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Shot noise and general jump processes in strong laser-atom interactions

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We discuss relations between different jump processes used by us and other workers for the description of external electromagnetic field fluctuations. We show that the shot noise, the random telegraph signal, and the Poisson process are examples of a general class of fluctuations described by the relevant Burshtein-Chapman-Kolmogorov-Smoluchowski equation. We establish general conditions for which all these processes lead to the same atomic response in strong field-atom interactions.

I. INTRODUCTION

In several recent papers[1–3] strong field-atom interactions with external noises described by random jump processes have been discussed. Phase, frequency, or amplitude fluctuations of the driving electromagnetic field were described by us[1,2] using the forward Chapman-Kolmogorov-Smoluchowski (CKS) equation for the joint probability distribution of the noise

$$\frac{\partial}{\partial t}p(\alpha t\mid\alpha_0 t_0)=-\frac{1}{T}p(\alpha t\mid\alpha_0 t_0)+\frac{1}{T}\int d\beta f(\alpha\mid\beta)p(\beta t\mid\alpha_0 t_0)\ . \qquad (1.1)$$

Accordingly, any dynamical variable $\hat{V}(t)$ which satisfied an evolution equation $\dot{\hat{V}}=-iM\hat{V}$ with a matrix M locally dependent on external noises could be averaged exactly, i.e., solutions of the following Burshtein-Chapman-Kolmogorov-Smoluchowski (BCKS) equation:[1,4–8]

$$\frac{dV_\alpha}{dt}=\left[-iM(\alpha)-\frac{1}{T}\right]V_\alpha+\frac{1}{T}\int d\beta f(\alpha\mid\beta)V_\beta \qquad (1.2)$$

for the marginal average $V_\alpha(t)$ of $\hat{V}$ could be obtained. The properties of the jump processes are completely defined by $1/T$ (frequency of jump), $f(\alpha\mid\beta)$ (conditional probability of jump from α to β), and the number of involved states α.

In our work[1–3] we have discussed in detail analytically and numerically various applications of Eq. (1.2) for a two-state ($\alpha=\pm a$) Markov chain with the transition function

$$f(\alpha\mid\beta)=\delta(\alpha+\beta)\ . \qquad (1.3)$$

This two-state jump process (random telegraph signal) was used by us to construct more complicated stochastic processes with pre-Gaussian[3] statistics and with non-Lorentzian band shapes.[2]

Apart from Eq. (1.3) there are at least two obvious possible choices of the transition function $f(\alpha\mid\beta)$ with interesting physical consequences. The first possibility assumes that the transitions $f(\alpha\mid\beta)$ are functions of the state difference only

$$f(\alpha\mid\beta)=g(\alpha-\beta)\ . \qquad (1.4)$$

Some examples of phase fluctuations with a transition function $f(\alpha\mid\beta)$ given by Eq. (1.4) were discussed in Refs. 4–6.

The second case defines $f(\alpha\mid\beta)$ as independent from its initial state prior to the jump, i.e.,

$$f(\alpha\mid\beta)=g(\beta)\ . \qquad (1.5)$$

Examples of phase fluctuations and atomic response based on selection (1.5) were discussed in Refs. 7 and 8. In both cases, $g(\alpha)$ can be an arbitrary function of its argument.

It is the purpose of this paper to establish relations between all these apparently very different models of phase fluctuations described by the CKS equation with functions (1.3), (1.4), and (1.5), respectively. In fact we show under what conditions these differing jump processes lead to the same physical results already discussed by us in the framework of random telegraph signals. This establishes a general framework for all these jump processes and shows the importance of the random telegraph signals used by us in our previous discussions of atomic response to external fluctuations.

In Sec. II of this paper we show that the CKS equation with $f(\alpha\mid\beta)=g(\alpha-\beta)$ is equivalent to a shot-noise description of the fluctuating phase or frequency. The shot-noise model known in electronic devices turns out to be very closely related to the telegraph noise description of laser phase or frequency fluctuations. We establish the proper conditions under which the shot-noise model leads to physical results already obtained by us for random telegraph signals.

In Sec. III we discuss phase fluctuations described by the CKS equations with the transition function given by Eq. (1.5). Again a very close connection to random telegraph jumps of the phase is established. Finally, some concluding remarks are given in the Summary.

II. SHOT-NOISE PHASE FLUCTUATIONS

The shot-noise model of phase fluctuations assumes that the instantaneous phase $\Phi(t)$ of the electromagnetic field consists of a sum of statistically independent pulses

$$\Phi(t)=\sum_{i=1}^{n}\alpha_i h(t-t_i)\ , \tag{2.1}$$

where $h(t-t_i)$ is a causal pulse-shape function $[h(t)=0$ for $t<0]$, generated at a random time t_i with amplitude α_i. The probability distribution function of the independent amplitudes is given by the function $g(\alpha_i)=g(\alpha)$. The number n, of accumulated pulses in a time interval Δt, is given by a Poisson distribution: $P_n=e^{-\bar{n}}(\bar{n})^n/n!$ with $\bar{n}=(1/T)\Delta t$, where $1/T$ is a fixed parameter.

The generating function of shot noise is very well known since the early investigations of noise in electric currents.[9] For an arbitrary smooth function $\mathscr{f}(\tau)$, we have the following result:

$$\left\langle\exp\left[i\int_0^t ds\,\mathscr{f}(s)\Phi(s)\right]\right\rangle=\exp\left\{\frac{1}{T}\int_0^t ds\left[\int d\alpha\, g(\alpha)\exp\left[i\alpha\int_s^t d\tau\,\mathscr{f}(\tau)h(\tau-s)\right]-1\right]\right\}. \tag{2.2}$$

From this relation we can calculate all the correlation functions of $\Phi(t)$ by a repeated differentiation of Eq. (2.2), with respect to the arbitrary function $\mathscr{f}(t)$. As an example we obtain Cambell's formulas[10]

$$\langle\Phi(t)\rangle=\frac{1}{T}\int d\alpha\,\alpha g(\alpha)\int_0^\infty d\tau\, h(t-\tau)\ , \tag{2.3a}$$

$$\langle\Phi(t_1)\Phi(t_2)\rangle=\frac{1}{T}\int d\alpha\,\alpha^2 g(\alpha)\times\int_0^\infty d\tau\, h(t_1-\tau)h(t_2-\tau)\ . \tag{2.3b}$$

Even if it is possible to discuss the shot-noise phase fluctuations for an arbitrary pulse-shape function we shall assume for simplicity that $h(t)$ is a unit step function

$$h(t)=\Theta(t)\ . \tag{2.4}$$

From Cambell's formulas we then obtain

$$\langle\Phi(t_1)\Phi(t_2)\rangle=\frac{1}{T}\int d\alpha\,\alpha^2 g(\alpha)\min(t_1,t_2)\ . \tag{2.5}$$

This formula is equivalent in form to the Wiener-Levy correlation function of the phase-diffusion model.[11] The important difference between the shot noise and the Wiener-Levy stochastic process is that $\Phi(t)$ with the characteristic functional (2.2) is not a Gaussian stochastic process.

With all these descriptions of the shot-noise fluctuations we can derive the proper CKS equation for the joint probability distribution function of the phase defined as follows:

$$p(\alpha t\,|\,\alpha_0 t_0)=\int\frac{d\kappa}{2\pi}e^{i(\alpha-\alpha_0)\kappa}\langle e^{i\kappa[\Phi(t)-\Phi(t_0)]}\rangle\ . \tag{2.6}$$

With the help of Eq. (2.2) we obtain

$$p(\alpha t\,|\,\alpha_0 t_0)=\int\frac{d\kappa}{2\pi}e^{i(\alpha-\alpha_0)\kappa}\times\exp\left[\left[\int d\alpha\, g(\alpha)e^{i\alpha\kappa}-1\right]\frac{t-t_0}{T}\right]. \tag{2.7}$$

By simple time differentiation of Eq. (2.7) we obtain the following equation:

$$\frac{\partial}{\partial t}p(\alpha t\,|\,\alpha_0 t_0)=-\frac{1}{T}p(\alpha t\,|\,\alpha_0 t_0)+\frac{1}{T}\int d\beta\, g(\alpha-\beta)p(\beta t\,|\,\alpha_0 t_0)\ . \tag{2.8}$$

This important result indicates that the probability distribution function of shot-noise phase fluctuations satisfies the CKS equation with a transition function given by condition (1.4). From the CKS equation (2.8) we calculate very easily the phase-dependent part of the electric field correlation function. Using Eq. (2.7) we obtain

$$\langle e^{-i\Phi(t+\tau)+\Phi(t)}\rangle=\exp\left[-\frac{|\tau|}{T}\left[1-\int d\alpha\, g(\alpha)e^{i\alpha}\right]\right], \tag{2.9}$$

i.e., the power spectrum has a Lorentzian band shape with an effective line width γ given by

$$\gamma=(1-\tilde{g})/T\ , \tag{2.10}$$

where we have denoted

$$\tilde{g}=\int d\alpha\, g(\alpha)e^{i\alpha}\ . \tag{2.11}$$

The shot-noise model of phase fluctuations leads to the BCKS equation (1.2) with a transition function $f(\alpha\,|\,\beta)=g(\alpha-\beta)$. In all of the examples below, we take g to be an even function of its argument, for simplicity.

In order to calculate the two-level atom response to such phase fluctuations we proceed in a similar way as in the random telegraph signal case,[1] i.e., we solve the BCKS equation (1.2) with condition (1.4) for the optical Bloch equations. For example, the stochastic average of the two-level inversion $w(t)=\langle\hat{w}(t)\rangle$ satisifies the following set of equations:

$$\dot{w}=-\frac{1}{T_1}(1+w)+i\Omega_0\sigma_{-1}^{+}-i\Omega_0^*\sigma_{+1}\ , \tag{2.12a}$$

$$\dot{\sigma}_{-1}^{+}=-\left[\frac{1}{T_2}-i\Delta-\frac{1}{T}(\tilde{g}-1)\right]\sigma_{-1}^{+}+\frac{i}{2}\Omega_0^* w\ , \tag{2.12b}$$

$$\dot{\sigma}_{+1}=-\left[\frac{1}{T_2}+i\Delta-\frac{1}{T}(\tilde{g}-1)\right]\sigma_{+1}-\frac{i}{2}\Omega_0 w\ , \tag{2.12c}$$

where we have used the notation

$$\sigma_{-1}^{+}=\int d\alpha\,\sigma_{\alpha}^{+}e^{-i\alpha},\quad \sigma_{+1}=\int d\alpha\,\sigma_{\alpha}e^{i\alpha} \tag{2.13}$$

for the integrated marginal averages of the dipole operators $\hat{\sigma}$ and $\hat{\sigma}^{+}$. For simplicity we have also $\tilde{g}=\int d\alpha\, e^{\pm i\alpha}g(\alpha)$ [see Eq. (2.11)].

By a repeated differentiation of Eqs. (2.12) we can obtain the following third-order differential equation satisfied by the stochastic expectation value of the atomic inversion operator:

$$\dddot{w}+\left[\frac{1}{T_1}+\frac{2}{T_2}+2\gamma\right]\ddot{w}+\left[\frac{2}{T_1T_2}+\frac{2\gamma}{T_1}+\left[\frac{1}{T_2}+\gamma\right]^2+\Delta^2+\Omega_0^2\right]\dot{w}+\left[\left[\frac{1}{T_2}+\gamma\right]^2\frac{1}{T_1}+\frac{\Delta^2}{T_1}+\frac{\Omega_0^2}{T_2}+\Omega_0^2\gamma\right]w+\frac{\Delta^2+\left[\dfrac{1}{T_2}+\gamma\right]^2}{T_1}=0\,. \tag{2.14}$$

Because we are mostly interested here in effects of phase fluctuations on strong-field atomic responses, we take $1/T \gg 1/T_1, 1/T_2$ and $\Omega_0 \gg 1/T_1, 1/T_2$. At exact resonance ($\Delta=0$), we obtain from Eq. (2.14)

$$\dddot{w}+\frac{2(1-\tilde{g})}{T}\ddot{w}+\left[\frac{(1-\tilde{g})^2}{T^2}+\Omega_0^2\right]\dot{w}+\Omega_0^2\frac{(1-\tilde{g})}{T}w=0\,. \tag{2.15}$$

A closer investigation of the system of Eqs. (2.12) indicates that at exact resonance, Eq. (2.15) can be reduced to a second-order differential equation of the following form:

$$\ddot{w}+\frac{(1-\tilde{g})}{T}\dot{w}+\Omega_0^2 w=0\,. \tag{2.16}$$

Equation (2.15) was derived for the first time in Ref. 4 without any reference to the shot-noise model of phase fluctuations. For a uniform distribution of $g(\alpha)$ in the interval $[0,2\pi]$, we have $\tilde{g}=0$ and Eq. (2.15) is equivalent to a random telegraph phase signal[1] with the amount of jump $a=\pi/2$, and also leads to the same dynamical equation[2] for frequency fluctuations with $a=0$. The explanation of these facts is very simple. The shot noise with $\tilde{g}=0$, the phase telegraph with jump $\pi/2$, and the frequency telegraph[3] with $a=0$ all lead to a Lorentzian power spectrum of the electric field correlation function. The obvious difference between shot-noise and the telegraph signal is in the replacement of $2/T$ by $1/T$. This means that the frequency of any change from a state α is half of the frequency of a telegraph signal. This is of course due to the simple fact that the two-state telegraph signal has to change its sign with probability 1 in contrast to the shot-noise case where such a probability is $\frac{1}{2}$ (because the phase can always jump up or down).

From Eqs. (2.1) and (2.4) we calculate that the frequency

$$\omega(t)=\frac{d\Phi}{dt}=\sum_{i=1}^{n}\alpha_i\delta(t-t_i) \tag{2.17}$$

is also a shot noise but with a δ-type pulse and its autocorrelation function has the white-noise form. Models of shot-noise frequency with different pulses can be easily obtained by a simple convolution of Eq. (2.17) with a proper shape function.

III. GENERAL JUMP PROCESS

The last example of phase fluctuations that we discuss in this paper is given by condition (1.5), i.e., $f(\alpha\mid\beta)=g(\beta)$. With such a transition the CKS equation (1.1) can be solved exactly leading to

$$P(\alpha t\mid\alpha_0 t_0)=e^{-|t-t_0|/T}\delta(\alpha-\alpha_0)+g(\alpha)\times\left[1-\exp\left[-\frac{|t-t_0|}{T}\right]\right]. \tag{3.1}$$

With this explicit form for the probability distribution function of the phase we calculate easily the phase-dependent part of the electric field correlation function

$$\langle e^{-i[\Phi(t)-\Phi(t_0)]}\rangle=\tilde{g}^2+(1-\tilde{g}^2)e^{-|t-t_0|/T}\,. \tag{3.2}$$

where, as in the previous section, $\tilde{g}=\int d\alpha\, e^{\pm i\alpha}g(\alpha)$.

Expression (3.12) has a very clear resemblance to the proper correlation function of phase fluctuations described by a random telegraph signal which we have derived in Ref. 1. For a distribution of phase which is peaked around two possible values $\pm a$, i.e., if

$$g(\alpha)=\tfrac{1}{2}[\delta(\alpha+a)+\delta(\alpha-a)]\,, \tag{3.3}$$

we obtain $\tilde{g}=\cos a$ and the correlation function (3.2) is precisely equal to the random telegraph model of phase fluctuations.[1]

For a uniform distribution of phase given by $g(\alpha)=1/2\pi$, we obtain $\tilde{g}=0$ and the spectrum has a Lorentzian band shape with line width $1/T$. This corresponds[1] in form to a jump process with a jump size $a=\pi/2$.

For a more complicated distribution of α given for example by a Lorentzian profile with band width a, we obtain $\tilde{g}=e^{-a}$. For very small a, i.e., if $\tilde{g}=1$, we obtain the telegraph model with the jump size equal to zero.[1] For very large values of a, i.e., if $\tilde{g}=0$, we recover the uniform distribution of phase already given. Note that in expression (3.2) the characteristic line width $1/T$ is precisely half of the random telegraph frequency $2/T$, as has been noted in the previous section.

As in the previous section, we derive from the BCKS equation (1.2) with condition (1.5) the following exact set of equations of motion involving the two-level atomic inversion (for $\Delta=0$ and $1/T_1=1/T_2=0$):

$$\ddot{w}+\Omega_0^2 w=-\frac{1}{T}\dot{w}+\frac{i}{T}\tilde{g}\Omega_0 B\ , \qquad (3.4a)$$

$$\dot{B}=\frac{i}{2}\Omega_0 C\ , \qquad (3.4b)$$

$$\dot{C}=i\Omega(A+B)-\frac{1}{T}C+\frac{2}{T}\tilde{g}_2 w\ , \qquad (3.4c)$$

$$\dot{A}=\frac{i}{2}\Omega_0 C-\frac{1}{T}A+\frac{1}{T}\tilde{g}_2 B\ , \qquad (3.4d)$$

where we have used the following notation to denote the integrated marginal averages of the Bloch vector components:

$$B=\int d\alpha\, g(\alpha)(\sigma_\alpha^+-\sigma_\alpha)=\sigma^+-\sigma\ , \qquad (3.5a)$$

$$C=\int d\alpha\, g(\alpha)(e^{-i\alpha}w_{-\alpha}+e^{i\alpha}w_\alpha)=w_{-1}+w_1\ , \qquad (3.5b)$$

$$A=\int d\alpha\, g(\alpha)(e^{-2i\alpha}\sigma_\alpha^+-e^{2i\alpha}\sigma_\alpha)=\sigma_{-2}^+-\sigma_2\ , \qquad (3.5c)$$

and

$$\tilde{g}_2=\int d\alpha\, g(\alpha)e^{\pm 2i\alpha}\ . \qquad (3.6)$$

Note that only for the two-peaked distribution function given by Eq. (3.3) α_0 we have from (3.5b)

$$C=w_{-1}+w_{+1}=2\cos w\ , \qquad (3.7)$$

i.e., the two equations involving higher "harmonics" (σ_{-2}^+ and σ_2) are decoupled from (3.4a) and (3.4b).

As a result we obtain the following third-order differential equation for the stochastic expectation value of the inversion operator:

$$\dddot{w}+\frac{1}{T}\ddot{w}+\Omega_0^2\dot{w}+\frac{1}{T}\Omega_0^2\cos^2 a w=0\ ,$$

which is precisely[1] the random telegraph equation with jump size equal to a and with the change of $2/T$ to $1/T$.

For an arbitrary function $g(\alpha)$, we have to solve the entire set of equations (3.4). By a repeated differentiation we can obtain the following fifth-order differential equation satisfied only by the averaged atomic inversion:

$$w^{(V)}+\frac{3}{T}w^{(IV)}+\left[2\Omega_0^2+\frac{3}{T^2}\right]w^{(iii)}+\left[\frac{7}{2}\frac{\Omega_0^2}{T}+\frac{1}{T^3}+\frac{\Omega_0^2}{2T}\tilde{g}_2\right]\ddot{w}+\left[\Omega_0^4+\frac{\Omega_0^2}{T^2}\left[\frac{3}{2}+\frac{\tilde{g}_2}{2}+\tilde{g}\right]\right]\dot{w}$$
$$+\left[\frac{\Omega_0^4}{2T}\tilde{g}_2+\frac{\Omega_0^4}{2T}+\frac{\Omega_0^2}{T^3}\tilde{g}_2\right]w=0\ .$$

Note that the evolution of the averaged atomic inversion operator is governed by two line widths, $\tilde{g}=\int d\alpha\, e^{\pm i\alpha}g(\alpha)$ and $\tilde{g}_2=\int d\alpha\, e^{\pm 2i\alpha}g(\alpha)$, as predicted in Ref. 4.

IV. SUMMARY

In this paper we have discussed several examples of generalized Poisson processes described by the CKS equation with different transition functions $f(\alpha\,|\,\beta)$. We have shown that the case of $f(\alpha\,|\,\beta)=g(\alpha-\beta)$ is equivalent to a shot-noise model of external fluctuation, and that the case $f(\alpha\,|\,\beta)=g(\beta)$ corresponds very closely to a two-state Markov chain description of phase fluctuations. The exact equations obtained for the atomic response variables indicate that the different versions of telegraph signals discussed by us in Refs. 1–3 are general enough to take into account the cases discussed. This paper also shows the universality of the jump processes and their atomic responses regardless of the details of the stochastic models employed.

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