New form of the time-energy uncertainty relation

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A new form of the time-energy uncertainty principle is presented. It has the form $\tau\,\Delta E \geq 3\pi 5^{1/2}/25$, where τ is the average lifetime of a decaying state, and ΔE is the energy spread of the state computed from $\Delta E^2 = \langle E^2\rangle - \langle E\rangle^2$. The particular state for which the above relationship becomes an equality is identified, and it is proved that all other states give a greater $\tau\,\Delta E$ product. This general result should have a wide range of applications.

I. INTRODUCTION

The usual form of the Heisenberg uncertainty principle[1] states that it is not possible at a particular instant of time t to measure two incompatible observables (that is, observables whose operators do not commute) to arbitrary accuracy. For a state described by the ket $|\psi(t)\rangle$, the uncertainty ΔA in the measurement of an observable A is normally computed as

$$\Delta A = (\langle A^2\rangle - \langle A\rangle^2)^{1/2} , \tag{1a}$$

$$\langle A^n\rangle = \langle \psi(t) \,|\, A^n \,|\, \psi(t)\rangle . \tag{1b}$$

The uncertainty principle for position x and momentum p_x is then[2]

$$\Delta x\, \Delta p_x \geq \tfrac{1}{2}\, |\langle [x,p_x]\rangle| = \tfrac{1}{2} , \tag{2}$$

where we are using atomic units ($\hbar = 1$).

An analogous principle for time and energy was suggested by Heisenberg, but it is clear that a time-energy uncertainty principle must take a form different from Eq. (2). In nonrelativistic quantum mechanics time is a parameter, rather than an observable with a corresponding Hermitian operator. The fundamental problem[2,3] with attempting to construct a satisfactory time operator is that a continuous spectrum for t $(-\infty < t < \infty)$ would imply a similar result for the spectrum of H. But the spectrum of H is known to be bounded from below. A number of formulations of the time-energy uncertainty principle have been presented, and we shall briefly review that work here.

One of the earliest versions[4,5] stated that if the energy of a system is measured in a time Δt, the uncertainty ΔE in the result must satisfy

$$\Delta E\, \Delta t \geq \tfrac{1}{2} . \tag{3}$$

In 1961 Aharonov and Bohm[6] disputed this conclusion, and a spirited controversy ensued between Fock[7] and those authors.[8] The controversy has been reviewed by Vorontsov.[9] Today it is generally agreed that Aharonov and Bohm were correct, and the energy of a system can be measured with arbitrary precision and speed.[10] Perhaps the final word on this subject is the recent paper by Aharonov and Safko.[11]

A more satisfactory version of the uncertainty principle is that of Mandelstam and Tamm.[12–14] If the Hamiltonian H is not an explicit function of time, then for any Hermitian operator A it can be shown that

$$\Delta A\, \Delta E \geq \tfrac{1}{2}\, |\langle [A,H]\rangle| = \tfrac{1}{2}\, |d\langle A\rangle/dt| . \tag{4}$$

Here we have followed standard notation in writing $\Delta E = \Delta H$. This can be rewritten in a form resembling Eq. (3):

$$\tau_A\, \Delta E \geq \tfrac{1}{2} , \tag{5a}$$

$$\tau_A = \Delta A / |d\langle A\rangle/dt| . \tag{5b}$$

The interpretation of Eq. (5) is that the state $|\psi(t)\rangle$ has a distribution of A values, and τ_A is the time required for the center $\langle A\rangle$ of the distribution to move an amount equal to its width ΔA. Note that τ_A does not represent a dispersion in t values; in fact, τ_A may vary as t changes.

The time-of-arrival version[15,16] of the uncertainty principle is also widely known. An observer located at a fixed value of x watches a wave packet pass. If the energy spread of the packet is ΔE, it will pass the observer over a time period Δt such that Eq. (3) is obeyed. Wigner[16] has given a simple mathematical formulation of this version. The energy spread ΔE is computed from Eq. (1) in the usual way. To compute Δt the expectation values of t and t^2 are calculated from the expression

$$\langle t^n\rangle = \frac{\int\int\int |\psi(x,y,z,t)|^2 t^n\, dy\, dz\, dt}{\int\int\int |\psi|^2 dy\, dz\, dt} \tag{6}$$

and then Eq. (1a) is used. Note that the integrals are carried out over the full range of y, z, and t for a fixed value of x.

A number of other formulations of the time-energy uncertainty principle have also been published. These include versions based upon the construction of a time operator,[6,17–20] or an inverse time operator,[21,22] as well as ones based upon the equivalent-width method,[23] the diffraction in time phenomenon,[24] and relativistic quantum mechanics.[25,26] One other version, which is closely related to the work in this paper, involves the lifetime of a decaying state. This is normally stated[27] that the average lifetime τ of a state with an energy spread ΔE obeys the equality $\tau \Delta E = \frac{1}{2}$. (We shall review this relationship in detail in the following section.) The problem with this version is that ΔE is not computed from Eq. (1), but rather represents the half-width at half maximum of a Lorentzian distribution. For certain applications of the Heisenberg time-energy uncertainty principle it would be useful to have a version which relates the average lifetime τ to an energy spread ΔE given by Eq. (1). We construct such a version in this paper. The result is a much more general form of the time-energy uncertainty principle based upon the lifetime of a decaying state.

II. THEORY

We consider a system with Hamiltonian H. For simplicity we assume that the energy spectrum of H is continuous (the result derived here is valid if H also has a discrete spectrum). The complete set of energy eigenfunctions $|E\rangle$ of H has the properties that $H|E\rangle = E|E\rangle$ and

$$\langle E' | E \rangle = \delta(E-E') \ . \tag{7}$$

A decaying state $|\phi\rangle$ can be expanded in terms of the complete set $|E\rangle$ as

$$|\phi\rangle = \int_{-\infty}^{\infty} dE\, f(E) |E\rangle \ , \tag{8a}$$

$$f(E) = \langle E | \phi \rangle \ . \tag{8b}$$

[The spectrum of H is bounded from below, so the integral in Eq. (8a) should properly run from $E_{\min}$ to ∞. We shall write all such integrals from $-\infty$ to ∞, with the understanding that $f(E)=0$ for $E < E_{\min}$.] Since $|\phi\rangle$ is normalized to 1, it must be true that

$$1 = \int_{-\infty}^{\infty} dE\, P(E) \ , \tag{9a}$$

$$P(E) = |f(E)|^2 \ . \tag{9b}$$

The function $P(E)$ is a probability density function; it is properly normalized and $P(E) \geq 0$ for all E. The time dependence of the kets $|E\rangle$ is well known, and it is straightforward to show that

$$\langle \phi | \phi(t) \rangle = \int_{-\infty}^{\infty} dE\, P(E) \exp(-iEt) \ , \tag{10}$$

where we shall use the notation $|\phi\rangle = |\phi(t=0)\rangle$. The observable time behavior of the system is given by the "nondecay probability"

$$\begin{aligned} Q(t) &= |\langle \phi | \phi(t) \rangle|^2 \\ &= |\langle \phi | \exp(-iHt) | \phi \rangle|^2 \ . \end{aligned} \tag{11}$$

A number of properties of $Q(t)$ have been reviewed by Fonda *et al.*[28] We note that $Q(t)$ is uniquely determined by $P(E)$.

The function $Q(t)$ decays from one at $t=0$ to zero at $t=\infty$. The probability that the system has not decayed at time t is $Q(t)$, and the probability that the system decays between t and $t+dt$ is given by $-Q'(t)dt$. Consequently, the *average* lifetime[29] of the decaying state is

$$\begin{aligned} \tau &= -\int_0^{\infty} dt\, tQ'(t) \\ &= \int_0^{\infty} Q(t) dt \ , \end{aligned} \tag{12}$$

where the last line was obtained by an integration by parts.

The standard form of the time-energy relationship based upon a decaying state[27] is obtained by assuming that $P(E)$ is a Lorentzian distribution,

$$P(E) = (\Gamma/2\pi)/[E^2 + (\Gamma/2)^2] \ , \tag{13}$$

with a half-width at half maximum of $\delta E = \Gamma/2$. For this case $Q(t) = \exp(-\Gamma t)$ and $\tau = 1/\Gamma$. Consequently,

$$\tau \delta E = \tfrac{1}{2} \ . \tag{14}$$

It must be emphasized that the energy spread δE is not computed from Eq. (1). In fact, for the probability density function (PDF) given in Eq. (13) it is not possible to compute ΔE from Eq. (1) because the integral for $\langle E^2 \rangle$ does not converge.[4] It is also important to note that the Lorentzian distribution in Eq. (13) is not consistent with the fact that the spectrum of H is bounded from below. The consequence of this for any real system is that the exponential decay law breaks down at very short and very long times.[28]

We wish to look at a more general class of PDF's for which $\langle E^2 \rangle$ exists and examine the product $\tau \Delta E$, computed from Eqs. (12) and (1), respectively. We shall see that this product gives a new form of the time-energy uncertainty principle. We note that if $P(E)$ is not Lorentzian, $Q(t)$ cannot be purely exponential.[28,30] In addition, the nondecay probability $Q(t)$ computed from some of the PDF's considered here oscillates with time. Thus, there are "unphysical" regions where $Q'(t) > 0$. This behavior is seen in the well-known phenomenon of quantum beats.[31]

The energy matrix elements can be computed directly from $P(E)$:

$$\begin{aligned} \langle E^n \rangle &= \langle \phi | H^n | \phi \rangle \\ &= \int_{-\infty}^{\infty} dE\, E^n P(E) \ . \end{aligned} \tag{15}$$

It is also possible to obtain an expression for the average lifetime τ in terms of $P(E)$. Combining Eqs. (10) and (11) we obtain

$$Q(t) = \int_{-\infty}^{\infty} dE' \int_{-\infty}^{\infty} dE\, P(E') P(E) \cos[(E-E')t] \ . \tag{16}$$

If we substitute this into Eq. (12) and carry out the integral over t first, we obtain

$$\tau = \pi \int_{-\infty}^{\infty} dE\, P(E)^2 \ , \tag{17}$$

where we have used the identity[13]

$$\int_0^{\infty} \cos(kt) dt = \pi \delta(k) \ . \tag{18}$$

We are now in a position to write down the product $\tau \Delta E$ in terms of $P(E)$. For simplicity we assume that the zero of energy is chosen so that $\langle E \rangle = 0$. Then $\Delta E = \langle E^2 \rangle^{1/2}$, and

$$I(P) = \tau \Delta E = \pi \int_{-\infty}^{\infty} dE\, P(E)^2 \left[\int_{-\infty}^{\infty} dE\, P(E) E^2 \right]^{1/2} \ . \tag{19}$$

It is interesting that the problem has reduced to one involving the probability density function (PDF) $P(E)$ rather than the probability amplitude $f(E)$ defined in Eq. (8). We shall prove below that there is a function $P(E)$ which gives a minimum value for $I(P)$.

First, however, it is possible to simplify Eq. (19) somewhat. Consider the function $T(E)=\lambda P(\lambda E)$. This is a properly normalized PDF with mean zero which has the properties

$$\langle E^2\rangle_T=\lambda^{-2}\langle E^2\rangle_P , \tag{20a}$$

$$I(T)=I(P) . \tag{20b}$$

Consequently, if we choose $\lambda=\langle E^2\rangle_P^{1/2}$, then $\langle E^2\rangle_T=1$. Thus, it is possible to restrict our search to PDF's with mean zero and variance one.

The problem can be restated as follows. We wish to find the PDF $P(E)$ which minimizes the functional

$$I(P)=\int_{-\infty}^{\infty} dE\,[P(E)]^2 , \tag{21}$$

subject to the conditions $P(E)\geq 0$,

$$\int_{-\infty}^{\infty} dE\,P(E)=1 , \tag{22a}$$

$$\int_{-\infty}^{\infty} dE\,P(E)E=0 , \tag{22b}$$

$$\int_{-\infty}^{\infty} dE\,P(E)E^2=1 . \tag{22c}$$

This minimization problem also occurs in nonparametric statistics, and the solution has been published.[32]

TABLE I. Simple examples of $P(E)$. The $P(E)$ functions satisfy the conditions given in Eq. (22). $Q(t)$ was computed from $P(E)$ using Eqs. (10) and (11), and the product $\tau\Delta E$ was computed from Eq. (21).

Gaussian

$$P(E)=(2\pi)^{-1/2}\exp(-E^2/2), \quad -\infty<E<\infty$$

$$Q(t)=\exp(-t^2)$$

$$\tau\Delta E=(\pi)^{1/2}/2=0.886$$

Step function

$$P(E)=\begin{cases}(3)^{1/2}/6, & |E|\leq 3^{1/2}\\ 0, & |E|>3^{1/2}\end{cases}$$

$$Q(t)=\sin^2(3^{1/2}t)/(3^{1/2}t)^2$$

$$\tau\Delta E=\pi 3^{1/2}/6=0.907$$

Truncated parabola

$$P(E)=\begin{cases}[3(5)^{1/2}/20](1-E^2/5), & |E|\leq 5^{1/2}\\ 0, & |E|>5^{1/2}\end{cases}$$

$$Q(t)=(9/z^6)(\sin z-z\cos z)^2 \quad \text{where } z=5^{1/2}t$$

$$\tau\Delta E=3\pi 5^{1/2}/25=0.843$$

Before presenting the result it is instructive to consider a few simple examples for $P(E)$. These are given in Table I along with the corresponding nondecay probability and the product $\tau\Delta E$. It is seen that (contrary to our expectation) the Gaussian form of $P(E)$ does not give the minimum product. In fact, the functional $I(P)$ is minimized by the truncated parabola shown as the third example.

We give here a simple proof that the truncated parabola $P(E)$ given in Table I is the unique continuous function which minimizes $I(P)$ and satisfies the conditions in Eq. (22). Suppose some other function $R(E)$ satisfies Eq. (22). We shall show that $I(P)\leq I(R)$ with equality only if $P(E)=R(E)$ except on a set of measure zero (so that a continuous R equals P everywhere). Let

$$f(E)=R(E)-P(E) . \tag{23}$$

Then

$$I(R)=I(P+f)=I(P)+I(f)+2\int_{-\infty}^{\infty} dE\,P(E)f(E) . \tag{24}$$

From Eq. (21) it is clear that $I(f)\geq 0$, and, in fact, $I(f)>0$ unless $f(E)=0$ for all E outside a set of measure zero. Hence it is only necessary to show that the integral in Eq. (24) is nonnegative. To do this let $g(E)$ denote the parabola without truncation

$$g(E)=[3(5)^{1/2}/20](1-E^2/5), \quad -\infty<E<\infty . \tag{25}$$

Note that $g(E)=P(E)$ for $|E|\leq(5)^{1/2}$. Since both $R(E)$ and $P(E)$ obey the conditions in Eq. (22), it must be true that

$$\int_{-\infty}^{\infty} dE\,f(E)E^n=0 \quad \text{for } n=0,1,2 . \tag{26}$$

Consequently,

$$\begin{aligned}\int_{-\infty}^{\infty} dE\,f(E)g(E) &= 0\\ &=\int_{-5^{1/2}}^{5^{1/2}} dE\,f(E)g(E)+\int_{-\infty}^{-5^{1/2}} dE\,f(E)g(E)\\ &\quad+\int_{5^{1/2}}^{\infty} dE\,f(E)g(E) .\end{aligned} \tag{27}$$

The last two integrals are less than or equal to zero, because in the range $|E|\geq 5^{1/2}$, $g(E)\leq 0$ and $f(E)=R(E)\geq 0$. Therefore, the first integral, which is identical to the integral in Eq. (24), is nonnegative. This completes the proof.

III. DISCUSSION

The time-energy uncertainty principle derived here can be written

$$\tau\Delta E\geq 3\pi 5^{1/2}/25 , \tag{28}$$

where the energy spread ΔE is computed from Eq. (15) or Eq. (1), and the average lifetime τ can be calculated from Eq. (12) or Eq. (17). This very general result can be used to estimate the lifetime of decaying states. It should be particularly useful when continuum states are approximated by a technique such as box normalization. For such a case the calculation of ΔE is straightforward. We are presently using this procedure in our calculation[33] of the lowest H_2^- resonance to obtain estimates of the energy width and lifetime of the resonance.

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