Particles in spherical electromagnetic radiation fields

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If the time dependence of a Hamiltonian can be eliminated by an appropriate symmetry transformation, the corresponding quantum-mechanical problem can be reduced to an effectively stationary one. With this result we investigate the behavior of nonrelativistic particles in a spherical radiation field produced by a rotating source—then the symmetry transformation corresponds to a rotation. We calculate the transition probabilities in the Born approximation. The extension to problems involving an additional Coulomb potential is briefly discussed.

I. INTRODUCTION

The behavior of nonrelativistic charged particles in classical electromagnetic fields is of importance for many applications. The Schrödinger equation with time-independent external fields has been studied extensively since the invention of quantum mechanics. In recent years time-dependent Hamiltonians have also attracted considerable interest. Abstract investigations concerning the existence of unitary evolution operators and a scattering theory have been carried out by various authors.[1–3] Concrete results are available for temporally periodic Hamiltonians and some situations involving plane-wave fields.[4,5]

In this paper we consider spherical electromagnetic waves coupled minimally to the Schrödinger equation. One possible application concerns the motion of particles in intense laser fields. Special fusion-laser systems consist of many laser beams pointing toward a center. In a certain region around the origin (but slightly away from it) the electromagnetic field approximates an incoming spherical wave.[6] One can hope to obtain results in cases where the external field has a simple multipole expansion consisting of only a few terms, but also then the problem is obscured by the time dependence of the Hamiltonian and the absence of spherical symmetry (remember that the lowest order of multipole expansion of electromagnetic fields is the dipole radiation). Nevertheless, under some additional assumptions on the symmetry properties of the external field one may obtain detailed information about the behavior of a particle subjected to this field. If, for example, the source of the electromagnetic field is a uniformly rotating charge and current distribution, the time dependence of the problem can be eliminated completely. The reason for this is explained in Secs. II and III: The Hamiltonian $H(t)$ is connected to $H(0)$ by a simple symmetry transformation, namely a rotation through an angle ωt (where ω is the angular velocity of the rotating source). Therefore it is possible to derive the Floquet form[3,4] of the time-evolution operator by a transition to a rotating frame. It must be stressed that, for all physical systems for which the time dependence of the Hamiltonian can be expressed with the aid of some symmetry transformation, the problem can be reduced to a time-independent one (Sec. II). $H(t)$ does not even have to be periodic in time. If, for example, $H(t)$ is the Dirac operator with an external (nonperiodic) plane electromagnetic wave, the corresponding symmetry transformation is a translation by ct. This again implies a Floquet-like form of the time evolution which, together with the solution of the reduced time-independent problem, finally leads to the well-known Volkov states.[7]

In Sec. IV we consider as a special example the field of a rotating magnetic dipole. Since this field has a simple expansion in terms of vector spherical harmonics, it seems to be useful to expand also the Schrödinger wave function in terms of spherical harmonics. It is then possible to obtain a stationary system of coupled radial Schrödinger equations. This is shown to be a general feature for the external fields with this type of symmetry. The coupling of radial equations with different values of angular-momentum quantum numbers l and m is, of course, a consequence of the fact that the external field is not spherically symmetric. As the number of terms contained in the multipole expansion of the radiation field increases, the number of different values of l and m that will be coupled by the interaction also increases.

The expansion of the wave function in terms of angular-momentum eigenstates is of interest for a more detailed description of the possible transitions that might occur. In Sec. V we develop the basic principles of a scattering theory for the quasistationary states (i.e., the solutions of the corresponding time-independent problem). We calculate in the Born approximation the possible transitions of an electron that is subjected to the radiation field of a rotating source.

II. TIME-DEPENDENT HAMILTONIANS

Consider a quantum-mechanical system which is described by a time-dependent Hamiltonian $H(t)$ in some Hilbert space $\mathscr{H}$. The time evolution of the physical states is described by the Schrödinger equation

$$i\frac{d}{dt}\psi(t)=H(t)\psi(t) \; . \tag{2.1}$$

Given an initial state ψ_0 at time $t=t_0$, one can write a solution of (2.1) under some general mathematical conditions[1] in the form

$$\psi(t)=U(t,t_0)\psi_0 \ . \tag{2.2}$$

Here, $U(t,t_0)$ is a two-parameter family of unitary operators, which is sometimes called the propagator, and which has the properties

(i) $U(t,t)=1$,

(ii) $U(t,t_1)U(t_1,t_0)=U(t,t_0)$.

For simplicity we set $t_0=0$ and write $U(t,0)\equiv U(t)$ from now on.

Without further assumptions it is hard to make any precise statements about the concrete form of the time evolution. Therefore we suppose that the system shows additional symmetry in the following sense: Let $S(\alpha)=\exp(-iA\alpha)$ be a one-parameter Lie group of symmetry transformations. The Hamiltonian $H(t)$ should depend on time in such a way that waiting for a time t amounts to an application of the symmetry transformation $S(\alpha)$ with $\alpha=\omega t$ to the system. More precisely,

$$H(t)=S(\omega t)H(0)S^{-1}(\omega t) \ . \tag{2.3}$$

This form of time dependence is indeed typical for many situations in quantum mechanics. A simple example is an external scalar plane-wave field $w(z-ct)$ moving in the z direction with propagation speed c. In nonrelativistic quantum mechanics the Hamiltonian is given by

$$H(t)=-\Delta+w(z-ct) \tag{2.4}$$

($2m=\hbar=1$). We see that after a time t the system looks like it has undergone a translation by ct in the z direction. Therefore we may choose A to be the momentum operator $P_z=-id/dz$, and $\omega=c$ in this case.

Now let $\psi(t)$ be a solution of (2.1) with $H(t)$ fulfilling (2.3). Then we have

$$i\frac{d}{dt}[e^{i\omega At}\psi(t)]=-\omega A\, e^{i\omega At}\psi(t)+e^{i\omega At}H(t)\psi(t) \ .$$

With the definition $\phi(t)\equiv\exp(i\omega At)\psi(t)$ and using Eq. (2.3), we can rewrite this in the following form:

$$i\frac{d}{dt}\phi(t)=[H(0)-\omega A]\phi(t) \ . \tag{2.5}$$

Thus we have obtained a Schrödinger equation with a time-independent Hamiltonian. With the initial condition $\phi(0)=\psi(0)=\psi_0$, its solution can be written as

$$\phi(t)=\exp\{-i[H(0)-\omega A]t\}\psi_0 \ ,$$

from which we conclude that

$$U(t)=e^{-i\omega At}e^{-i[H(0)-\omega A]t} \ . \tag{2.6}$$

[In order to define $H(0)-\omega A$, one should require some additional domain properties in a more mathematical proof of Eq. (2.6).] Following Zel'dovich,[4] we call the eigenvalues E of $H(0)-\omega A$ quasienergies and the corresponding eigenstates ψ_E quasistationary states, because their time evolution is similar to the one of the usual stationary states in quantum mechanics:

$$\psi_E(t)=e^{-iEt}S(\omega t)\psi_E \ . \tag{2.7}$$

In many cases it is almost trivial to evaluate the action of $S(\omega t)$ on ψ_E (for instance, if it amounts to a simple rotation or translation). Thus, problem (2.4) involving the external plane wave will be solved completely if one can solve the following eigenvalue problem (obtained after a separation of variables):

$$\left[-\frac{d^2}{dz^2}+i\omega\frac{d}{dz}+w(z)\right]\psi_E=E\psi_E \ . \tag{2.8}$$

The ansatz

$$\psi_E(z)=\exp[-i(\omega/2)z]\chi_E(z)$$

turns Eq. (2.8) into

$$\left[-\frac{d^2}{dz^2}+w(z)\right]\chi_E=E'\chi_E \ , \tag{2.9}$$

where $E'=E+3\omega/4$. This shows that the time-dependent Schrödinger equation with Hamiltonian (2.4) can be reduced to a stationary eigenvalue problem if one exploits the symmetries of the system.

A warning should be added. It might happen (as in the example) that $H(0)-\omega A$ displays a continuous quasienergy spectrum (cf. also Ref. 8). Then ψ_E is not contained in the Hilbert space [i.e., $\psi_E(t,\mathbf{x})$ is not square integrable] and, as usual, we must form wave packets as superpositions of eigensolutions with different quasienergies:

$$\psi(t,\mathbf{x})=\int dE\, g(E)\psi_E(t,\mathbf{x}) \ , \tag{2.10}$$

where $g(E)$ is some appropriate function describing the quasienergy distribution in the wave packet. Whenever $H(0)-\omega A$ is self-adjoint, the solutions ψ_E can be chosen to form a complete orthonormal set of eigenfunctions (in a generalized sense). Then

$$\int d^3x\,|\psi(t,\mathbf{x})|^2=\int dE\,|g(E)|^2 \ , \tag{2.11}$$

where the integral on the right-hand side extends over the quasienergy spectrum of $H(0)-\omega A$. Very often there is also an additional difficulty, namely that $H(0)-\omega A$ is not semibounded like ordinary Schrödinger operators but rather has a continuous spectrum extending over $(-\infty,+\infty)$ (cf. Sec. V).

It is interesting to compare Eq. (2.6) with the Floquet form of $U(t)$, which can be derived for periodically time-dependent Hamiltonians, i.e., if $H(t+\tau)=H(t)$, $\tau>0$. In this case one can write[3,4]

$$U(t)=P(t)e^{-iGt} \ , \tag{2.12}$$

where $P(t)$ is unitary, periodic with period τ, $P(0)=1$, and where G is a self-adjoint operator not depending on time. Observe that our solution (2.6) has this form even in cases where $H(t)$ is not periodic in time [e.g., (2.4), with nonperiodic w).

III. THE FIELD OF A ROTATING SOURCE

As an important application of the general theory presented in Sec. II, we now consider the classical electromagnetic radiation field of a uniformly rotating source.

Without loss of generality, we assume the x_3 axis is the axis of rotation. Throughout this section $\underline{D}(\alpha)$ denotes the orthogonal (3×3) matrix corresponding to a rotation around the x_3 axis through an angle α, and $S(\alpha)$ denotes the unitary operator representing the same rotation in the Hilbert space of a nonrelativistic spinless particle, i.e., $S(\alpha)=\exp(-iL_3\alpha)$, where L_3 is the third component of orbital angular momentum.

If the charge and current densities rotate with angular velocity ω, we can write

$$\begin{aligned}\phi(\mathbf{x},t)&=\phi(\underline{D}^{-1}(\omega t)\mathbf{x},0)\ ,\\ \mathbf{j}(\mathbf{x},t)&=\underline{D}(\omega t)\mathbf{j}(\underline{D}^{-1}(\omega t)\mathbf{x},0)\ .\end{aligned} \tag{3.1}$$

It is easy to see that these properties carry over to the potentials Φ and $\mathbf{A}$. In the Lorentz gauge we have, for the retarded magnetic vector potential,

$$\begin{aligned}\mathbf{A}(\mathbf{x},t)&=\int d^3x'\frac{\mathbf{j}(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|/c}\\ &=\int d^3x'\frac{\underline{D}(\omega t)\mathbf{j}(\underline{D}^{-1}(\omega t)\mathbf{x}',-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|/c}\\ &=\underline{D}(\omega t)\int d^3y\frac{\mathbf{j}(\mathbf{y},-|\mathbf{x}-\underline{D}(\omega t)\mathbf{y}|/c)}{|\mathbf{x}-\underline{D}(\omega t)\mathbf{y}|/c}\\ &=\underline{D}(\omega t)\mathbf{A}(\underline{D}^{-1}(\omega t)\mathbf{x},0)\ .\end{aligned} \tag{3.2}$$

After a similar calculation, we obtain, for the potential,

$$\begin{aligned}\Phi(\mathbf{x},t)&=\int d^3x'\frac{\rho(\mathbf{x},t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}\\ &=\Phi(\underline{D}^{-1}(\omega t)\mathbf{x},0)\ .\end{aligned} \tag{3.3}$$

The interaction of a charged particle with the external electromagnetic field $(\Phi,\mathbf{A})$ is described by the Schrödinger equation (2.1) and the principle of minimal coupling which leads to the following time-dependent Hamilton operator:

$$\begin{aligned}H(t)&=[-i\nabla-\mathbf{A}(\mathbf{x},t)]^2+\Phi(\mathbf{x},t)\\ &=-\Delta+i(\nabla\cdot\mathbf{A})(\mathbf{x},t)+i\,\mathbf{A}(\mathbf{x},t)\cdot\nabla\\ &\quad+\mathbf{A}^2(\mathbf{x},t)+\Phi(\mathbf{x},t)\ .\end{aligned} \tag{3.4}$$

Consider first the system described by the Hamiltonian $H(0)$. It is well known that a rotation through an angle α turns the system into a new one described by a Hamiltonian of the same form, but with rotated external fields

$$\mathbf{A}'(\mathbf{x},0)=\underline{D}(\alpha)\mathbf{A}(\underline{D}^{-1}(\alpha)\mathbf{x},0)$$

and

$$\Phi'(\mathbf{x},0)=\Phi(\underline{D}^{-1}(\alpha)\mathbf{x},0)\ .$$

For potentials satisfying the properties (3.2) and (3.3), it is therefore clear that the Hamiltonian fulfills

$$e^{-i\omega L_3 t}H(0)e^{i\omega L_3 t}=H(t)\ . \tag{3.5}$$

Since time evolution is, according to Eq. (2.6), given by the expression

$$U(t)=e^{-i\omega L_3 t}e^{-i[H(0)-\omega L_3]t}\ , \tag{3.6}$$

we have again succeeded in reducing the time-dependent problem to a stationary eigenvalue problem, i.e., to that of the differential equation

$$[H(0)-\omega L_3]\psi_E=E\psi_E\ . \tag{3.7}$$

Unfortunately, nonstatic electromagnetic fields cannot be spherically symmetric. Therefore the detailed treatment of Eq. (3.7) is very difficult because this problem cannot be further simplified. Nevertheless, one might obtain additional information in some cases by expanding the wave function in terms of spherical harmonics:

$$\psi_E(\mathbf{x})=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{f_{lm}(E,r)}{r}Y_{lm}(\theta,\phi)\ . \tag{3.8}$$

Since the Y_{lm} span the angular-momentum eigenspaces where L_3 corresponds to multiplication by the magnetic quantum number m, we obtain, with (3.6) for the time evolution of ψ_E, the following expression:

$$\psi_E(t,\mathbf{x})=\sum_{l,m}\frac{f_{lm}(E,r)}{r}e^{-i(E+\omega m)t}Y_{lm}(\theta,\phi)\ . \tag{3.9}$$

In order to illustrate this point and to obtain a better understanding of the problems arising in practical calculations, we consider a concrete physical system in the next section.

IV. THE RADIAL EQUATIONS

Let us assume that the charge and current distribution of the source may be approximately described by a magnetic dipole of strength μ rotating in the x-y plane with frequency ω. We characterize the direction of the magnetic moment by a unit vector $\mathbf{n}(t)$,

$$\mathbf{n}(t)=(\cos(\omega t),\,\sin(\omega t),\,0)\ . \tag{4.1}$$

The retarded magnetic vector potential in the Lorentz gauge is then given by

$$\mathbf{A}(\mathbf{x},t)=\mu g(r)\left[\mathbf{n}(\tau)+\frac{r}{c}\frac{d\mathbf{n}(\tau)}{d\tau}\right]\times\frac{\mathbf{x}}{r}\ , \tag{4.2}$$

with $r=|\mathbf{x}|$ and $\tau=t-r/c$. Here, $g(r)$ is a phenomenological factor describing a spatially extended source. For a point dipole we have $g(r)=1/r^2$. Later we shall implicitly introduce a smooth cutoff for the singularity of g at $r=0$ in order to have no problems in formulating the Born approximation. This cutoff is, however, not necessary to guarantee the self-adjointness of the Hamiltonian (3.4). It is well known that an *attractive* singularity may destroy the possibility of defining a unique self-adjoint Hamiltonian on some suitable domain (if the perturbation of $-\Delta$ is more singular than $-1/4r^2$; an example is given by the electric point dipole[9]). However, in the case of a magnetic vector potential the singularities of the *repulsive* term $\mathbf{A}^2$ will always dominate, and thus there are no problems with respect to self-adjointness.[10]

Expanding (4.2) in terms of vector spherical harmonics,[11]

$$\mathbf{Y}_{l,m}^{(+)}(\theta,\phi)\equiv\frac{-i(\mathbf{x}\times\nabla)}{\sqrt{l(l+1)}}Y_{lm}(\theta,\phi)\ , \tag{4.3}$$

we obtain, for the magnetic dipole field $\mathbf{A}(\mathbf{x},t)=\mathbf{A}(r,\theta,\phi,t)$,

$$\mathbf{A}(r,\theta,\phi,t)=(4\pi/3)^{1/2}h(\omega,r)\mathbf{Y}_{1,1}^{(+)}(\theta,\phi)e^{-i\omega t}+\text{c.c.}\ , \tag{4.4}$$

where c.c. is the complex conjugate and (setting $\mu=1$)

$$h(\omega,r)=g(r)[(\omega/c)r+i]e^{i(\omega/c)r}\ .$$

More explicitly, the angular dependence is given by

$$\mathbf{Y}_{1,1}^{(+)}(\theta,\phi)=\langle\mathbf{Y}_{1,1}^{(+)}(\theta,\phi)\rangle$$
$$=(3/16\pi)^{1/2}(\cos\theta,\,i\cos\theta,\,-\sin\theta\,e^{i\phi})\ . \tag{4.5}$$

Since (i) the field consists only of a vector potential $\mathbf{A}$, and (ii) $\mathbf{A}$ fulfills $\nabla\cdot\mathbf{A}=0$ (i.e., in our case the Lorentz gauge coincides with the radiation gauge), expression (3.4) for the Hamiltonian $H(t)$ can be simplified. Furthermore, it is easy to see that the vector potential (4.4) indeed has the symmetry property (3.2). Thus, we can use the spherical harmonics expansion (3.9) for the solutions of the time-dependent Schrödinger equation (2.1). After a lengthy but elementary calculation using well-known properties of spherical harmonics, we obtain, for the radial functions $f_{lm}(E,r)$, the following system of coupled radial Schrödinger equations:

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-\omega m\right]f_{lm}-\frac{i}{r}hC(l,m)f_{l,m-1}+\frac{i}{r}\bar{h}C(l,-m)f_{l,m+1}$$
$$+h^2[A(l,m)f_{l-2,m-2}-B(l,m)f_{l,m-2}+A(l+2,-m+2)f_{l+2,m-2}]$$
$$+2|h|^2[D(l,m)f_{l-2,m}+E(l,m)f_{l,m}+D(l+2,m)f_{l+2,m}]$$
$$+\bar{h}^2[A(l,-m)f_{l-2,m+2}-B(l,-m)f_{l,m+2}+A(l+2,m+2)f_{l+2,m+2}]=Ef_{lm}\ . \tag{4.6}$$

The coefficients $A(l,m)$ to $E(l,m)$ are defined in the Appendix. Here we only mention the symmetry properties

$$B(l,m)=B(l,-m+2),\quad C(l,m)=C(l,-m+1)\ . \tag{4.7}$$

Of course, it is the missing spherical symmetry which is responsible for the coupling of the different partial waves. One observes that the Hamiltonian $H(0)$ couples l to $l\pm2$ and m to $m\pm1, m\pm2$.

Now we want to show that any external radiation field satisfying (3.2) and (3.3) leads to a system of ordinary differential equations which, like Eq. (4.6), has the general form

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-\omega m\right]f_{lm}(E,r)+\sum_{l',m'}V_{lm,l'm'}(r)f_{l'm'}(E,r)=Ef_{lm}(E,r)\ . \tag{4.8}$$

For the following we find it more convenient to work in the radiation gauge. Then $\Phi(\mathbf{x},t)$ is the instantaneous Coulomb potential of the rotating source and $\nabla\cdot\mathbf{A}=0$. Furthermore, we can write

$$\mathbf{A}(\mathbf{x},t)=\mathbf{A}_\omega(\mathbf{x})e^{-i\omega t}+\mathbf{A}_\omega^*(\mathbf{x})e^{+i\omega t}\ , \tag{4.9}$$

and derive for $\mathbf{A}_\omega$ the multipole expansion $[\xi=(\omega/c)r]$

$$\mathbf{A}_\omega(r,\theta,\phi)=\sum_{l,m}\left\{a^{\text{el}}(l,m)\left[\left[\frac{d}{dr}\frac{h_l^{(\pm)}(\xi)}{\xi}+h_l^{(\pm)}(\xi)\right]\mathbf{Y}_{lm}^{(-)}(\theta,\phi)-\sqrt{l(l+1)}h_l^{(\pm)}(\xi)\mathbf{Y}_{lm}^{(0)}(\theta,\phi)\right]\right.$$
$$\left.+a^{\text{mag}}(l,m)h_l^{(\pm)}(\xi)\mathbf{Y}_{lm}^{(+)}(\theta,\phi)\right\}\ , \tag{4.10}$$

where we have introduced the vector spherical harmonics

$$\mathbf{Y}_{lm}^{(0)}(\theta,\phi)=-i\frac{\mathbf{x}}{r}Y_{lm}(\theta,\phi),\quad \mathbf{Y}_{lm}^{(-)}(\theta,\phi)=\frac{ir}{\sqrt{l(l+1)}}\nabla Y_{lm}(\theta,\phi)$$

[for $\mathbf{Y}_{lm}^{(+)}$, see Eq. (4.3)], and where

$$\xi h_l^{(\pm)}(\xi)=\hat{h}_l^{(\pm)}(\xi)=\hat{n}_1(\xi)\pm i\hat{j}_l(\xi),\quad \hat{n}_l(\xi)=-(\pi\xi/2)^{1/2}Y_{l+1/2}(\xi),\quad \hat{j}_l(\xi)=(\pi\xi/2)^{1/2}J_{l+1/2}(\xi)$$

are the Riccati-Bessel functions (Y and J are defined in Ref. 12). The upper sign describes outgoing waves; the lower sign describes incoming waves, which are also solutions of the Maxwell equations and which correspond to the case realized in fusion-laser systems (cf. the remarks in the Introduction).

Because of the symmetry properties of $\mathbf{A}$, it suffices to consider $H(0)$:

$$H(0)=-\Delta+V(\mathbf{x},i\nabla),\quad V(\mathbf{x},i\nabla)=i\,\mathbf{A}(\mathbf{x},0)\cdot\nabla+\mathbf{A}^2(\mathbf{x},0)+\phi(\mathbf{x},0)\ . \tag{4.11}$$

If $\mathbf{A}$ is of magnetic type, only the coefficients of $\mathbf{Y}_{lm}^{(+)}$ are nonzero. Because $\mathbf{x}\cdot\mathbf{Y}_{lm}^{(+)}=0$, the term $\mathbf{Y}_{lm}^{(+)}\cdot\nabla$, and thus

$\mathbf{A}\cdot\nabla$, contains only angular derivatives, i.e., we can write

$$V(\mathbf{x},i\nabla)\equiv V\left[r,\theta,\phi,\frac{\partial}{\partial\theta},\frac{\partial}{\partial\phi}\right]$$

in polar coordinates. If we now transform $H(0)$ to the angular-momentum representation, the potentials $V_{lm,l'm'}$ are just given by the matrix elements of V between angular-momentum eigenstates, and since V contains no radial derivative we can write

$$V_{lm,l'm'}(r)=\int \sin\theta\, d\theta\, d\phi\, Y^*_{lm}(\theta,\phi)V\left[r,\theta,\phi,\frac{\partial}{\partial\theta},\frac{\partial}{\partial\phi}\right]Y_{l'm'}(\theta,\phi)$$

$$=V^*_{l'm',lm}(r)\ . \qquad (4.12)$$

Otherwise, the term $\mathbf{Y}^{(0)}_{lm}(\theta,\phi)$ in (4.10) has the consequence that V contains also a radial derivative. The additional term $\sim d/dr$, which then appears in (4.8), can, however, be eliminated by an ansatz similar to the one used for Eq. (2.8). Therefore it is possible to obtain a coupled system of radial equations for any rotating source. Of course, the coupling induced by the coefficients (4.12) is rather complicated.

It seems to be rather difficult to find exact solutions of Eq. (4.6) or (4.8). Therefore we shall look for approximate solutions in the next section.

V. SOME REMARKS ON SCATTERING THEORY

An electromagnetic radiation field with spherical wavefronts vanishes at infinity. In the system (4.6), appropriate for magnetic dipole radiation, all potential terms vanish at infinity as $1/r^2$. Since higher multipole terms in $\mathbf{A}$ vanish even faster, we can assume that the particles become asymptotically free. Then the complete time evolution $U(t)$ should be asymptotically given by the free evolution $\exp(-iH_0t)$ with $H_0=-\Delta$.

In scattering theory one tries to obtain expressions for the solutions of a given problem by comparing it with a simpler problem which is exactly solvable. Instead of comparing $U(t)$ and $\exp(-iH_0t)$ directly, we find it more convenient to first formulate a scattering theory for the Hamiltonians $H_0-\omega L_3$ and $H(0)-\omega L_3$. This is useful since the free evolution can be written in the form

$$e^{-iH_0t}=e^{-i\omega L_3t}e^{-i(H_0-\omega L_3)t}\ , \qquad (5.1)$$

which is similar to expression (3.6) for $U(t)$. Since the eigenfunctions of $H_0-\omega L_3$ are known exactly, we obtain in this way expressions for the eigenfunctions of $H(0)-\omega L_3$ and thus for the entire problem. Observe that $H_0-\omega L_3$ shows a continuous quasienergy spectrum on the entire real axis because in every partial-wave subspace with quantum numbers (l,m) the quasienergy may range in $[-m\omega,\infty)$. For simplicity, we assume that $H(0)-\omega L_3$ also has a continuous spectrum $(-\infty,+\infty)$ without eigenvalues embedded in it. In the following we also require that the external fields ϕ and $\mathbf{A}$ (and thus the potentials $V_{lm,l'm'}$) are sufficiently regular at the origin, so that the integrals appearing in Eqs. (5.9) and (5.11) below can be defined without difficulties.

We begin by defining the Green's function

$$G_{lm}(E;r,r')\equiv\frac{1}{p_m}\times\begin{cases}\hat{j}_l(p_mr)\hat{h}^{(+)}_l(p_mr'), & r<r'\\ \hat{j}_l(p_mr')\hat{h}^{(+)}_l(p_mr), & r>r'\end{cases} \qquad (5.2)$$

with $p_m\equiv\sqrt{E+\omega m}$. The Riccati Bessel functions [defined after Eq. (4.10)] are solutions of the "free" radial equation

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-\omega m\right]f^{(0)}_{lm}(E,r)=Ef^{(0)}_{lm}(E,r)\ . \qquad (5.3)$$

Because of (5.1), any solution of the differential equation

$$i\frac{d}{dt}\psi^{(0)}(t)=H_0\psi^{(0)}(t) \qquad (5.4)$$

can be written as a superposition of the eigenfunctions $f^{(0)}_{lm}(E,r)$ of $H_0-\omega L_3$:

$$\psi^{(0)}(\mathbf{x},t)=\sum_{l,m}\int_{-\omega m}^{\infty}dE\frac{f^{(0)}_{lm}(E,r)}{r}e^{-i(E+\omega m)t}Y_{lm}(\theta,\phi)\ . \qquad (5.5)$$

For instance, a spherical wave of definite angular momentum l_0,m_0 and energy k^2 is given by (5.5), and

$$f^{(0)}_{lm}(E,r)=\delta^{l_0m_0}_{lm}\delta(k^2-\omega m-E)\hat{j}_l(\sqrt{E+\omega m}\,r)\ , \qquad (5.6)$$

whereas for a plane wave $\exp(i\mathbf{k}\cdot\mathbf{x}-ik^2t)$ with wave vector $\mathbf{k}=(k,\theta',\phi')$, we must use (5.5) with

$$f^{(0)}_{lm}(E,r)=4\pi i^lY_{lm}(\theta',\phi')\frac{1}{k}\delta(k^2-\omega m-E)\hat{j}_l(\sqrt{E+\omega m}\,r). \qquad (5.7)$$

Square-integrable solutions may be formed, e.g., by replacing the δ functions in (5.6) and (5.7) by smooth energy distributions.

The Green's function (5.2) satisfies

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-\omega m-E\right]G_{lm}(E,r,r')=\delta(r-r')\ , \qquad (5.8)$$

which allows us to rewrite Eq. (4.8) as a system of integral equations

$$f_{lm}(E,r)=f_{lm}^{(0)}(E,r)-\int_0^\infty dr'\,G_{lm}(E,r,r')\sum_{l',m'}V_{lm,l'm'}(r')f_{l'm'}(E,r')\ , \tag{5.9}$$

where the $f_{lm}^{(0)}$ are solutions of Eq. (5.3).

A scattering state described by the time-dependent Schrödinger equation is then given by [cf. Eqs. (3.9) and (5.5)]

$$\psi(\mathbf{x},t)=\sum_{l,m}\int_{-\omega m}^{\infty}dE\frac{f_{lm}(E,r)}{r}e^{-i(E+\omega m)t}Y_{lm}(\theta,\phi)\ , \tag{5.10}$$

with f_{lm} given by (5.9) [i.e., as a superposition of eigenfunctions of $H(0)-\omega L_3$]. Using Eqs. (5.9) and (5.5), we can write $\psi(\mathbf{x},t)$ as a sum of a free solution $\psi^{(0)}(\mathbf{x},t)$ and a remainder which comes from the interaction with the external field. According to our choice of the Green's function (5.2), this remainder consists only of outgoing waves $\hat{h}_l^{(+)}(p_m r)\sim e^{ip_m r}$ for large r, in agreement with one's intuition for scattering systems.

It is useful to consider the Born approximation $f_{lm}^{(1)}(E,r)$, which is obtained by replacing $f_{l'm'}(E,r)$ on the right-hand side of Eq. (5.9) by $f_{l'm'}^{(0)}(E,r)$. As an example, we consider the scattering of a spherical wave with angular momentum l_0,m_0 and energy k^2, given by Eq. (5.6),

$$\psi^{(0)}(\mathbf{x},t)\equiv\psi^{(0)}(r,\theta,\phi,t)=\frac{\hat{j}_{l_0}(kr)}{r}Y_{l_0m_0}(\theta,\phi)e^{-ik^2t}\ . \tag{5.6'}$$

Observe that all initial states $\psi^{(0)}$ can be written as an appropriate superposition of the states (5.6′). For $\psi(\mathbf{x},t)$ the Born approximation reads

$$\psi^{(1)}(\mathbf{x},t)=\psi^{(0)}(\mathbf{x},t)-\sum_{l,m}\int_{-\omega m}^{\infty}dE\frac{1}{r}\int_0^\infty dr'\,G_{lm}(E,r,r')V_{lm,l_0m_0}(r')\delta(k^2-\omega m_0-E)\hat{j}_{l_0}(\sqrt{E+\omega m_0}\,r)$$
$$\times e^{-i(E+\omega m_0)t}Y_{lm}(\theta,\phi)\ .$$

Because of the δ function, only the terms with $k^2-\omega m_0>-\omega m$ contribute to the sum over l and m. Thus we obtain

$$\psi^{(1)}(\mathbf{x},t)=\psi^{(0)}(\mathbf{x},t)-\sum_{l,m}\Theta(k^2+\omega(m-m_0))\frac{1}{r}\int_0^\infty dr'\,G_{lm}(k^2-\omega m_0,r,r')V_{lm,l_0m_0}(r')\hat{j}_{l_0}(kr')e^{-i[k^2+\omega(m-m_0)]t}Y_{lm}(\theta,\phi)\ . \tag{5.11}$$

We see that in the Born approximation the scattered state contains only values of l and m for which

(i) $V_{lm,l_0m_0}\neq 0$, (ii) $k^2+\omega(m-m_0)>0$.

If the incoming particles have energy k^2, observers far away from the scattering center see only states with energies $k^2+\omega(m-m_0)$ in the Born approximation. Transitions to states with the same magnetic quantum number m_0 do not change the energy. For the rotating magnetic dipole this means that, in first order, only transitions to angular momenta $(l_0\,|\,m_0,m_0\pm1,m_0\pm2)$ or $(l_0\pm2\,|\,m_0,m_0\pm2)$ and to energies k^2, $k^2\pm\omega$, $k^2\pm2\omega$ (as long as k is sufficiently large) are possible.

For the behavior of the Born solution at large distances from the origin, we obtain, from Eq. (5.11),

$$\psi^{(1)}(r,\theta,\phi,t)\to Y_{l_0m_0}(\theta,\phi)e^{-ik^2t}\frac{1}{r}\sin(kr-1\pi/2)-\sum_{l,m}Y_{lm}(\theta,\phi)e^{-ip_m^2t}\frac{1}{r}e^{i(p_mr-l\pi/2)}A_{lm,l_0m_0}(k)\quad\text{as } r\to\infty\ , \tag{5.12}$$

where $p_m=[k^2+\omega(m-m_0)]^{1/2}$ and

$$A_{lm,l_0m_0}(k)=\frac{1}{p_m}\int_0^\infty dr\,\hat{j}_l(p_mr)V_{lm,l_0m_0}(r)\hat{j}_{l_0}(kr)\ . \tag{5.13}$$

The functions $\Theta(p_m^2)$ can be omitted in (5.12) because, for $p_m^2<0$, the second term is damped exponentially for large r and vanishes in the asymptotic expression for $\psi^{(1)}$ [we have to choose $p_m=+i(-p_m^2)^{1/2}$, so that $\psi^{(1)}$ is regular at infinity]. From (5.12) we conclude that the probability for the transition from a state with the quantum numbers l_0,m_0,k to a state l,m,p_m in the Born approximation is proportional to the square of (5.13),

$$P(l_0,m_0\to l,m)\sim|A_{lm,l_0m_0}(k)|^2\ . \tag{5.14}$$

Since for the magnetic point dipole $g(r)=1/r^2$ [cf. Eq. (4.2)], most of the V_{lm,l_0m_0} contain a singularity of the form $1/r^4$. In order to make all the A_{lm,l_0m_0} finite, one would have to introduce a cutoff at $r=0$. If, however, the angular momentum and hence the impact parameter is sufficiently large, the particle becomes insensitive to the singularity at $r=0$ or to the explicit form of the function $g(r)$ in a neighborhood of $r=0$, and, indeed, since $\hat{j}_l(kr)\sim r^{l+1}$ as $r\to0$, the integral (5.13) will exist in the point-dipole case if and only if $l+l_0>1$. This means that only the Born approximation for the elastic s-wave-scattering amplitude diverges.

For the rotating magnetic dipole (4.4), the transition amplitudes (5.13) can be expressed by the integrals $[p(\mu)\equiv(k^2+\omega\mu)^{1/2}]$

$$I_\beta^{\lambda\mu}(k;l,m) \equiv \int_0^\infty dr\, J_{l+1/2+\lambda}(p(\mu)r) \times J_{l+1/2}(kr)\frac{e^{-i\mu(\omega/c)r}}{r^\beta}\,, \qquad (5.15)$$

which, in turn, can be evaluated in terms of hypergeometric functions (cf., e.g., Eqs. 6.574 and 6.626 of Ref. 13). For elastic scattering these expressions can be further simplified, and we obtain (since $I_1^{\pm 2,0}=0$)

$$A_{lm;lm}(k)=\pi E(l,m)\left[\frac{\omega^2}{c^2}I_1^{00}+I_3^{00}\right] = \frac{\pi E(l,m)}{2l+1}\left[\frac{\omega^2}{c^2}+\frac{2k^2}{(2l-1)(2l+3)}\right] \quad (l\geq 1)\,,$$

$$A_{l-2,m;lm}(k)=\pi D(l,m)I_3^{-2,0} = \frac{k^2\pi D(l,m)}{(2l-3)(2l-1)(2l+1)} \quad (l\geq 2)\,, \qquad (5.16)$$

$$A_{l+2,m;lm}(k)=\pi D(l+2,m)I_3^{2,0} = \frac{k^2\pi D(l+2,m)}{(2l+1)(2l+3)(2l+5)} \quad (l\geq 0)\,.$$

Only the diagonal term contains an ω-dependent contribution. The remainder is identical to the contributions one would obtain for a static source.

Finally, we want to indicate how to obtain information about plane-wave scattering. Inserting (5.7) in (5.9) and (5.10), we obtain a solution $\psi(\mathbf{x},t)$ of the Schrödinger equation which initially corresponds to a plane wave $\exp(i\mathbf{k}\cdot\mathbf{x}-ik^2t)$. Since plane waves are a superposition of spherical waves, we obtain, for the Born solution, the asymptotic form

$$\psi^{(1)}(\mathbf{x},t)\to\exp(i\mathbf{k}\cdot\mathbf{x}-ik^2t)-\sum_{m,m'} f_{mm'}(\mathbf{k},\theta,\phi)\frac{1}{r}\exp\{ip(m-m')r-i[p(m-m')]^2t\} \quad \text{as } r\to\infty \qquad (5.17)$$

$(m,m'=0,\pm 1,\pm 2,\ldots)$, where

$$f_{mm'}(\mathbf{k},\theta,\phi)=\frac{4\pi}{k}\sum_{l=|m|}^\infty\sum_{l'=|m'|}^\infty i^{l'-l}Y_{lm}(\theta,\phi)A_{lm,l'm'}(k)Y_{l'm'}(\theta';\phi')\,. \qquad (5.18)$$

The coefficients $A_{lm,l'm'}$ in (5.18) are given by (5.3), and θ',ϕ' denotes the direction of the initial wave vector $\mathbf{k}$. The dependence of the amplitude (5.18) on two directions $(\theta,\phi,\theta',\phi')$ is due to the fact that the external field $\mathbf{A}$ introduces an independent direction (the rotation axis of the source in the case considered here) which we have chosen as the reference axis for the angular-momentum decomposition. Therefore we are not allowed to choose $\mathbf{k}$ as this axis (in contrast to scattering by a scalar potential). As a consequence, the scattering amplitude (5.18) is rather complicated and cannot be represented in terms of the usual phases. By the usual (heuristic) arguments we may define, nevertheless, a differential cross section for the scattering of particles (characterized by $\mathbf{k}$) into the solid angle (θ,ϕ) and to energy $k^2+\omega\mu$, which reads, in Born approximation

$$\frac{d\sigma}{d\Omega}(\mathbf{k},\theta,\phi)=\left|\sum_{\substack{m,m'\\ m-m'=\mu}} f_{mm'}(\mathbf{k},\theta,\phi)\right|^2\,. \qquad (5.19)$$

VI. CONCLUDING REMARKS

The aim of this work was to understand quantum-mechanical systems with a special form of time dependence that is related to symmetry transformations. We considered highly nontrivial examples such as time-dependent external fields, which are realistic solutions of the free Maxwell equations. Exploiting the symmetries of the system, we were able to reduce the problem to a time-independent one, and even to develop something like a stationary scattering theory. It was our goal to stress physical intuition and not to derive the best possible mathematical conditions for admitted potentials. The present paper should mainly serve as a guide to future investigations; we have left many problems open for discussion.

One of these problems concerns the development of a general scattering theory for the evolution groups $\exp[-i(H_0-\omega L_3)t]$ and $\exp[-i(H-\omega L_3)t]$, where $H=H_0+H_1$, and the precise formulation of the relations between the time-independent scattering theory [e.g., Eq. (5.8)] and the time-dependent formalism with Møller operators.

Another problem arises if we want to generalize the scattering theory outlined in Sec. V. We have considered here a particle that is only subjected to the external multipole field. In practice, one would like to consider a particle in the presence of an additional Coulomb potential γ/r in order to deal with atoms in radiation fields. Since the Coulomb potential is long ranged, the wave functions cannot be approximated asymptotically by plane waves. Possible ways to treat $H(t)$ with additional Coulomb forces are to use distorted plane waves or to take, instead of (5.4),

$$\left[\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+\frac{\gamma}{r}-\omega m\right]f_{lm}^C(E,r)=Ef_{lm}^C(E,r) \qquad (6.1)$$

as the modified "free" equation for comparison with the full problem [(6.1) can be solved exactly]. For scattering states (i.e., free-free transitions) most of the calculations of Sec. V remain valid if one only replaces the Riccati Bessel functions by regular Coulomb wave functions f_l^C. For example, the amplitude for the transition from a Coulomb scattering state with quantum numbers (k,l_0,m) to another Coulomb state (p_m,l,m) is given by

$$A^C_{lm,l_0m_0}(k)=\frac{1}{p_m}\int_0^\infty dr\, f_l^C(p_m r)V_{lm,l_0m_0}(r)f_{l_0}^C(kr) \quad (6.2)$$

in the Born approximation.

To study the influence of Coulomb bound states (e.g., bound-free transitions, ionization) is certainly more interesting. Naively, one could try to use a bound-state wave function f_l^C in (6.2). The question to be asked before doing this is, however, whether $H(t)$ has any bound states (in the sense of states which stay near the origin for all times) at all, so that an ionization probability can be defined. In a mathematical context this is certainly a question involving geometric scattering theory,[14] which provides appropriate characterizations of bound and scattering states even for time-dependent external fields.

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APPENDIX

For the sake of completeness, we list the coefficients appearing in Eq. (4.6):

$$A(l,m)=\left[\frac{(l+m-3)(l+m-2)(l+m-1)(l+m)}{(2l-3)(2l-1)^2(2l+1)}\right]^{1/2},$$

$$B(l,m)=2\frac{[(l+m-1)(l+m)(l-m+1)(l-m+2)]^{1/2}}{(2l-1)(2l+3)},$$

$$C(l,m)=[(l+m)(l-m+1)]^{1/2},$$

$$D(l,m)=\left[\frac{(l+m-1)(l-m-1)(l+m)(l-m)}{(2l-3)(2l-1)^2(2l+1)}\right]^{1/2},$$

$$E(l,m)=2\frac{(l+m)(l-m)+(l+2)(2l-1)}{(2l-1)(2l+3)}+1 .$$

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