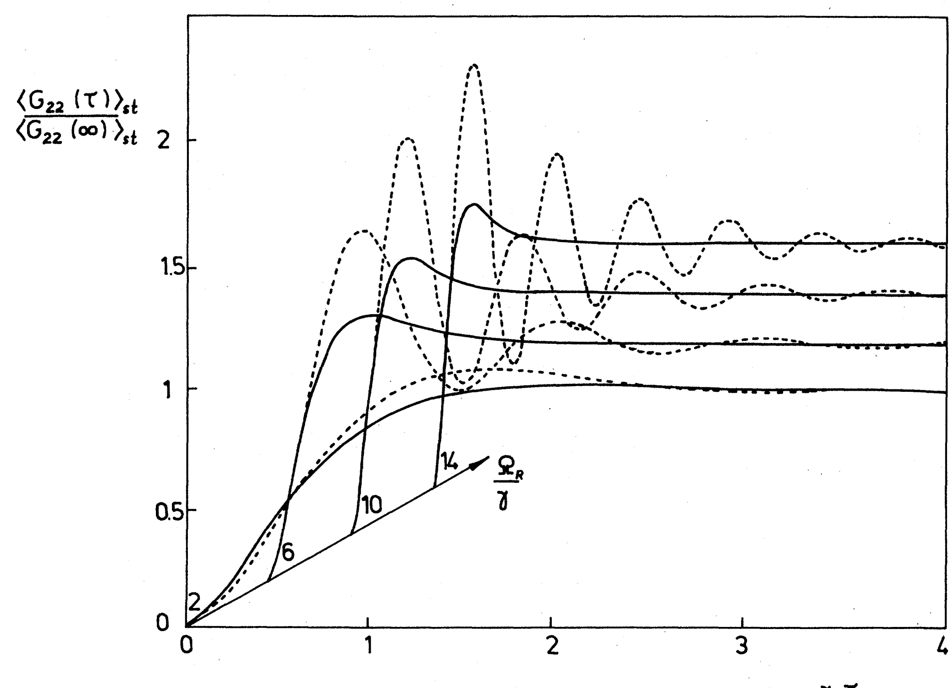


FIG. 1. Time development of the normalized stationary intensity correlation function of the fluorescent light for various values of the Rabi frequency $\Omega_R$. Radiative damping ($\Gamma_2=\gamma$, $\Gamma_1=2\gamma$) and small laser linewidth ($\Gamma_L=0.01\gamma$) are assumed. The amplitude correlation decay rate is chosen to be $\Gamma_A=2\gamma$. Behavior in the case of a realistic laser with small relative amplitude fluctuations ($\langle \delta\Omega_R\rangle_{st}/\Omega_R=0.1$, solid lines) is compared with the behavior in the case without amplitude fluctuations ($\langle \delta\Omega_R\rangle_{st}=0$, dashed curves).

overshoot peak at time $t \lesssim \pi\omega^{-1}$ can be observed because the new oscillations frequency and the new damping rate, which are given by Eq. (3.23), are equal.

Let us now consider more general cases. In Fig. 1 the steady-state intensity correlation function of the fluorescent light is shown for certain values of the Rabi frequency and for relatively slow correlation decay of the amplitude fluctuations of the driving light field. All of the parameters are chosen in such a way that the condition (2.27) is fulfilled. In particular, since the value of the relative amplitude fluctuations is assumed to be small ($\langle \delta\Omega_R \rangle_{st}/\Omega_R = 0.1$) the curves correspond to the case of weak amplitude fluctuations rather than to the case of strong amplitude fluctuations. As in the case of one-photon resonant excitation the amplitude fluctuations are seen to be responsible for damping the Rabi oscillations. Another feature common to both cases is that for a fixed (small) value of the relative amplitude fluctuations $\langle \delta\Omega_R \rangle_{st}/\Omega_R$ this damping effect vanishes with increasing value of $\Gamma_A$, that is, when the correlation decay of the amplitude fluctuations becomes sufficiently fast, as can be seen from Fig. 2. At this point it is worth noting that even in the case when the relative amplitude fluctuations are small, significant effects can occur due to their finite correlation length, since the strength $\langle \delta\Omega_R \rangle_{st}/\Gamma_A$ of the fluctuations depends on $\Gamma_A$.

As mentioned above, strong amplitude fluctuations of the driving light field lead to drastically different effects in the intensity and the intensity correlation of the scattered light in the cases of one- and two-photon resonant excitations. A detailed comparison between both situations is given in Figs. 3–5 for the case when the atomic relaxation is sufficiently slow. For weak amplitude fluctuations ($\langle \delta\Omega_R \rangle_{st}/\Gamma_A \ll 1$) the results of the two cases are qualitatively identical. The only difference is in the value of $\tilde{\Gamma}$. Since in the case of two-photon resonant excitation it is larger than in the case of one-photon resonant excitation [cf. Eq. (3.15)] the Rabi oscillations are less pronounced in the former case. With increasing strength of the amplitude fluctuations in both cases the damping of the Rabi oscillations becomes stronger, and hence the time scale of intensity anticorrelations of the scattered light decreases. In the case of one-photon resonant excitation the consequence of this tendency is that when the amplitude fluctuations of the exciting field are sufficiently strong, the Rabi oscillations in the intensity and the intensity correlation of the scattered light are completely "washed out" and both functions develop into step functions over time. In the case of two-photon resonant excitation however, the situation is quite different. The usual Rabi oscillations are not simply damped but they disappear in principle. In their place, a new overshoot peak is seen to appear in the intensity and the intensity correlation of the scattered light, the relevant time scale being $(0.5\Gamma_A \langle \delta\Omega_R \rangle_{st})^{-1/2}$.

## ACKNOWLEDGMENTS

Two of us (W.V. and D.-G.W.) would like to thank K. Wódkiewicz for discussions about the problem of averag-

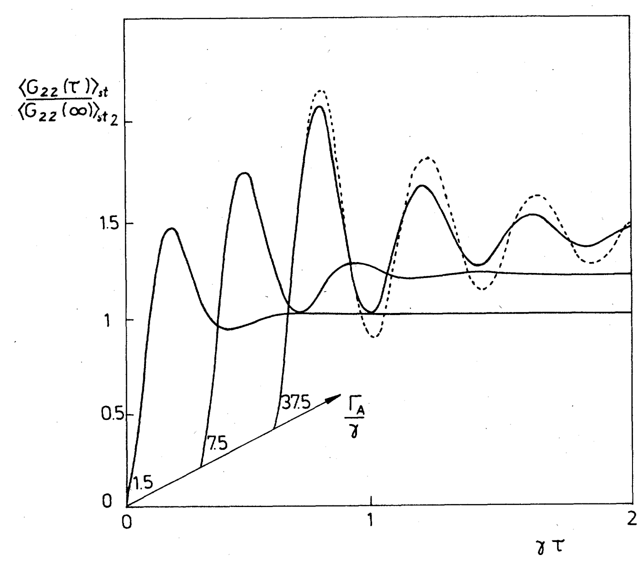


FIG. 2. Time development of the normalized stationary intensity correlation function of the fluorescent light for various values of the amplitude correlation decay rate $\Gamma_A$. Radiative damping ($\Gamma_2 = \gamma$, $\Gamma_1 = 2\gamma$) and small laser linewidth ($\Gamma_L = 0.01\gamma$) are assumed, and the Rabi frequency $\Omega_R = 15\gamma$. Behavior for small relative amplitude fluctuations ($\langle \delta\Omega_R \rangle_{st}/\Omega_R = 0.03$, solid lines) and for the case without amplitude fluctuations ($\langle \delta\Omega_R \rangle_{st} = 0$, dashed curve) are given.

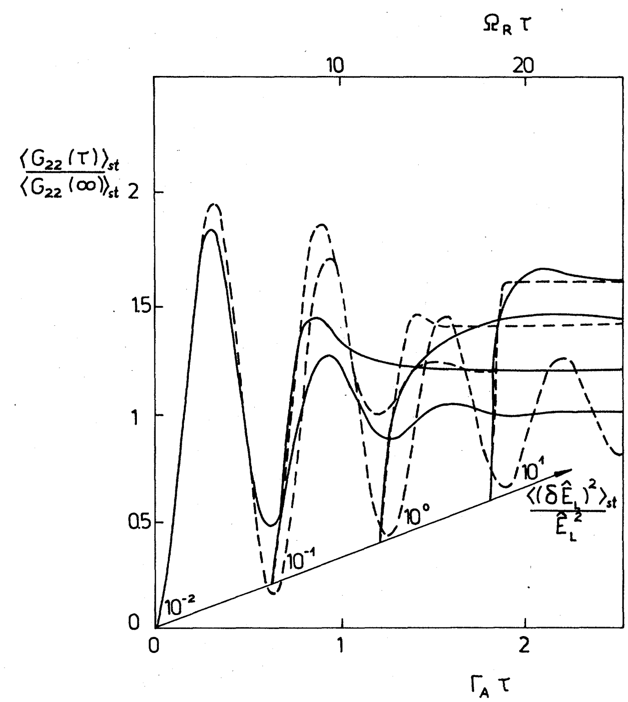


FIG. 3. Time development of the normalized stationary intensity correlation function of the fluorescent light for the Rabi frequency $\Omega_R = 10\Gamma_A$ and various values of the relative laser amplitude fluctuations $\langle (\delta\hat{E}_L)^2 \rangle_{st}/\hat{E}_L^2$. Slow atomic relaxation ($\Gamma_1, \Gamma_2, \Gamma_L \to 0$) is assumed. The case of two-photon resonant excitation (solid lines) is compared with the case of one-photon resonant excitation (dashed lines).

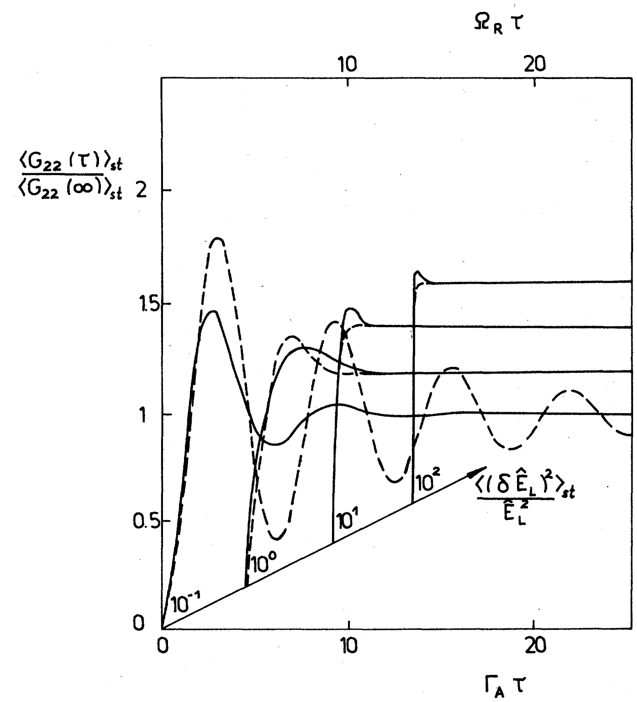


FIG. 4. Time development of the normalized stationary intensity correlation function of the fluorescent light for the Rabi frequency $\Omega_R = \Gamma_A$ and various values of the relative laser amplitude fluctuations $\langle(\delta\hat{E}_L)^2\rangle_{st}/\hat{E}_L^2$. Slow atomic relaxation $(\Gamma_1, \Gamma_2, \Gamma_L \to 0)$ is assumed. The case of two-photon resonant excitation (solid lines) is compared with the case of one-photon resonant excitation (dashed lines).

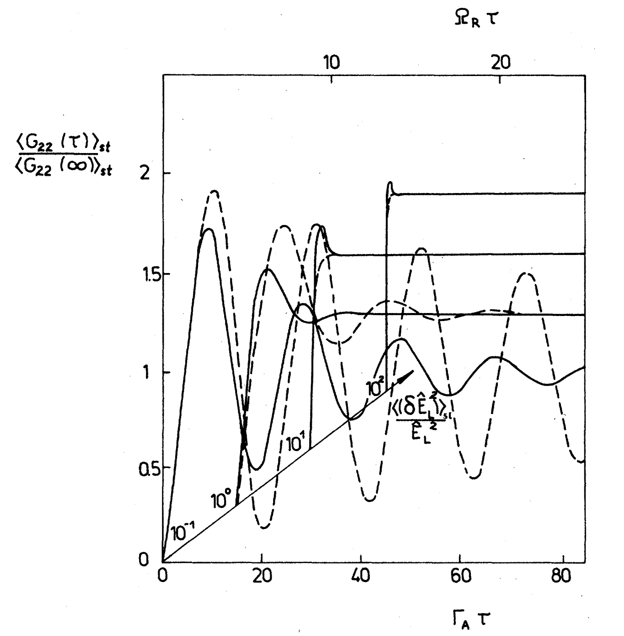


FIG. 5. Time development of the normalized stationary intensity correlation function of the fluorescent light for the Rabi frequency $\Omega_R = 0.3\Gamma_A$ and various values of the relative laser amplitude fluctuations $\langle(\delta\hat{E}_L)^2\rangle_{st}/\hat{E}_L^2$. Slow atomic relaxation $(\Gamma_1, \Gamma_2, \Gamma_L \to 0)$ is assumed. The case of two-photon resonant excitation (solid lines) is compared with the case of one-photon resonant excitation (dashed lines).

ing a Langevin equation with nonlinear multiplicative noise.

## APPENDIX

Let us consider the function

$$\tilde{\varphi}(t,t') = \exp\left[a \int_0^t d\tau[cx(\tau)+x^2(\tau)] + b\int_t^{t'} d\tau[cx(\tau)+x^2(\tau)]\right], \quad t' \geq t \tag{A1}$$

which with regard to the time argument $t'$ obeys the differential equation

$$\frac{d}{dt'}\tilde{\varphi}(t,t') = b[cx(t')+x^2(t')]\tilde{\varphi}(t,t') \tag{A2}$$

with the initial condition

$$\tilde{\varphi}(t,t')\,|_{t'=t} = \varphi(t) = \exp\left[a\int_0^t d\tau[cx(\tau)+x^2(\tau)]\right]. \tag{A3}$$

The function $\varphi(t)$ itself satisfies a differential equation of the type of Eq. (A2) as well:

$$\frac{d}{dt}\varphi(t) = a[cx(t)+x^2(t)]\varphi(t)\,, \tag{A4}$$

where the initial condition reads

$$\varphi(t)\,|_{t=0} = 1\,. \tag{A5}$$

We now assume that $x(t)$ is the stochastic variable of an Ornstein-Uhlenbeck process, which can be described by a Langevin equation with a Gaussian white-noise driving term $F(t)$:

$$\frac{d}{dt}x(t) = -\gamma x(t) + F(t)\,, \tag{A6}$$

$$\langle F(t)F(t')\rangle_{st} = 2q\delta(t-t')\,. \tag{A7}$$

The correlation function $\langle x(t)x(t')\rangle_{st}$ is given by the following expression:

$$\langle x(t)x(t')\rangle_{st} = D\gamma e^{-\gamma|t-t'|}\,, \tag{A8}$$

where the constant $D$ is related with the diffusion constant $q$ of the white noise as follows:

$$D = \frac{q}{\gamma^2}\,. \tag{A9}$$

From Eq. (A8) the power spectrum of the correlation function $\langle x(t)x(t')\rangle_{st}$ is easily seen to have a Lorentzian shape with full width at half maximum $2\gamma$. In the limit $\gamma \to \infty$ the Ornstein-Uhlenbeck process obviously tends to a Gaussian white-noise process provided that $q$ goes to infinity as well and $q/\gamma^2$ remains finite:

$$\langle x(t)x(t')\rangle_{st} \to 2D\delta(t-t')\,. \tag{A10}$$

In order to calculate the stochastic expectation value of the function $\tilde{\varphi}(t,t')$ we make use of the methods of generating functions.[15,17,18] For this purpose we introduce the function

$$\tilde{\Psi}(t,t',z)=e^{zx(t')}\tilde{\varphi}(t,t') , \quad \text{(A11)}$$

which gives $\tilde{\varphi}(t,t')$ at the point $z=0$. Hence, if the stochastic averaging of $\tilde{\Psi}(t,t',z)$ can be performed the stochastic expectation value of $\tilde{\varphi}(t,t')$ is simply found by choosing $z=0$ in $\langle\tilde{\Psi}(t,t',z)\rangle_{\rm st}$:

$$\langle\tilde{\varphi}(t,t')\rangle_{\rm st}=\langle\tilde{\Psi}(t,t',z)\rangle_{\rm st}\,|_{z=0} . \quad \text{(A12)}$$

Differentiating $\tilde{\Psi}(t,t',z)$ with respect to $z$ we obtain from Eq. (A11) the relations

$$\frac{\partial}{\partial z}\tilde{\Psi}(t,t',z)=x(t')\tilde{\Psi}(t,t',z) , \quad \text{(A13)}$$

$$\frac{\partial^2}{\partial z^2}\tilde{\Psi}(t,t',z)=x^2(t')\tilde{\Psi}(t,t',z) . \quad \text{(A14)}$$

Differentiating Eq. (A11) with respect to $t'$ and using Eqs. (A6), (A13), and (A14) we find the following stochastic differential equation for $\tilde{\Psi}(t,t',z)$:

$$\frac{\partial}{\partial t'}\tilde{\Psi}(t,t',z)=\left[(bc-z\gamma)\frac{\partial}{\partial z}+b\frac{\partial^2}{\partial z^2}+zF(t')\right]\tilde{\Psi}(t,t',z) . \quad \text{(A15)}$$

From Eqs. (A11) and (A12) the initial condition is derived to be

$$\tilde{\Psi}(t,t',z)\,|_{t'=t}\equiv\Psi(t,z)=e^{zx(t)}\varphi(t) . \quad \text{(A16)}$$

In an analogous way the function $\Psi(t,z)$ can be shown to obey the stochastic differential equation

$$\frac{\partial}{\partial t}\Psi(t,z)=\left[(ac-z\gamma)\frac{\partial}{\partial z}+a\frac{\partial^2}{\partial z^2}+zF(t)\right]\Psi(t,z) \quad \text{(A17)}$$

with the initial condition

$$\psi(t,z)\,|_{t=0}=e^{zx(0)} . \quad \text{(A18)}$$

Since $F$ is assumed to be a Gaussian white-noise force we can exactly average Eq. (A15). The result is

$$\frac{\partial}{\partial t}\langle\tilde{\Psi}(t,t',z)\rangle_{\rm st}=\left[(bc-z\gamma)\frac{\partial}{\partial z}+b\frac{\partial^2}{\partial z^2}+z^2q\right]\langle\tilde{\Psi}(t,t',z)\rangle_{\rm st} \quad \text{(A19)}$$

with the initial condition according to Eq. (A16):

$$\langle\tilde{\Psi}(t,t',z)\rangle_{\rm st}\,|_{t'=t}=\langle\Psi(t,z)\rangle_{\rm st} . \quad \text{(A20)}$$

Analogously the averaging of Eq. (A17) yields

$$\frac{\partial}{\partial t}\langle\Psi(t,z)\rangle_{\rm st}=\left[(ac-z\gamma)\frac{\partial}{\partial z}+a\frac{\partial^2}{\partial z^2}+z^2q\right]\langle\Psi(t,z)\rangle_{\rm st} . \quad \text{(A21)}$$

Taking into account that $x$ is a Gaussian stochastic variable and making use of Eqs. (A8) and (A9), we obtain from Eq. (A18) the initial condition

$$\langle\Psi(t,z)\rangle_{\rm st}\,|_{t=0}=e^{qz^2/2\gamma} . \quad \text{(A22)}$$

In analogy to the solution of the Schrödinger equation by normal ordering[20,21] we solve Eq. (A19) by means of the following ansatz:

$$\langle\tilde{\Psi}(t,t',z)\rangle_{\rm st}=\exp[\tilde{G}_0(t,t')+z\tilde{G}_1(t,t')+z^2\tilde{G}_2(t,t')] . \quad \text{(A23)}$$

Inserting Eq. (A23) into Eq. (A19) we obtain the system of coupled differential equations

$$\frac{d}{dt'}\tilde{G}_0(t,t')=bc\tilde{G}_1(t,t')+2b\tilde{G}_2(t,t')+b\tilde{G}_1^2(t,t') , \quad \text{(A24)}$$

$$\frac{d}{dt'}\tilde{G}_1(t,t')=-\gamma\tilde{G}_1(t,t')+2bc\tilde{G}_2(t,t')+4b\tilde{G}_1(t,t')\tilde{G}_2(t,t') , \quad \text{(A25)}$$

$$\frac{d}{dt'}\tilde{G}_2(t,t')=-2\gamma\tilde{G}_2(t,t')+4b\tilde{G}_2^2(t,t')+q . \quad \text{(A26)}$$

The initial conditions

$$\tilde{G}_i(t,t')\,|_{t'=t}\equiv G_i(t),\quad i=0,1,2 \quad \text{(A27)}$$

have to be calculated from the solution of Eq. (A21). Applying the above technique to Eq. (A21) we obtain

$$\langle\Psi(t,z)\rangle_{\rm st}=\exp[G_0(t)+zG_1(t)+z^2G_2(t)] , \quad \text{(A28)}$$

$$\frac{d}{dt}G_0(t)=acG_1(t)+2aG_2(t)+aG_1^2(t) , \quad \text{(A29)}$$

$$\frac{d}{dt}G_1(t)=-\gamma G_1(t)+2acG_2(t)+4aG_1(t)G_2(t) , \quad \text{(A30)}$$

$$\frac{d}{dt}G_2(t)=-2\gamma G_2(t)+4a_2G_2^2(t)+q . \quad \text{(A31)}$$

It is easily seen that Eq. (A22) implies the following initial conditions:

$$G_0(t)\,|_{t=0}=0 , \quad \text{(A32)}$$

$$G_1(t)\,|_{t=0}=0 , \quad \text{(A33)}$$

$$G_2(t)\,|_{t=0}=\frac{1}{2}\frac{q}{\gamma} . \quad \text{(A34)}$$

The differential equation (A26) is of the Riccati type and can be solved by standard methods. We obtain the result

$$\tilde{G}_2(t,t')=\frac{1}{4b}\frac{[\gamma^2-\tilde{u}(t)\tilde{w}]\cosh[\tilde{w}(t'-t)]+\gamma[\tilde{u}(t)-\tilde{w}]\sinh[\tilde{w}(t'-t)]}{\gamma\cosh[\tilde{w}(t'-t)]+\tilde{u}(t)\sinh[\tilde{w}(t'-t)]} , \quad \text{(A35)}$$

where

$$\widetilde{w}=\gamma\left[1-4b\frac{q}{\gamma^2}\right]^{1/2}, \tag{A36}$$

$$\widetilde{u}(t)=\frac{\gamma^2}{\widetilde{w}}\left[1-4b\frac{G_2(t)}{\gamma}\right]. \tag{A37}$$

For the following it is convenient to represent $\widetilde{G}_2(t,t')$ in the equivalent form

$$\widetilde{G}_2(t,t')=-\frac{1}{4b}\frac{d}{dt'}\ln[Z(t,t')], \tag{A38}$$

where

$$Z(t,t')=e^{-\gamma(t'-t)}\{\gamma\cosh[\widetilde{w}(t'-t)]+\widetilde{u}(t)\sinh[\widetilde{w}(t'-t)]\}. \tag{A39}$$

Formal integration of Eq. (A25) yields

$$\widetilde{G}_1(t,t')=G_1(t)\exp\left[-\gamma(t'-t)+4b\int_t^{t'}d\tau\,\widetilde{G}_2(t,\tau)\right]+2bc\int_t^{t'}d\tau\,\widetilde{G}_2(t,\tau)\exp\left[-\gamma(t'-\tau)+4b\int_\tau^{t'}d\tau'\widetilde{G}_2(t,\tau')\right]. \tag{A40}$$

Making use of Eqs. (A38) and (A39) we can perform the integrations in Eq. (A40) in a simple way. We cast $\widetilde{G}_1(t,t')$ in the form

$$\widetilde{G}_1(t,t')=-\frac{c}{2}\left[1-\frac{\gamma}{\widetilde{w}}\frac{\widetilde{v}(t)-\widetilde{u}(t)+\widetilde{u}(t)\cosh[\widetilde{w}(t'-t)]+\gamma\sinh[\widetilde{w}(t'-t)]}{\gamma\cosh[\widetilde{w}(t'-t)]+\widetilde{u}(t)\sinh[\widetilde{w}(t'-t)]}\right], \tag{A41}$$

where

$$\widetilde{v}(t)=\widetilde{w}\left[1+2\frac{G_1(t)}{c}\right]. \tag{A42}$$

Finally, we formally integrate Eq. (A24):

$$\widetilde{G}_0(t,t')=G_0(t)+b\int_t^{t'}d\tau[c\widetilde{G}_1(t,\tau)+\widetilde{G}_1^2(t,\tau)+2\widetilde{G}_2(t,\tau)]. \tag{A43}$$

Inserting Eqs. (A38), (A39), and (A41) into Eq. (A43) and calculating the $\tau$ integral we obtain after some mathematical manipulations

$$\begin{aligned}\widetilde{G}_0(t,t')={}&G_0(t)+\frac{1}{2}\left[\gamma-\frac{bc^2}{2}\left[1-\frac{\gamma^2}{\widetilde{w}^2}\right]\right](t'-t)-\frac{1}{2}\ln\left[\cosh[\widetilde{w}(t'-t)]+\frac{\widetilde{u}(t)}{\gamma}\sinh[\widetilde{w}(t'-t)]\right]\\&+\frac{1}{4}bc^2\frac{\gamma^2}{\widetilde{w}^3}\left[\frac{\widetilde{v}^2(t)}{\widetilde{u}(t)\gamma}-\frac{\left[\dfrac{\widetilde{v}^2(t)}{\widetilde{u}(t)}-2[\widetilde{v}(t)-\widetilde{u}(t)]\right]\cosh[\widetilde{w}(t'-t)]+\gamma\sinh[\widetilde{w}(t'-t)]+2[\widetilde{v}(t)-\widetilde{u}(t)]}{\gamma\cosh[\widetilde{w}(t'-t]+\widetilde{u}(t)\sinh[\widetilde{w}(t'-t)]}\right].\end{aligned} \tag{A44}$$

From a comparison of Eqs. (A24)–(A26) with Eqs. (A29)–(A31) we see that the functions $G_i(t)$ $(i=0,1,2)$ can be derived from the functions $G_i(t,t')$ by means of the substitutions $t'\to t$, $t\to 0$, and $b\to a$ in Eqs. (A35), (A41), and (A44). Making use of the explicit form of the initial values $G_i(0)$ as given in Eqs. (A32)–(A34) we can express the functions $G_i(t)$ as follows:

$$\begin{aligned}G_0(t)={}&\frac{1}{2}\left[\gamma-\frac{ac^2}{2}\left[1-\frac{\gamma^2}{w^2}\right]\right]t-\frac{1}{2}\ln\left[\cosh(wt)+\frac{u}{\gamma}\sinh(wt)\right]\\&+\frac{1}{4}ac^2\frac{\gamma^2}{w^3}\left[\frac{w^2}{u\gamma}-\frac{\left[\dfrac{w^2}{u}-2(w-u)\right]\cosh(wt)+\gamma\sinh(wt)+2(w-u)}{\gamma\cosh(wt)+u\sinh(wt)}\right],\end{aligned} \tag{A45}$$

$$G_1(t)=-\frac{c}{2}\left[1-\frac{\gamma}{w}\frac{w-u+u\cosh(wt)+\gamma\sinh(wt)}{\gamma\cosh(wt)+u\sinh(wt)}\right], \tag{A46}$$

$$G_2(t)=\frac{1}{4a}\frac{2aq\cosh(wt)+\gamma(u-w)\sinh(wt)}{\gamma\cosh(wt)+u\sinh(wt)}, \tag{A47}$$

where

$$w=\gamma\left[1-4a\frac{q}{\gamma^2}\right]^{1/2}, \tag{A48}$$

$$u=\frac{\gamma^2}{w}\left[1-2a\frac{q}{\gamma^2}\right]. \tag{A49}$$

Remembering Eq. (A12) we find the average of the function $\widetilde{\varphi}(t,t')$ by choosing $z=0$ in Eq. (A23):

$$\langle\widetilde{\varphi}(t,t')\rangle_{\mathrm{st}}=\exp[\widetilde{G}_0(t,t')], \tag{A50}$$

where $\widetilde{G}_0(t,t')$ is given in Eq. (A44).


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