PHYSICAL REVIEW A VOLUME 31, NUMBER 4 APRIL 1985

Relativistic Coulomb bremsstrahlung in soft-photon approximation

Leonard Rosenberg
Department of Physics, New York University, New York, New York 10003
(Received 8 November 1984)

A low-frequency approximation for the bremsstrahlung transition amplitude is derived for a relativistic electron scattered by a central potential which is Coulombic at great distances. The Coulomb tail is shown here to have the effect of introducing correction terms, not present in previous relativistic treatments, which depend logarithmically on the photon frequency. The approximation procedure, in the form presented here, is based on the fact that at low frequencies the dominant contribution to the matrix element comes from the domain where r, the electron distance from the scattering center, is large. An asymptotic representation of the Dirac wave function is derived which includes a correction term of relative order $1/r$ compared to the leading term. This enables us to include a correction to the matrix element of relative order k, where k is the photon wave number. The essential feature of the low-frequency approximation, present in previous versions, is preserved here. That is, the result is expressed in terms of the physical amplitude, A, for scattering in the absence of any radiative interaction. Another potentially useful feature of the present version is that it involves angular derivatives of A, not energy derivatives, and should therefore remain valid in a region where A varies rapidly with energy.

I. INTRODUCTION

In his derivation of the soft-photon approximation for single-photon bremsstrahlung, Low[1] explicitly excluded the case where both particles in the collision are charged. This allowed him to make use of certain analyticity properties of the transition amplitude which do not in fact hold in the presence of long-range Coulomb interactions. Here we shall examine the effect of the Coulomb tail on the form of the soft-photon approximation in the context of a very simple model, namely, photon emission by a Dirac electron scattered by a center of force. The scattering interaction is assumed to be described by a local, central potential $V(r)$ which behaves, for large values of r, like the Coulomb potential

$$V_C = -\frac{Ze^2}{r} . \tag{1.1}$$

For small r the potential $V(r)$ deviates from the purely Coulombic form. One may, for example, think of $V(r)$ as representing a local, energy-independent approximation to the effective potential for the scattering of electrons by atoms (neglecting screening effects) or ions. (Alternatively, one may keep in mind a model of proton-nucleus scattering, but for definiteness we shall refer to the projectile as an electron.) With such a simple, explicitly defined model we may take a quite different approach than that used by Low. While far less powerful and elegant, it does allow us to account for the effect of the Coulomb tail using our knowledge of the asymptotic form of the solution of the Dirac equation in the presence of a Coulomb potential. Here we recognize that the dominant contribution to the low-frequency bremsstrahlung matrix element comes from large values of r. From this point of view the most striking and useful feature of the soft-photon approximation—its dependence on only physically measurable radiation-free scattering parameters—is easily understood since the asymptotic form of the wave function depends on the scattering interaction through these parameters alone. A similar approach has been discussed previously in the context of the nonrelativistic scattering problem.[2] Substitution of the Dirac for the Schrödinger equation causes no essential difficulties. In addition to inclusion of spin degrees of freedom the method of derivation has been improved to allow for the effect of all multipoles, rather than just the dipole and quadrupole terms. This generalization is made necessary by the fact that v/c is not treated as a small parameter here.

The problem of relativistic Coulomb bremsstrahlung, in the potential scattering model, has of course received a considerable amount of theoretical attention in the past. However, none of this earlier work deals directly and comprehensively with the issue of particular concern to us here. Thus, in the early work of Bethe and Maximon[3] an analytic approximation is obtained which is valid for *all* photon frequencies but the scattering energy is assumed to be large compared to the electron rest energy and the effect of a short-ranged component to the potential is not included. To fill this gap large-scale numerical procedures were initiated subsequently[4] based on specific, simple representations of $V(r)$. These efforts, while extremely useful, lack the above-mentioned feature of the soft-photon approximation—model independence in a relatively simple analytic form. In addition to its practical role (however limited) in the analysis of atomic-field bremsstrahlung, results reported on here should be useful in further studies of the infrared radiation problem. Here we have in mind not only single-photon but also multiphoton radiation processes where, in the general context of relativistic collision physics, Coulomb-field effects exert a subtle and as yet not fully understood influence.

Section II of this paper contains a derivation of the first two terms in the asymptotic expansion of the wave function. The result may be of use in applications other than that considered here. (One possible application that comes to mind is the construction of a variational principle for relativistic Coulomb scattering.) Use of this asymptotic form in the derivation of the soft-photon approximation is described in Sec. III. Calculational strategy is discussed in general terms in Sec. III A, and the lowest-order version of the approximation is obtained in Sec. III B. It is of a fairly simple form, differing from the standard perturbative result only in the inclusion of a multiplicative factor accounting for the effect of the Coulomb tail. A correction term, of first order in the frequency, is derived in Sec. III C, with some of the calculational details left to an Appendix.

II. WAVE FUNCTION AT GREAT DISTANCES

A. Preliminaries: Angular momentum decomposition

To prepare the way for later developments we summarize, in this subsection, some well-known properties of the solution of the time-independent Dirac equation

$$(-i\boldsymbol{\alpha}\cdot\nabla+\beta+V)\psi=W\psi \tag{2.1}$$

for an electron in a local, central potential $V(r)$. We follow closely the treatment of the Coulomb wave function given in Sec. 32 of Rose's book[5] (to which the reader is referred for notational conventions). Here, however, we include the irregular as well as the regular solution, in appropriate linear combination, to allow for a phase-shifted asymptotic form reflecting the influence of the short-ranged component of the potential.

To begin we recall the form of the two-component spin-angle function

$$\mathscr{Y}^{\mu}_{jl(1/2)}(\hat{\mathbf{r}})=\sum_{m',\nu'} C(l\tfrac{1}{2}j,\nu'm'\mu)Y_l^{\nu'}(\hat{\mathbf{r}})\chi^{m'} \tag{2.2}$$

describing an electron of total angular momentum j, projection μ, and orbital angular momentum $l=j\pm\frac{1}{2}$. We note the relation

$$(\boldsymbol{\sigma}\cdot\mathbf{L}+1)\mathscr{Y}^{\mu}_{jl(1/2)}(\hat{\mathbf{r}})=-\kappa\mathscr{Y}^{\mu}_{jl(1/2)}(\hat{\mathbf{r}}) , \tag{2.3}$$

with $\mathbf{L}=-i\mathbf{r}\times\nabla$ and

$$-\kappa=j(j+1)-l(l+1)+\tfrac{1}{4} . \tag{2.4}$$

Suppose that the electron, in a continuum state of energy $W=(p^2+1)^{1/2}$, is initially in the spin state

$$\chi=c_{1/2}\chi^{1/2}+c_{-1/2}\chi^{-1/2} \tag{2.5}$$

with momentum $\mathbf{p}$. We allow for scattered waves which are either outgoing (+) or incoming (−) [as required by the form of the matrix element shown in Eq. (3.1) below]. The full wave function may be expanded as

$$\psi^{(\pm)}_{\mathbf{p}}=4\pi\left[\frac{\pi}{2Wp}\right]^{1/2}\sum_{j,l,\mu}\sum_{m,\nu}c_m i^l C(l\tfrac{1}{2}j,\nu m\mu)Y_l^{\nu *}(\hat{\mathbf{p}})\begin{bmatrix} g^{(\pm)}(r)\mathscr{Y}^{\mu}_{jl(1/2)}(\hat{\mathbf{r}}) \\ -if^{(\pm)}(r)\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\mathscr{Y}^{\mu}_{jl(12)}(\hat{\mathbf{r}})\end{bmatrix} . \tag{2.6}$$

The radial wave functions are defined by a set of coupled differential equations whose asymptotic solutions are of interest to us here. Accordingly, we replace $V(r)$ by $-Ze^2/r$ in which case the solutions can be expressed in terms of the confluent hypergeometric functions

$$F(a,b,z)=W_1(a,b,z)+W_2(a,b,z) , \tag{2.7a}$$

$$G(a,b,z)=iW_1(a,b,z)-iW_2(a,b,z) . \tag{2.7b}$$

The asymptotic forms of W_1 and W_2 are given, for example, in Schiff's text.[6] The regular solutions, g_R and f_R, of the radial equations can be expressed as

$$rg_R=(W+1)^{1/2}(\Phi_R+\Phi_R^*) , \tag{2.8a}$$

$$rf_R=i(W-1)^{1/2}(\Phi_R-\Phi_R^*) , \tag{2.8b}$$

with

$$\begin{aligned}\Phi_R=&(2pr)^{\gamma}(\gamma+iy)e^{-ipr+i\eta}e^{\pi y/2}\,|\Gamma(\gamma+iy)|\\ &\times[2(\pi p)^{1/2}\Gamma(2\gamma+1)]^{-1}\\ &\times F(\gamma+1+iy,2\gamma+1,2ipr) .\end{aligned} \tag{2.9}$$

Here we have

$$y=e^2ZW/p , \tag{2.10a}$$

$$\gamma^2=\kappa^2-e^4Z^2 , \tag{2.10b}$$

$$e^{2i\eta}=-\frac{\kappa-iy/W}{\gamma+iy} . \tag{2.10c}$$

The irregular solutions are given by

$$rg_I=(W+1)^{1/2}(\Phi_I+\Phi_I^*) , \tag{2.11a}$$

$$rf_I=i(W-1)^{1/2}(\Phi_I-\Phi_I^*) . \tag{2.11b}$$

Φ_I is obtained from Eq. (2.9) by replacing F with G. Linear combinations of regular and irregular radial functions which satisfy the proper outgoing-wave or incoming-wave boundary conditions may be chosen as

$$g^{(\pm)}=e^{\pm i\bar{\delta}_\kappa}(g_R\cos\bar{\delta}_\kappa-g_I\sin\bar{\delta}_\kappa) , \tag{2.12a}$$

$$f^{(\pm)}=e^{\pm i\bar{\delta}_\kappa}(f_R\cos\bar{\delta}_\kappa-f_I\sin\bar{\delta}_\kappa) , \tag{2.12b}$$

where $\bar{\delta}_\kappa$ represents the contribution to the total phase shift δ_κ due to the deviation of the potential from purely Coulombic form. These phases are real; it follows that $g^{(-)*}=g^{(+)}$, $f^{(-)*}=f^{(+)}$.

From the known asymptotic expansions of the hypergeometric functions we find that, for $r\to\infty$,

$$rg^{(+)}\sim\left[\frac{W+1}{\pi p}\right]^{1/2}(\mathscr{O}_g e^{2i\delta_\kappa}+\mathscr{O}_g^*)/2 , \tag{2.13}$$

where, ignoring terms of order $1/r^2$,

$$\mathcal{O}_g = \exp\{i[pr + y\ln(2pr) - (l+1)\pi/2]\} \times \left[1 - \frac{y}{2pWr} + \frac{i}{2pr}(\kappa + \gamma^2 + y^2)\right] . \quad (2.14)$$

We also have

$$rf^{(+)} \sim i\left[\frac{W-1}{\pi p}\right]^{1/2}(\mathcal{O}_f e^{2i\delta_\kappa} - \mathcal{O}_f^*)/2 \quad (2.15)$$

with

$$\mathcal{O}_f = \exp\{i[pr + y\ln(2pr) - (l+1)\pi/2]\} \times \left[1 + \frac{y}{2pWr} + \frac{i}{2pr}(-\kappa + \gamma^2 + y^2)\right] . \quad (2.16)$$

In arriving at these forms we have used the relation

$$\delta_\kappa = \bar{\delta}_\kappa + \eta - \pi\gamma/2 - \arg\Gamma(\gamma + iy) + (l+1)\pi/2 , \quad (2.17)$$

expressing the total phase shift as the sum of components associated with the short range and Coulomb interactions. The $1/r$ correction terms which have been retained in the spherical waves shown in Eqs. (2.14) and (2.16) lead to higher-order corrections in the low-frequency expansion of the bremsstrahlung matrix element, as shown below in Sec. III C.

B. Asymptotic expansion of the full wave function

Let us now insert the asymptotic forms of the radial wave functions into the expansion (2.6) and attempt to carry out the sums over angular momentum states. It will be convenient to consider separately the contributions to the asymptotic wave function arising from the incoming and outgoing spherical waves in Eq. (2.13) for the "large"-component wave function; Eq. (2.15) provides a similar decomposition of the "small"-component contribution. That is, we write

$$\psi_{\mathbf{p}}^{(\pm)} = \tfrac{1}{2}(1+\beta)\psi_{\mathbf{p},\mathrm{in}}^{(\pm)} + \tfrac{1}{2}(1+\beta)\psi_{\mathbf{p},\mathrm{out}}^{(\pm)} + \tfrac{1}{2}(1-\beta)\psi_{\mathbf{p},\mathrm{in}}^{(\pm)} + \tfrac{1}{2}(1-\beta)\psi_{\mathbf{p},\mathrm{out}}^{(\pm)} \quad (2.18)$$

and examine these eight terms individually.

Consider first the angular momentum summations in the components $\frac{1}{2}(1\pm\beta)\psi_{\mathbf{p},\mathrm{in}}^{(\pm)}$; these are built up from the incoming waves $\mathcal{O}_g^*$ and $\mathcal{O}_f^*$. The dependence of these functions on the angular momentum quantum numbers presents an apparent obstacle to the performance of the summations. However, noting first that $\gamma^2 + y^2 = \kappa^2 + y^2/W^2$ we may replace the number κ by the operator $K = -(\boldsymbol{\sigma}\cdot\mathbf{L} + 1)$ since it multiplies the spin-angle eigenfunction in Eq. (2.6). The summations over j and μ may now be performed using the orthogonality property

$$\sum_{j,\mu} C(l\tfrac{1}{2}j, \nu m\mu)C(l\tfrac{1}{2}j, \nu' m'\mu) = \delta_{m'm}\delta_{\nu'\nu} . \quad (2.19)$$

Turning to the summation over l we observe that the phase factor $\exp(il\pi/2)$ in the incoming spherical wave function is multiplied by the identical factor i^l appearing in the expansion (2.6) and that the product may be absorbed by writing

$$(-1)^l Y_l^{\nu'}(\hat{\mathbf{r}}) = Y_l^{\nu'}(-\hat{\mathbf{r}}) . \quad (2.20)$$

The closure property then gives

$$\sum_{l,\nu} Y_l^{\nu}(-\hat{\mathbf{r}}) Y_l^{\nu *}(\hat{\mathbf{p}}) = \delta(\Omega_{-\hat{\mathbf{r}}} - \Omega_{\hat{\mathbf{p}}}) . \quad (2.21)$$

In this way we obtain the asymptotic forms

$$\tfrac{1}{2}(1+\beta)\psi_{\mathbf{p},\mathrm{in}}^{(+)} \sim \left[\frac{W+1}{2W}\right]^{1/2}\left[\frac{2\pi i}{pr}\right] \times \exp\{-i[pr + y\ln(2pr)]\} \times \left[1 - \frac{y/W}{2pr} - \frac{i}{2pr}(K + K^2 + y^2/W^2)\right] \times \delta(\Omega_{-\hat{\mathbf{r}}} - \Omega_{\hat{\mathbf{p}}})\chi \quad (2.22)$$

and

$$\tfrac{1}{2}(1-\beta)\psi_{\mathbf{p},\mathrm{in}}^{(+)} \sim \left[\frac{W-1}{2W}\right]^{1/2}\left[\frac{-2\pi i}{pr}\right]\exp\{-i[pr + y\ln(2pr)]\}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\left[1 + \frac{y/W}{2pr} - \frac{i}{2pr}(-K + K^2 + y^2/W^2)\right] \times \delta(\Omega_{-\hat{\mathbf{r}}} - \Omega_{\hat{\mathbf{p}}})\chi . \quad (2.23)$$

The appearance of angular derivatives acting on the δ function will not lead to complications since, in the matrix element of interest, the Hermitian operator K will be allowed to act to the left and this will enable us to perform the angular integrations immediately.

Analysis of the large and small components of $\psi_{\mathbf{p},\mathrm{in}}^{(-)}$ proceeds in a similar way. Writing out the result in the adjoint form required later on we have, with

$$\chi' = c'_{1/2}\chi^{1/2} + c'_{-1/2}\chi^{-1/2} \quad (2.24)$$

representing the final spin state of the electron, and with $\mathbf{p}'$ and $W' = (p'^2 + 1)^{1/2}$ representing its momentum and energy, respectively,

$$[\tfrac{1}{2}(1+\beta)\psi^{(-)}_{\mathbf{p}',\mathrm{in}}]^\dagger \sim \chi'^\dagger \left[\frac{W'+1}{2W'}\right]^{1/2} \left[\frac{2\pi i}{p'r}\right] \exp\{-i[p'r+y'\ln(2p'r)]\}$$
$$\times \delta(\Omega_{\hat{\mathbf{r}}}-\Omega_{\hat{\mathbf{p}}'}) \left[1-\frac{y'/W'}{2p'r}-\frac{i}{2p'r}(K+K^2+y'^2/W'^2)\right], \tag{2.25}$$

$$[\tfrac{1}{2}(1-\beta)\psi^{(-)}_{\mathbf{p}',\mathrm{in}}]^\dagger \sim \chi'^\dagger \left[\frac{W'-1}{2W'}\right]^{1/2} \left[\frac{2\pi i}{p'r}\right] \exp\{-i[p'r+y'\ln(2p'r)]\}$$
$$\times \delta(\Omega_{\hat{\mathbf{r}}}-\Omega_{\hat{\mathbf{p}}'}) \left[1+\frac{y'/W'}{2p'r}-\frac{i}{2p'r}(-K+K^2+y'^2/W'^2)\right] \boldsymbol{\sigma}\cdot\hat{\mathbf{r}} . \tag{2.26}$$

It will be convenient here to think of K as operating to the right.

The outgoing spherical waves in Eqs. (2.13) and (2.15) are accompanied by S matrix factors. Of course, one cannot then perform the angular momentum summations explicitly; rather, the result may be expressed in terms of the physical scattering amplitude. Following closely the treatment in Ref. 5 we introduce the 2×2 matrix

$$A(\hat{\mathbf{p}}',\hat{\mathbf{p}};p)=F(p,\cos\theta)+G(p,\cos\theta)\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} , \tag{2.27}$$

where $\cos\theta=\hat{\mathbf{p}}'\cdot\hat{\mathbf{p}}$ and where[7]

$$\hat{\mathbf{n}}=\hat{\mathbf{p}}\times\hat{\mathbf{p}}'/|\hat{\mathbf{p}}\times\hat{\mathbf{p}}'| , \tag{2.28}$$

$$F(p,\cos\theta)=-\frac{i}{4p}\sum_\kappa (e^{2i\delta_\kappa}-1)(2j+1)P_l(\cos\theta) , \tag{2.29}$$

$$G(p,\cos\theta)=-\frac{1}{2p}\sum_\kappa \frac{\kappa}{|\kappa|}(e^{2i\delta_\kappa}-1)P_l^1(\cos\theta) . \tag{2.30}$$

We then find the asymptotic forms

$$\tfrac{1}{2}(1+\beta)\psi^{(+)}_{\mathbf{p},\mathrm{out}} \sim \left[\frac{W+1}{2W}\right]^{1/2} \frac{1}{r}\exp\{i[pr+y\ln(2pr)]\} \left[1-\frac{y/W}{2pr}+\frac{i}{2pr}(K+K^2+y^2/W^2)\right] A(\hat{\mathbf{r}},\hat{\mathbf{p}};p)\chi \tag{2.31}$$

and

$$\tfrac{1}{2}(1-\beta)\psi^{(+)}_{\mathbf{p},\mathrm{out}} \sim \left[\frac{W-1}{2W}\right]^{1/2} \frac{1}{r}\exp\{i[pr+y\ln(2pr)]\}\boldsymbol{\sigma}\cdot\hat{\mathbf{r}} \left[1+\frac{y/W}{2pr}+\frac{i}{2pr}(-K+K^2+y^2/W^2)\right] A(\hat{\mathbf{r}},\hat{\mathbf{p}};p)\chi . \tag{2.32}$$

In obtaining these results we have replaced the factor $\exp(2i\delta_\kappa)$ by $\exp(2i\delta_\kappa)-1$ in order to arrive at the standard forms (2.29) and (2.30). It is readily verified that the error in this procedure appears only in the direction $\hat{\mathbf{r}}=\hat{\mathbf{p}}$. Anticipating that the expressions (2.31) and (2.32) will be required only for $\hat{\mathbf{r}}=\hat{\mathbf{p}}'$ we see that the above-mentioned replacement is valid provided we restrict our analysis to scattering away from the forward direction $\hat{\mathbf{p}}'=\hat{\mathbf{p}}$.

A very similar analysis leads to the asymptotic form of the scattered part of the final-state wave function. We find that

$$[\tfrac{1}{2}(1+\beta)\psi^{(-)}_{\mathbf{p}',\mathrm{out}}]^\dagger \sim \chi'^\dagger \left[\frac{W'+1}{2W'}\right]^{1/2} A(\hat{\mathbf{p}}',-\hat{\mathbf{r}};p') \left[1-\frac{y'/W'}{2p'r}+\frac{i}{2p'r}(K+K^2+y'^2/W'^2)\right]$$
$$\times\frac{1}{r}\exp\{i[p'r+y'\ln(2p'r)]\} \tag{2.33}$$

and

$$[\tfrac{1}{2}(1-\beta)\psi^{(-)}_{\mathbf{p}',\mathrm{out}}]^\dagger \sim -\chi'^\dagger \left[\frac{W'-1}{2W'}\right]^{1/2} A(\hat{\mathbf{p}}',-\hat{\mathbf{r}};p') \left[1+\frac{y'/W'}{2p'r}+i(-K+K^2+y'^2/W'^2)\right] \boldsymbol{\sigma}\cdot\hat{\mathbf{r}}$$
$$\times\frac{1}{r}\exp\{i[p'r+y'\ln(2p'r)]\} . \tag{2.34}$$

Here again we have assumed that $\hat{\mathbf{p}}'\neq\hat{\mathbf{p}}$ and have anticipated that Eqs. (2.33) and (2.34) will be evaluated below for the particular direction $-\hat{\mathbf{r}}=\hat{\mathbf{p}}$.

III. ASYMPTOTIC EVALUATION OF THE BREMSSTRAHLUNG MATRIX ELEMENT

A. Discussion of procedure

The matrix element of interest is

$$M=\int d^3r\,\psi_{\mathbf{p}'}^{(-)\dagger}\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}\psi_{\mathbf{p}}^{(+)}e^{-i\mathbf{k}\cdot\mathbf{r}}\,, \tag{3.1}$$

appropriate to the emission of a photon of wave number $\mathbf{k}$ and polarization $\boldsymbol{\lambda}$, accompanying the scattering of an electron with initial and final momenta $\mathbf{p}$ and $\mathbf{p}'$, respectively. Assuming the photon energy to be small compared to the energy of the incident electron the dominant contribution to the integral in Eq. (3.1) will come from the region far from the scattering center, where the potential is well approximated by the Coulombic form (1.1). The wave functions in Eq. (3.1) may then be replaced by their asymptotic forms, derived above in Sec. II B. The dominance of the asymptotic contribution in the soft-photon limit is made apparent by the observation that the matrix element is singular in the limit $k\rightarrow 0$, and a singularity can only develop in this case from an infinite range of integration. We may in fact make use of this feature to systematize the approximation procedure. That is, we retain only those contributions to the matrix element which are singular in the limit $k\rightarrow 0$. Noting that singularities develop from those terms in the integrand with slowly oscillating exponentials we may write

$$M\cong M^{(1)}+M^{(2)}\,, \tag{3.2}$$

with

$$M^{(1)}=\int_R d^3r\,\psi_{\mathbf{p}',\mathrm{in}}^{(-)\dagger}\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}\psi_{\mathbf{p},\mathrm{out}}^{(+)}e^{-i\mathbf{k}\cdot\mathbf{r}}\,, \tag{3.3}$$

$$M^{(2)}=\int_R d^3r\,\psi_{\mathbf{p}',\mathrm{out}}^{(-)\dagger}\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}\psi_{\mathbf{p},\mathrm{in}}^{(+)}e^{-i\mathbf{k}\cdot\mathbf{r}}\,. \tag{3.4}$$

The domain of integration is here understood to be confined to the region $r>R$, with R chosen large enough so that the asymptotic forms developed in Sec. II B are valid. For convenience we impose the requirement that our approximation be independent of R. This will be achieved if we keep only those terms in the asymptotic expansions of the wave functions which give rise to radial integrations convergent at the origin. The integration domain can then be enlarged to include the region $r<R$ thereby introducing an error which remains finite in the zero-frequency limit. Since errors of this order are ignored in our approximation scheme we see that dependence on the parameter R is removed by this procedure. The drawback is that by rejecting higher-order terms in the asymptotic expansion (because of their more singular behavior at the origin) we lose higher-order correction terms, in powers of the frequency, in the soft-photon approximation. We shall return to this point later on.

The integral $M^{(1)}$ in Eq. (3.3) may be interpreted as the contribution corresponding to photon emission in the final state. Keeping in mind the results of standard perturbation theory one expects that the amplitude $M^{(2)}$, corresponding to photon emission in the initial state, could be generated from $M^{(1)}$ by carrying out a fairly simple set of transformations. The appropriate rule, now to be stated, may be verified directly by examination of the form taken by the matrix element after substitution of the asymptotic wave functions. Thus, let us write

$$M^{(i)}=\chi'^{\dagger}\mathscr{M}^{(i)}\chi,\ \ i=1,2\,. \tag{3.5}$$

One finds that after the required integration is performed $\mathscr{M}^{(1)}$ takes the form of a sum of products

$$\mathscr{M}^{(1)}=\sum_s T_{1s}T_{2s}\cdots T_{n_s s}\,, \tag{3.6a}$$

with $\mathscr{M}^{(2)}$ taking a similar form

$$\mathscr{M}^{(2)}=\sum_s \widetilde{T}_{n_s s}\cdots \widetilde{T}_{1s}\,. \tag{3.6b}$$

To construct the factor $\widetilde{T}_{is}$ given T_{is} one carries out the transformations $\mathbf{k}\rightarrow-\mathbf{k}$, $\mathbf{p}'\rightarrow\mathbf{p}$, $\mathbf{p}\rightarrow\mathbf{p}'$. Furthermore in the expression for $A(\hat{\mathbf{p}}',\hat{\mathbf{p}};p)$ given in Eq. (2.27), one replaces $\boldsymbol{\sigma}$ by $-\boldsymbol{\sigma}$ in addition to interchanging $\mathbf{p}'$ and $\mathbf{p}$. The net effect of the transformation on $A(\hat{\mathbf{p}}',\hat{\mathbf{p}};p)$ is to replace p by p' in the scalar functions F and G in Eq. (2.27). The forms (3.6a) and (3.6b) involve products of spin matrices which can be simplified by repeated use of the identity

$$\boldsymbol{\sigma}\cdot\mathbf{a}\,\boldsymbol{\sigma}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{b}+i\boldsymbol{\sigma}\times(\mathbf{a}\times\mathbf{b})\,, \tag{3.7}$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors whose components commute with those of $\boldsymbol{\sigma}$ but not necessarily with each other.

Continuing with our discussion of the calculational procedure let us observe that due to the appearance of the directional δ functions in the asymptotic wave functions the angular integrations required in the expressions (3.3) and (3.4) are readily performed. With regard to the radial integrals we recall that only those which converge at the origin will appear and these are of the form

$$I(s;a,b)=\lim_{\epsilon\rightarrow 0+}\int_0^\infty e^{-(\epsilon+ia)r}r^{b+s}dr \tag{3.8}$$

with $s=0$ or 1 and

$$a=-(p-p'-\mathbf{k}\cdot\hat{\mathbf{p}}')\,, \tag{3.9a}$$

$$b=i(y-y')\,. \tag{3.9b}$$

We have

$$I(s;a,b)=(ia)^{-1-s-b}\Gamma(1+s+b)\,; \tag{3.10}$$

recall that $\Gamma(2+b)=(1+b)\Gamma(1+b)$ and

$$\Gamma(1+b)\cong 1-b\gamma,\ \ b\ll 1\,, \tag{3.11}$$

where $\gamma=0.5772\ldots$ is the Euler-Mascheroni constant.

B. Result of lowest-order calculation

As a first approximation we keep only the leading term in the asymptotic form of each wave function. The contribution from the correction terms, of order r^{-1} compared to the leading term, will be included later on. The influence of the Coulomb tail is then contained in the logarithmic contribution to the phase of the wave function.

This in turn leads to a modification of the radial integral through the appearance of the factor

$$B(p',p)=e^{-|y-y'|\pi/2}\left[\frac{|p-p'|}{2p}\right]^{i(y'-y)}\times\left[\frac{p'}{p}\right]^{-iy'}\Gamma(1+i(y-y'))\ . \qquad (3.12)$$

Introducing the small parameter $\delta=(p-p')/p$ we have the expansion

$$B(p',p)=1+iy\delta\ln(\tfrac{1}{2}|\delta|)+(y\pi/2)|\delta| +iy\delta(1+\gamma)-y^2\delta^2\ln^2(\tfrac{1}{2}|\delta|) +O(\delta^2\ln\delta)\ . \qquad (3.13)$$

Our first approximation for $M^{(1)}$ can now be written as

$$M^{(1)}\cong-\frac{2\pi}{W'}B(p',p)(p-p'-\hat{\mathbf{p}}'\cdot\mathbf{k})^{-1}\times\hat{\mathbf{p}}'\cdot\boldsymbol{\lambda}\chi'^{\dagger}A(\hat{\mathbf{p}}',\hat{\mathbf{p}};p)\chi\ . \qquad (3.14)$$

Since terms of order δ^0 are ignored in this approximation only the first two terms in the expansion (3.13) for $B(p',p)$ need be retained. The expression for $M^{(2)}$ in this same approximation is

$$M^{(2)}=-\frac{2\pi}{W}B(p,p')(p'-p+\hat{\mathbf{p}}\cdot\mathbf{k})^{-1}\times\hat{\mathbf{p}}\cdot\boldsymbol{\lambda}\chi'^{\dagger}A(\hat{\mathbf{p}}',\hat{\mathbf{p}};p')\chi\ , \qquad (3.15)$$

in agreement with the transformation rule discussed above.

Note that to the required accuracy we have

$$\hat{\mathbf{p}}'\cdot\boldsymbol{\lambda}(p-p'-\hat{\mathbf{p}}\cdot\mathbf{k})^{-1}\cong\mathbf{p}'\cdot\boldsymbol{\lambda}(kW'-\mathbf{p}'\cdot\mathbf{k})^{-1}\ , \qquad (3.16a)$$

$$\hat{\mathbf{p}}\cdot\boldsymbol{\lambda}(p'-p+\hat{\mathbf{p}}\cdot\mathbf{k})^{-1}\cong-\mathbf{p}\cdot\boldsymbol{\lambda}(kW-\mathbf{p}\cdot\mathbf{k})^{-1}\ . \qquad (3.16b)$$

The above approximation for M is then seen to differ from the standard perturbative result in that here we have the Coulomb factor $B(p',p)\cong 1+iy\delta\ln(\frac{1}{2}|\delta|)$. [Of course, the scattering interaction is accounted for nonperturbatively through the appearance of the on-shell scattering amplitude $A(\hat{\mathbf{p}}',\hat{\mathbf{p}};p)$.] Thus the presence of the Coulomb tail changes the analytic form of the bremsstrahlung amplitude, introducing a correction of order $\ln\delta$ to the lowest-order contribution of order δ^{-1}. The deviation from unity of the factor $B(p',p)$ gives a measure of the numerical significance of the logarithmic terms in the Coulomb-modified soft-photon approximation. As a rough indication of the domain in which the modification may play a role we note that the correction term $y\delta\ln(\delta/2)$ is of order 0.1 for the set of parameters $Z=40$, $p/W=v/c=0.5$, and $kW/p^2=\delta=0.05$. [Of course such Coulomb corrections are accounted for in treatments based on numerical solution of the Dirac equation, valid for the full range of photon frequencies.[4] The noteworthy feature of the soft-photon approximation (emphasized earlier) is that within its limited domain of validity it bypasses the need to assume a particular form of potential.]

C. A more accurate approximation

It is possible to include still higher-order corrections, of order δ^0 and $\delta\ln^2(\delta)$, with coefficients which are known in terms of the on-shell amplitude $A(\hat{\mathbf{p}}',\hat{\mathbf{p}};p)$. In order to do so, however, it will be necessary to introduce a slight modification of the procedure described above in Sec. III A since that procedure ignores corrections which remain finite in the limit $\delta\to 0$. We see, in fact, that the integral (3.1) must be transformed in a way which suppresses the contribution from the interior region ($r<R$). To do so we make use of the fact that the potential V has been assumed to be local so that the commutator of $\mathbf{r}$ with the Hamiltonian

$$H=-i\boldsymbol{\alpha}\cdot\nabla+\beta+V \qquad (3.17)$$

is

$$[H,\mathbf{r}]=-i\boldsymbol{\alpha}\ . \qquad (3.18)$$

Then, writing

$$\boldsymbol{\alpha}\cdot\boldsymbol{\lambda}=i[H,\mathbf{r}\cdot\boldsymbol{\lambda}] \qquad (3.19)$$

in the matrix element (3.1), and making use of the eigenvalue equation (2.1) for the initial and final scattering states, we find that Eq. (3.1) may be rewritten as

$$M=i(W'-W)\int d^3r\,\psi_{\mathbf{p}'}^{(-)\dagger}\mathbf{r}\cdot\boldsymbol{\lambda}\psi_{\mathbf{p}}^{(+)}e^{-i\mathbf{k}\cdot\mathbf{r}} +i\int d^3r\,\psi_{\mathbf{p}'}^{(-)\dagger}(\mathbf{r}\cdot\boldsymbol{\lambda})(\boldsymbol{\alpha}\cdot\mathbf{k})\psi_{\mathbf{p}}^{(+)}e^{-i\mathbf{k}\cdot\mathbf{r}}\ . \qquad (3.20)$$

In this form interior contributions are suppressed, to an extent sufficient for our present purposes, through the appearance of an additional factor of $\mathbf{r}$ in each integrand. The transformation leading to Eq. (3.20) is of course the starting point for the introduction of the dipole approximation in nonrelativistic treatments. Our motivation has been quite different, however. We do not make the dipole approximation since v/c is not assumed to be a small parameter here.

It may now be seen that corrections of higher order in the frequency may be obtained by applying the approximation procedure described in Sec. III A to each of the integrals in Eq. (3.20) after removing the overall factor of first order in the frequency which appears in each term. That is, we retain only those contributions to these integrals which are singular in the zero-frequency limit. This allows us to include the $1/r$ correction terms in the asymptotic wave functions. Due to the appearance of the additional factor of $\mathbf{r}$ in the integrand the radial integrals will converge when extended in to the origin.

Once again writing $M=M^{(1)}+M^{(2)}$, where $M^{(1)}$ and $M^{(2)}$ correspond to photon emission after and before the scattering, respectively, we find that the rule for constructing $M^{(2)}$ from $M^{(1)}$ remains just as it was stated in Sec. III A. We therefore confine our attention in the following to $M^{(1)}$ alone. In writing down the results of the calculation it will be convenient to introduce the abbreviation

$$C=(p-p'-\hat{\mathbf{p}}'\cdot\mathbf{k})^{-1}\hat{\mathbf{p}}'\cdot\boldsymbol{\lambda}\ . \tag{3.21}$$

Actually, the unit vector $\hat{\mathbf{p}}'$ in this function, as well as in $A(\hat{\mathbf{p}}',\hat{\mathbf{p}};p)$, should be thought of as a variable $\hat{\mathbf{r}}$ when acted upon by the operator $\mathbf{L}=-i\mathbf{r}\times\boldsymbol{\nabla}$, after which we set $\hat{\mathbf{r}}=\hat{\mathbf{p}}'$. More simply, we shall interpret $\mathbf{L}$, which appears in the operator $K=-(\boldsymbol{\sigma}\cdot\mathbf{L}+1)$, as $-i\mathbf{p}'\times\boldsymbol{\nabla}_{\mathbf{p}'}$, acting on functions of $\hat{\mathbf{p}}'$. Then, with

$$M^{(1)}=-\frac{2\pi}{W'}B(p',p)\chi'^{\dagger}\mathcal{N}^{(1)}A(\hat{\mathbf{p}}',\hat{\mathbf{p}};p)\chi \tag{3.22}$$

we find

$$\mathcal{N}^{(1)}=C-\frac{k}{2p^2}\left\{(p-p'-\hat{\mathbf{p}}'\cdot\mathbf{k})^{-1}\left[\frac{\mathbf{k}\cdot\mathbf{p}'}{W}+i\boldsymbol{\sigma}\cdot\mathbf{k}\times\hat{\mathbf{p}}'\right]C+[K,C]+W[K^2,C]-[K,i\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}\times\mathbf{p}'C]\right.$$
$$\left.-[K^2,\hat{\mathbf{k}}\cdot\mathbf{p}'C]-2i\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}\times\mathbf{p}'CK\right\}. \tag{3.23}$$

Note that this expression is free of long-range Coulomb effects; these are contained entirely in the multiplicative factor B in Eq. (3.22). The first term on the right-hand side (rhs) of order k^{-1}, corresponds, when combined with Eq. (3.22), to the first approximation (3.14). The remaining terms in Eq. (3.23) provide corrections of order k relative to the first term.

Further reduction of this fairly complicated expression is carried out in the Appendix. Let us remark here that in terms of the energy loss parameter $\delta=(p-p')/p$ we have derived an approximation for the bremsstrahlung matrix which contains, in addition to the usual terms of order δ^{-1} and δ^0, terms of order $\ln\delta$ and $\delta\ln^2(\delta)$ representing the effect of the Coulomb tail. These terms are given exactly, the error being of order $\delta\ln\delta$. In the nonrelativistic limit, $v/c \ll 1$, the result derived here reduces to one derived earlier[2] using the Schrödinger equation in the dipole approximation. It should perhaps be emphasized that the result involves the scattering amplitude and its angular derivatives. No derivatives with respect to the energy variable appear so that the approximation remains useful even in the neighborhood of a resonance. In this sense the result derived here bears a closer analogy to the Feshbach-Yennie[8] than to the Low[1] version of the soft-photon approximation.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. PHY-83-05186 and the U.S. Office of Naval Research under Contract No. N00014-76-C-0317.

APPENDIX

The expression (3.23) may be put in more explicit form using the properties of the angular momentum operators and the functional form (3.21) of C. In this Appendix we evaluate the various commutators which appear in Eq. (3.23), starting with $[K,C]$. Since $K=-(1+\boldsymbol{\sigma}\cdot\mathbf{L})$ we have

$$[K,C]=i\boldsymbol{\sigma}\cdot\mathbf{X}\ , \tag{A1}$$

where $\mathbf{X}=(\mathbf{p}'\times\boldsymbol{\nabla}_{\mathbf{p}'})C$. Since we are dealing with a correction term the approximation (3.16a) for C is correct to the required order. Using the abbreviation $(k\cdot p')=kW'-\mathbf{p}'\cdot\mathbf{k}$ we readily find

$$\mathbf{X}(k\cdot p')^{-1}[\mathbf{p}'\times\boldsymbol{\lambda}+(\mathbf{p}'\cdot\boldsymbol{\lambda})(\mathbf{p}'\times\mathbf{k})(k\cdot p')^{-1}]\ . \tag{A2}$$

The expression $i\boldsymbol{\sigma}\cdot\mathbf{X}$ is a spin operator acting on the amplitude A. This operation can be effected using the identity (3.7) and the form (2.27) for A.

Noting that $K^2=L^2+\boldsymbol{\sigma}\cdot\mathbf{L}$ and that

$$[L^2,C]=[L^2C]+2[\mathbf{L}C]\cdot\mathbf{L} \tag{A3}$$

we are left to evaluate $L^2C=-(\mathbf{p}'\times\boldsymbol{\nabla}_{\mathbf{p}'})\cdot\mathbf{X}\equiv Y$. Carrying out this operation we obtain, with $\mathbf{k}\cdot\boldsymbol{\lambda}=0$,

$$Y=2(k\cdot p')^{-2}[\,2(\mathbf{p}'\cdot\mathbf{k})(\mathbf{p}'\cdot\boldsymbol{\lambda})+(k\cdot p')(\mathbf{p}'\cdot\boldsymbol{\lambda})$$
$$-2(k\cdot p')^{-1}(\mathbf{p}'\cdot\boldsymbol{\lambda})(\mathbf{p}\times\mathbf{k})^2] \tag{A4}$$

and

$$[K^2,C]=Y-2i\mathbf{X}\cdot\mathbf{L}-i\mathbf{X}\cdot\boldsymbol{\sigma}\ . \tag{A5}$$

Turning now to the evaluation of $[\boldsymbol{\sigma}\cdot\mathbf{L},i\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}\times\mathbf{p}'C]$ we note that for any vector function $\mathbf{V}$ we have, from the identity (3.7),

$$[\boldsymbol{\sigma}\cdot\mathbf{L},i\boldsymbol{\sigma}\cdot\mathbf{V}]=[\mathbf{L}\cdot\mathbf{V}]+i\boldsymbol{\sigma}\cdot[\mathbf{L}\times\mathbf{V}]\ , \tag{A6}$$

where $\mathbf{L}$ is understood to operate only on $\mathbf{V}$. Carrying out these operations we obtain

$$[\boldsymbol{\sigma}\cdot\mathbf{L},i\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}\times\mathbf{p}'C]=\mathbf{X}\cdot(\hat{\mathbf{k}}\times\mathbf{p}')+2C\mathbf{p}'\cdot\hat{\mathbf{k}}$$
$$+i\boldsymbol{\sigma}\cdot[\mathbf{X}\times(\hat{\mathbf{k}}\times\mathbf{p}')+C\mathbf{p}'\times\hat{\mathbf{k}}]\ . \tag{A7}$$

Finally, we examine the commutator

$$[K^2,\hat{\mathbf{k}}\cdot\mathbf{p}'C]=\hat{\mathbf{k}}\cdot\mathbf{p}'[K^2,C]+[K^2,\hat{\mathbf{k}}\cdot\mathbf{p}']C\ . \tag{A8}$$

Having previously worked out the first term on the rhs in Eq. (A8) we turn to the second and write

$$[K^2,\hat{\mathbf{k}}\cdot\mathbf{p}']=[L^2\hat{\mathbf{k}}\cdot\mathbf{p}']+2[\mathbf{L}\hat{\mathbf{k}}\cdot\mathbf{p}']\cdot\mathbf{L}+i\boldsymbol{\sigma}\cdot\hat{\mathbf{k}}\times\mathbf{p}'\ . \tag{A9}$$

We note the relations $L^2\hat{\mathbf{k}}\cdot\mathbf{p}'=2\hat{\mathbf{k}}\cdot\mathbf{p}'$, $\mathbf{L}\hat{\mathbf{k}}\cdot\mathbf{p}'=-i\mathbf{p}'\times\hat{\mathbf{k}}$ and

$$\mathbf{L}C=[\mathbf{L},C]+C\mathbf{L}$$
$$=-i\mathbf{X}+C\mathbf{L}\ . \tag{A10}$$

Combining these results we have

$$[K^2,\hat{\mathbf{k}}\cdot\mathbf{p}']C=(2\hat{\mathbf{k}}\cdot\mathbf{p}'+i\sigma\cdot\hat{\mathbf{k}}\times\mathbf{p}')C+2(\hat{\mathbf{k}}\times\mathbf{p}')\cdot\mathbf{X}+2iC(\hat{\mathbf{k}}\times\mathbf{p}')\cdot\mathbf{L}\ . \quad \text{(A11)}$$

To proceed further one would have to specify the form of the function $A(\hat{\mathbf{p}}',\hat{\mathbf{p}};p)$ and evaluate the angular derivatives in the operation of $\mathbf{L}$ on A, but we shall not do so here.

[1]F. E. Low, Phys. Rev. **110**, 974 (1958).
[2]L. Rosenberg, Phys. Rev. A **27**, 1879 (1983).
[3]H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 738 (1954).
[4]See, for example, H. K. Tseng and R. H. Pratt, Phys. Rev. A **3**, 100 (1971).
[5]M. E. Rose, *Relativistic Electron Theory* (Wiley, New York, 1961).
[6]L. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), p. 140.
[7]There is a misprint in Eq. (5.107) of Ref. 5. The correct version is given in our Eq. (2.29).
[8]H. Feshbach and D. R. Yennie, Nucl. Phys. **37**, 150 (1962).