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Nonuniqueness of solutions to the Lippmann-Schwinger equation in a soluble three-body model

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It is generally accepted that with suitable boundary conditions the Lippmann-Schwinger (LS) integral equation is equivalent to the Schrödinger equation, even in systems of more than two particles. It also is generally accepted that although in two-particle systems the scattering solutions to the LS equation are uniquely defined by that equation without need for boundary conditions, the same cannot be said of systems containing more than two particles; in such many-particle systems, it is generally agreed, unique scattering solutions to the LS equation are not obtained unless boundary conditions are imposed. However, these assertions have not been verified heretofore in any system of three or more particles, because exact closed-form solutions of the Schrödinger or LS equation for such systems are so difficult to achieve. We have examined these questions in a one-dimensional three-body model first discussed by McGuire, wherein three equal-mass particles interact via equal and finite strength attractive δ-function potentials. With this model, the scattering solutions to the Schrödinger equation can be written exactly, in closed form. Thereby we are able to demonstrate explicitly that in this model the scattering solutions to the Schrödinger equation do satisfy the LS equation; we also demonstrate explicitly that the LS equation's scattering solutions really are nonunique unless boundary conditions are imposed. This latter result strongly suggests that, in three-particle systems at any rate, recent criticisms of the aforementioned generally accepted nonuniqueness thesis are not well taken.

I. INTRODUCTION

The Lippmann-Schwinger (LS) integral equation at real energies E is

$$\Psi=\psi_i-G_i^{(+)}V_i\Psi \;, \tag{1.1}$$

where the "incident wave" $\psi_i(E)$ and the outgoing Green's function $G_i^{(+)}(E)$ satisfy, respectively,

$$(H_i-E)\psi_i=0 \;, \tag{1.2a}$$

$$(H_i-E)G_i^{(+)}=I \;. \tag{1.2b}$$

Here I is the unit operator in the multidimensional configuration space of the particles comprising the system, whose initial motion is described by ψ_i.

More than two decades ago, Foldy and Tobocman[1] showed that for a system of more than two particles the solutions $\Psi(E)$ to the LS integral equation (1.1) at energies E corresponding to scattering states need not be unique. In particular, Foldy and Tobocman recognized that the nonuniqueness was implied by the existence of nontrivial solutions Ψ to the homogeneous LS equation

$$\Psi=-G_i^{(+)}V_i\Psi \;. \tag{1.3}$$

However, Foldy and Tobocman's proof of nonuniqueness was based solely on operator algebra manipulations; as such their "proof" was both nonintuitive and highly suspect from the standpoint of mathematical rigor. Similar remarks pertain to other operator algebra analyses of the nonuniqueness problem, e.g., by Epstein.[2]

These deficiencies largely were remedied by Gerjuoy,[3] who used conventional mathematical operations in configuration space to prove the nonuniqueness for systems of nonrelativistic spinless distinguishable particles. Gerjuoy's treatment indicates that solutions to Eq. (1.1) are not unique because when particle rearrangement can occur the "scattered part"

$$\Phi=\Psi-\psi_i \tag{1.4}$$

of the solution to Eq. (1.1) need not be everywhere outgoing, even though $G_i^{(+)}$ is an "everywhere outgoing" Green's function. When particle rearrangement can occur, therefore, unique scattering solutions to the LS equation are not obtained without imposition of the "boundary condition" that Φ defined by Eq. (1.4) must be everywhere outgoing. But Gerjuoy's analysis, though based on straightforward mathematical operations combined with very plausible assumptions about the asymptotic behavior of $G_i^{(+)}$ and other relevant functions, scarcely was mathematically rigorous in the strict sense. It generally is agreed[4] that the necessary rigor is not easily achieved, but has been supplied by Faddeev, especially in his monograph.[5] Various authors have shown that the nonuniqueness of solutions to the many-particle LS equation is connected to the noncompactness of the many-particle LS integral equation kernel, which in turn is related to the presence of disconnected diagrams in iterative

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expansions for that kernel.[6] Faddeev himself has explained that he developed the Faddeev equations to avoid *inter alia* the nonuniqueness property of the many-particle LS equation.[7]

Nevertheless, Mukherjee, in a series of recently published papers,[8–12] claims that the many-particle LS equation *does* have unique solutions. Although Mukherjee[8] refers to Faddeev, he does not directly challenge the purportedly mathematically rigorous investigations of Faddeev *et al.*[5–7] Instead, Mukherjee focuses on alleged deficiencies in Gerjuoy's analysis,[3] whose correction (Mukherjee claims) leads to the conclusion that the multiparticle LS equation's solution *are* unique. In particular, according to Mukherjee, the homogeneous LS equation (1.3) is incorrect,[8–10] and[9] the scattered part of the solution to the conventional LS equation (1.1) always is everywhere outgoing. Mukherjee's procedures (and his criticism of Gerjuoy's results) themselves have been criticized in the literature, notably by Adhikari and Glöckle,[13] Tobocman,[14] Lovitch,[15] and Levin and Sandhas.[16]

In this paper, we shed light on this nonuniqueness issue by examining the scattering solutions to the Schrödinger equation in an exactly soluble three-particle model system. We show that in this model the scattering solutions to the Schrödinger equation do satisfy the LS equation; we also show that the LS equation indeed does have nonunique solutions. We further show, in this same model, that the homogeneous LS equation (3) is correct, and that the scattered part of the solution to the conventional LS equation (1.1) is not necessarily everywhere outgoing. To our knowledge, there is no other many-particle model wherein scattering solutions solving the Schrödinger equation have been constructed and explicitly shown to satisfy either the inhomogeneous or homogeneous LS equation in their integral operator forms (1.1) and (1.3), respectively.

The model we employ is that of McGuire.[17] It involves three equal-mass spin-zero particles moving on the same line (the x axis say) and interacting via pairwise attractive δ-function potentials of equal strength. This model incorporates the mathematical and physical complexities of many-particle systems, but is much simpler to treat mathematically than the more conventional three-dimensional three-particle systems mirroring the real world. The same model has been used previously[18] to test scattering theory predictions, in an application rather different from ours, however.

We now conclude this introductory section of our paper with a brief summary of the contents of later sections. The details of McGuire's model, along with the notation we employ, are described in Sec. II; in Sec. II we also present the scattering solutions solving the Schrödinger equation in McGuire's model. Section III derives the Green's function which, in McGuire's model, appears in the LS equation. The scattering solutions and Green's function are used in Sec. IV to demonstrate the results we asserted in the penultimate paragraph. Our concluding remarks are contained in Sec. V.

II. McGUIRE'S MODEL

Although McGuire[17] treats systems of particles with unequal masses, we shall confine our attention to a system of three distinguishable spinless particles of equal mass m moving on the x axis and interacting via attractive pairwise δ-function interactions of equal strength. Denote the particle positions on the x axis by x_μ, $\mu=1,2,3$ $(-\infty<x_\mu<\infty)$. Then the Hamiltonian in the laboratory system is

$$H_L=-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right]-\sqrt{2}g\delta(x_1-x_2)-\sqrt{2}g\delta(x_2-x_3)-\sqrt{2}g\delta(x_3-x_1)\,, \tag{2.1}$$

where $g>0$; the factor $\sqrt{2}$ is introduced in the strength of the attractive δ-function interaction for future convenience.

Removal of the center-of-mass motion from the right-hand side of Eq. (2.1), to yield the Hamiltonian in the center-of-mass system, is accomplished by the coordinate transformation

$$Z=\tfrac{1}{3}(x_1+x_2+x_3)\,, \tag{2.2a}$$

$$X_\mu=\frac{1}{\sqrt{2}}(x_\nu-x_\sigma)\,, \tag{2.2b}$$

$$Y_\mu=(\tfrac{2}{3})^{1/2}[x_\mu-\tfrac{1}{2}(x_\nu+x_\sigma)]\,, \tag{2.2c}$$

where μ,ν,σ are any cyclic permutation of (1,2,3). In Eqs. (2.2) Z obviously denotes the position of the center of mass, X_μ is proportional to the relative displacement between particles ν and σ, and Y_μ is proportional to the relative displacement between particle μ and the center of mass of ν and σ. The transformation (2.2) is slightly different from the center-of-mass system transformation considered by McGuire.[17]

Using Eqs. (2.2) in Eq. (2.1), and dropping the center-of-mass kinetic energy $-(\hbar^2/6m)\partial^2/\partial Z^2$, the Hamiltonian in the center-of-mass system is seen to be

$$H_{\text{c.m.}}\equiv H=-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial X_\mu^2}+\frac{\partial^2}{\partial Y_\mu^2}\right]-g\delta(X_\mu)-g\delta\left[\frac{X_\mu+Y_\mu\sqrt{3}}{2}\right]-g\delta\left[\frac{X_\mu-Y_\mu\sqrt{3}}{2}\right]\,, \tag{2.3}$$

where $\mu=1$, 2, or 3. The three versions of H obtained from employment of these three independent possibilities for μ in Eq. (2.3) are equivalent. For the collisions on which we shall concentrate, however, namely collisions in which (in the laboratory system) beams of particle 3 are incident on beams of bound particle pairs 1,2, use of Eq. (2.3) with the choice $\mu=3$ is most convenient. Especially for such collisions, it is helpful to make the further change of notation

$$X_3 = x, \quad Y_3 = y \ . \tag{2.4a}$$

Now the dynamics of the actual three-particle collision (especially with particle 3 incident on bound pairs of particles 1,2) can be visualized as the motion in the horizontal plane $z=0$ of a single two-dimensional particle whose usual rectangular coordinates are (x,y). For similar reasons we also will use

$$X_1 = u, \quad Y_1 = v \tag{2.4b}$$

and

$$X_2 = r, \quad Y_2 = s \ . \tag{2.4c}$$

The three lines $x=0$ and $x=\pm\sqrt{3}y$, on which various δ functions in the Hamiltonian (2.3) are nonvanishing, divide the xy plane into six 60° sectors, as shown in Fig. 1. These three lines are the loci $x=0$, $r=0$, and $u=0$, as can be seen from the defining Eqs. (2.2) and (2.4). In fact,

$$u = -\frac{\sqrt{3}}{2}y - \frac{1}{2}x, \quad v = -\frac{1}{2}y + \frac{\sqrt{3}}{2}x \ ,$$
$$u = -\frac{1}{2}r + \frac{\sqrt{3}}{2}s, \quad v = -\frac{1}{2}s - \frac{\sqrt{3}}{2}r \ , \tag{2.5a}$$

$$r = \frac{\sqrt{3}}{2}y - \frac{1}{2}x, \quad s = -\frac{1}{2}y - \frac{\sqrt{3}}{2}x \ ,$$
$$r = -\frac{\sqrt{3}}{2}v - \frac{1}{2}u, \quad s = -\frac{1}{2}v + \frac{\sqrt{3}}{2}u \ , \tag{2.5b}$$

$$x = \frac{\sqrt{3}}{2}v - \frac{1}{2}u, \quad y = -\frac{1}{2}v - \frac{\sqrt{3}}{2}u \ ,$$
$$x = -\frac{1}{2}r - \frac{\sqrt{3}}{2}s, \quad y = \frac{\sqrt{3}}{2}r - \frac{1}{2}s \ , \tag{2.5c}$$

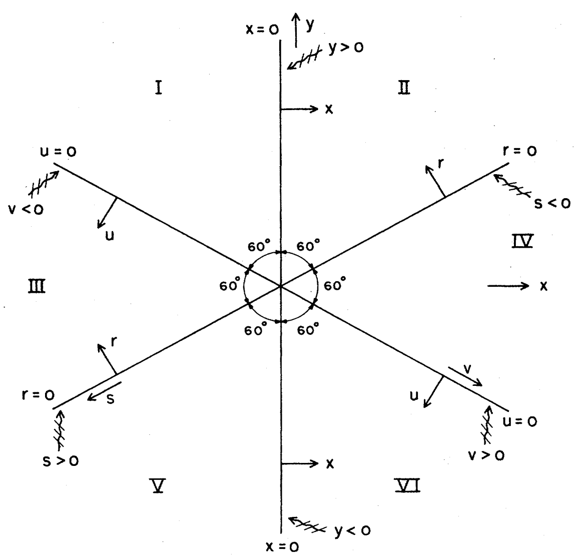

FIG. 1. Diagram showing the six 60° sectors I—VI into which the x,y plane is divided by the lines on which the δ-function interactions need not vanish, namely, the lines on which the original locations x_1,x_2,x_3 of the three particles are not all different. The arrows show the directions of positive x,r,u and y,s,v at these lines $x=0$, $r=0$, $u=0$, respectively. The signs of y,s,v at the opposing ends of these respective lines also are indicated.

which will be useful later.

The six 60° sectors in Fig. 1 are characterized by different relative values of the particle displacements x_1,x_2,x_3 along their original line of motion. With the six sectors labeled I—VI as shown in Fig. 1, the relative values of x_1,x_2,x_3 in the sectors are I, $x_1<x_2<x_3$; II, $x_2<x_1<x_3$; III, $x_1<x_3<x_2$; IV, $x_2<x_3<x_1$; V, $x_3<x_1<x_2$; and VI, $x_3<x_2<x_1$. The directions of increasing y, s, and v along the lines $x=0$, $r=0$, and $u=0$, respectively, are indicated by arrows; also indicated are the signs of y, s, and v, respectively, on the opposing halves (relative to the origin) of these lines. Figure 1 also shows the directions of increasing x, r, and u as these respective $x=0$, $r=0$, and $u=0$ lines are crossed.

The Schrödinger equation describing the motion of the three particles in the center-of-mass system is

$$(H-E)\Psi = 0 \ , \tag{2.6a}$$

where H from Eq. (2.3) now can be written as

$$H = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] - g\delta(x) - g\delta(u) - g\delta(r) \ . \tag{2.6b}$$

The equivalent two-dimensional particle moves freely (no interactions) in each of the six sectors I—VI. Along the boundaries of these sectors, i.e., along each of the lines $x=0$, $r=0$, and $u=0$, the wave function Ψ must obey the boundary condition

$$\Psi_+ = \Psi_- = \Psi_0 \ , \tag{2.7a}$$

$$\left.\frac{\partial \Psi}{\partial n}\right|_+ - \left.\frac{\partial \Psi}{\partial n}\right|_- = -\frac{2mg}{\hbar^2}\Psi_0 \equiv -2\alpha\Psi_0 \ , \tag{2.7b}$$

where n denotes the appropriate variable perpendicular to the line (namely x, r, and u at the lines $x=0$, $r=0$, and $u=0$, respectively); the derivative $\partial/\partial n$ is computed along the direction of increasing n; the subscripts + and − denote values at $n=0$, but are computed, respectively, on the $n>0$ and $n<0$ sides of the line; the subscript 0 merely makes explicit the fact that the wave function is continuous on the boundary lines; and Eq. (2.7b) provides the definition of α (which is >0).

For laboratory-system collisions in which beams of particles 3 are incident on beams of bound particles 1,2, the incident wave in the center-of-mass system of the three particles can be written as

$$\psi_i = e^{iky} w(x) \ , \tag{2.8a}$$

where

$$w(x) = \sqrt{\alpha} e^{-\alpha|x|} \ . \tag{2.8b}$$

Here $w(x)$ is the normalized wave function representing the bound pair of particles 1,2 in their own center-of-mass system, for the interaction $\sqrt{2}g\delta(x_1-x_2)$ appearing in Eq. (2.1). The bound-state energy is

$$E_b = -\frac{\hbar^2\alpha^2}{2m} \ . \tag{2.8c}$$

Correspondingly, ψ_i satisfies Eq. (1.2a) with

$$E = \frac{\hbar^2 k^2}{2m} + E_b \ , \tag{2.9a}$$

where, from Eq. (2.6b), the "initial" Hamiltonian in the center-of-mass system of the three particles is

$$H_i = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right] - g\delta(x) . \qquad (2.9b)$$

Evidently, as we have defined y, when $k>0$ the initial wave function ψ_i has particle 3 coming in from $-\infty$, and going out at $+\infty$, as seen from the center of mass of the bound pair 1,2.

With the incident wave ψ_i given by Eqs. (2.8), the solution Ψ_i to the Schrödinger equation (2.6a) whose scattered part (1.4) is everywhere outgoing now can be obtained quite readily, by the ray-tracing method McGuire himself employed,[17] or by applying the boundary conditions (2.7) to the most general forms of Ψ_i in each of the sectors I—VI consistent with the requirement that $\Psi_i - \psi_i$ is everywhere outgoing at infinity (and, of course, bounded). One finds (with $k>0$ henceforth)

$$\begin{aligned}
\Psi_{i\mathrm{I}} &= Ae^{iky}e^{\alpha x} ,\\
\Psi_{i\mathrm{II}} &= Ae^{iky}e^{-\alpha x} ,\\
\Psi_{i\mathrm{III}} &= Be^{iky}e^{\alpha x}+Ce^{iks}e^{-\alpha r} ,\\
\Psi_{i\mathrm{IV}} &= Be^{iky}e^{-\alpha x}+Ce^{ikv}e^{\alpha u} ,\\
\Psi_{i\mathrm{V}} &= \sqrt{\alpha}e^{iky}e^{\alpha x}+Ce^{iks}e^{\alpha r}+De^{ikv}e^{-\alpha u} ,\\
\Psi_{i\mathrm{VI}} &= \sqrt{\alpha}e^{iky}e^{-\alpha x}+De^{iks}e^{\alpha r}+Ce^{ikv}e^{-\alpha u} ,
\end{aligned} \qquad (2.10)$$

where

$$\begin{aligned}
A &= \sqrt{\alpha}\frac{ik\sqrt{3}-\alpha}{\sqrt{3}(\alpha\sqrt{3}+ik)} ,\\
B &= \sqrt{\alpha}\frac{ik\sqrt{3}-\alpha}{ik\sqrt{3}+\alpha} ,\\
C &= \frac{2\alpha^{3/2}(\alpha-ik\sqrt{3})}{\sqrt{3}(\alpha+ik\sqrt{3})(\alpha\sqrt{3}+ik)} ,\\
D &= -\frac{2\alpha^{3/2}}{(\alpha+ik\sqrt{3})} .
\end{aligned} \qquad (2.11)$$

It is easy to check that Eqs. (2.10) for Ψ_i obey Eqs. (2.7) when A,B,C,D are given by Eqs. (2.11). Equations (2.10) keep Ψ_i bounded at infinity because for each 60° sector the exponentials appearing in Eqs. (2.10) are damped; for example, referring to Fig. 1, it is apparent that in sector III $x<0$ and $r>0$, as required for damped $e^{\alpha x}$ and $e^{-\alpha r}$, respectively. Similarly, one sees that $\Psi_i - \psi_i$ is outgoing at infinity. For example, again referring to Fig. 1, in sector VI only the term $Ce^{ikv}e^{-\alpha u}$ in $\Psi_i - \psi_i$ is nonvanishing at infinity, and this term is nonvanishing only on the line $u=0$; on $u=0$ in sector VI $v>0$, so that (with $k>0$) the e^{ikv} factor in the nonvanishing term indeed is propagating outward at infinity.

Several interesting aspects of the solution (2.10) should be pointed out. First, as McGuire[17] has observed, there is no reflected direct scattering wave, i.e., there is no e^{-iky} wave in sectors V and VI. Second, and the reason we have undertaken to examine this model of McGuire's, the solution encompasses rearrangement. For example, $\Psi_{i\mathrm{IV}}$ and $\Psi_{i\mathrm{VI}}$ contain terms going out to ∞ along the $u=0$ line; these outgoing waves, proportional to e^{ikv}, represent particle 1 proceeding to ∞ relative to the bound pair 2,3, i.e., these waves represent exchange (during the collision) of particles 3 and 1. Similarly, the terms proportional to e^{iks} in $\Psi_{i\mathrm{III}}$ and $\Psi_{i\mathrm{IV}}$ manifest exchange of particles 3 and 2. These exchange waves we have identified carry particle 1 out along $v\to+\infty$ and particle 2 out along $s\to+\infty$; there are no corresponding exchange amplitudes along $v\to-\infty$ in $\Psi_{i\mathrm{I}}$ or $\Psi_{i\mathrm{III}}$, nor along $s\to-\infty$ in $\Psi_{i\mathrm{II}}$ and $\Psi_{i\mathrm{IV}}$. In other words, recalling the definitions (2.2) and (2.4), McGuire's model permits forward exchange but not backward exchange, i.e., the exchanged formerly bound particle is not reflected, but continues out to infinity along the same direction as was being traveled by the originally unbound particle 3. Of course, the absence of reflected waves, both in direct and exchange scattering, is an expected consequence of conservation of momentum in these collisions involving equal mass particles moving on a single one-dimension line. It also is noteworthy that the solution (2.10) does not manifest any breakup, i.e., no matter how large k^2 is, all terms in Eq. (2.10) represent a bound state of one of the three possible particle pairs.

In the foregoing, we have employed the subscript i to denote the incident wave (2.8a), which propagates in the initial channel (also denoted by the subscript i) wherein particles 1 and 2 are bound. In what follows we also shall require the solutions Ψ_f to Eq. (2.6a) associated with an incident wave ψ_f in a rearranged or "final" channel; we require, of course, that Ψ_f, like Ψ_i, be a "scattering solution," i.e., that $\Psi_f - \psi_f$ be everywhere outgoing at infinity. For specificity, let the final f channel correspond to propagation with particles 2 and 3 bound. Then an incident wave of energy E in the f channel is

$$\psi_f = e^{ikv}w(u) = \sqrt{\alpha}e^{ikv}e^{-\alpha|u|} , \qquad (2.12)$$

where E again satisfies Eq. (2.9a), and we again suppose $k>0$. Referring now to Eqs. (2.2) and (2.4), as well as to Fig. 1, it can be seen that the scattering solution Ψ_f associated with the incoming wave (2.12) is obtainable directly from Eqs. (2.10) provided the following changes are made: $(x,y)\to(u,v)$, $(u,v)\to(r,s)$, $(r,s)\to(x,y)$, V→I, VI→III, IV→V, II→VI, I→IV, III→II. More particularly, the formulas for Ψ_f in the various sectors of Fig. 1 are

$$\begin{aligned}
\Psi_{f\mathrm{I}} &= \sqrt{\alpha}e^{ikv}e^{\alpha u}+Ce^{iky}e^{\alpha x}+De^{iks}e^{-\alpha r} ,\\
\Psi_{f\mathrm{II}} &= Be^{ikv}e^{\alpha u}+Ce^{iky}e^{-\alpha x} ,\\
\Psi_{f\mathrm{III}} &= \sqrt{\alpha}e^{ikv}e^{-\alpha u}+De^{iky}e^{\alpha x}+Ce^{iks}e^{-\alpha r} ,\\
\Psi_{f\mathrm{IV}} &= Ae^{ikv}e^{\alpha u} ,\\
\Psi_{f\mathrm{V}} &= Be^{ikv}e^{-\alpha u}+Ce^{iks}e^{\alpha r} ,\\
\Psi_{f\mathrm{VI}} &= Ae^{ikv}e^{-\alpha u} ,
\end{aligned} \qquad (2.13)$$

with A,B,C,D still given by Eqs. (2.11). As with Ψ_i of Eqs. (2.10), the fact that Eqs. (2.13) guarantee Ψ_f is a scattering solution of Eq. (2.6a) consistent with the boundary conditions (2.7) easily can be checked.

III. OUTGOING GREEN'S FUNCTION FOR INCIDENT CHANNEL

In due course, we will verify that Ψ_i of Eqs. (2.10) satisfies the LS equation (1.1). First, however, we must construct the outgoing Green's function $G_i^{(+)}$ in the i channel. Everyone agrees that, with the stipulation $\epsilon > 0$,

$$G_i^{(+)}(E) = \lim_{\epsilon \to 0} G_i(E + i\epsilon) , \tag{3.1a}$$

where for complex λ

$$G_i(\lambda) = \frac{1}{H_i - \lambda} . \tag{3.1b}$$

In less symbolic notation, Eq. (3.1b) becomes[19]

$$G_i(x,y;x',y';\lambda) = \sum_n \frac{\psi_n(x,y)\psi_n^*(x',y')}{E_n - \lambda} , \tag{3.1c}$$

where ψ_n are a complete orthonormal set of eigenfunctions of H_i having energy E_n, and $\sum_n$ denotes the sum over all discrete eigenvalues plus integration over all continuous eigenvalues. The initial Hamiltonian H_i is given by Eq. (2.9b), as previously explained. Indeed, we see from Eq. (3.1c) that

$$\left[-\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] - g\delta(x) - \lambda \right] G_i(x,y;x',y';\lambda) = \delta(x - x')\delta(y - y') . \tag{3.2}$$

The definition (3.1a) is meaningful because for complex λ Eq. (3.2) has at most one quadratically integrable solution G_i; were there two such solutions, their difference would be a quadratically integrable eigenfunction of H_i for complex λ, impossible since H_i is Hermitian.

In the problem at hand, Eq. (3.1c) provides a convenient starting point for the computation of $G_i(\lambda)$, because finding the eigenfunctions and associated eigenvalues of H_i is straightforward. The eigenfunctions are plane waves in y multiplied by the easily obtained eigenfunctions of the operator $-(\hbar^2/2m)\partial^2/\partial x^2 - g\delta(x)$; the corresponding eigenvalues of H_i run from E_b of Eq. (2.8c) to ∞. However, it is necessary to be assured that the set ψ_n thus obtained is complete, i.e., that $\sum_n \psi_n \psi_n^*$ is a resolution of the identity;[20] otherwise Eq. (3.1c) is not equivalent to Eq. (3.1b), and G_i of Eq. (3.1c) does not satisfy Eq. (3.2). Although we have no doubt that the set ψ_n we have described is complete, the completeness might be questioned (we suppose) on the grounds that H_i involves the singular potential $g\delta(x)$. We certainly do not want to prove the completeness of the eigenfunctions ψ_n of H_i, which would take us far afield.

Consequently, because a major objective of this paper is to *remove* doubts about the nonuniqueness of solutions to the LS equation, we shall not construct G_i from Eq. (3.1c); we employ instead a method which is completely unexceptionable. Let $G_1(x;x';\lambda)$ be the Green's function satisfying

$$(H_1 - \lambda)G_1(x;x';\lambda) = \delta(x - x') , \tag{3.3a}$$

where

$$H_1 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - g\delta(x) . \tag{3.3b}$$

As with G_i for Eq. (3.2), for complex λ there can be only one quadratically integrable G_1 satisfying Eq. (3.3a).

Except at $x=0$ and $x=x'$, where the slope of G_1 is discontinuous, solutions to Eq. (3.3a) are linear combinations of $e^{\pm iKx}$, with

$$K = \left[\frac{2m}{\hbar^2} \right]^{1/2} \sqrt{\lambda} . \tag{3.4}$$

Specify the phase of complex λ by

$$0 < \arg\lambda < 2\pi , \tag{3.5a}$$

i.e.,

$$0 < \arg\sqrt{\lambda} < \pi , \qquad 0 < \arg K < \pi . \tag{3.5b}$$

Then for all λ constrained by Eq. (3.5a), the quadratically integrable G_1 must be proportional to e^{iKx} as $x \to \infty$, but must be proportional to e^{-iKx} as $x \to -\infty$. In view of this fact, introduce the functions φ_a and φ_b satisfying

$$(H_1 - \lambda)\varphi = 0 \tag{3.6a}$$

or

$$\left[\frac{\partial^2}{\partial x^2} + 2\alpha\delta(x) + K^2 \right] \varphi = 0 , \tag{3.6b}$$

where φ_b is proportional to e^{iKx} as $x \to \infty$, φ_a is proportional to e^{-iKx} as $x \to -\infty$. Evidently the solutions $\varphi(x)$ to Eq. (3.6b) satisfy the same boundary conditions (2.7) at $x=0$ as were previously stated for Ψ, with $\partial/\partial n$ in (2.7) now denoting $\partial/\partial x$. It is easily verified that solutions to Eq. (3.6b) satisfying these boundary conditions are

$$\begin{aligned} \varphi_a(x,K) &= \frac{iKe^{-iKx}}{\alpha + iK}, \quad x < 0 \\ \varphi_a(x,K) &= e^{-iKx} - \frac{\alpha}{\alpha + iK} e^{iKx}, \quad x > 0 \\ \varphi_b(x,K) &= e^{iKx} - \frac{\alpha}{\alpha + iK} e^{-iKx}, \quad x < 0 \\ \varphi_b(x,K) &= \frac{iK}{\alpha + iK} e^{iKx}, \quad x > 0 . \end{aligned} \tag{3.7}$$

The Green's function G_1 now is given by the formula

$$\begin{aligned} G_1(x;x';\lambda) &= -\left[\frac{2m}{\hbar^2} \right] \frac{\varphi_a(x,K)\varphi_b(x',K)}{W(\varphi_a,\varphi_b)}, \quad x < x' \\ G_1(x;x';\lambda) &= -\left[\frac{2m}{\hbar^2} \right] \frac{\varphi_b(x,K)\varphi_a(x',K)}{W(\varphi_a,\varphi_b)}, \quad x > x' \end{aligned} \tag{3.8}$$

where the Wronskian

$$W(\varphi_a,\varphi_b) = \varphi_a \frac{d\varphi_b}{dx} - \varphi_b \frac{d\varphi_a}{dx} \tag{3.9}$$

is a pure number, independent of x. Equations (3.7)—(3.9) yield the desired expression for G_1. However, it is necessary to use the appropriate forms from Eq. (3.7) for φ_a, φ_b in Eq. (3.8), depending on whether x and/or x' are >0 or <0. We find

$$G_1(x;x';\lambda)=\frac{mi}{\hbar^2 K}\left[e^{-iK(x-x')}-\frac{\alpha}{\alpha+iK}e^{-iK(x+x')}\right], \quad x<x'<0$$

$$G_1(x;x';\lambda)=\frac{mi}{\hbar^2 K}\left[e^{iK(x-x')}-\frac{\alpha}{\alpha+iK}e^{-iK(x+x')}\right], \quad x'<x<0$$

$$G_1(x;x';\lambda)=-\frac{m}{\hbar^2}\frac{e^{-iK(x-x')}}{\alpha+iK}, \quad x<0<x'$$

$$G_1(x;x';\lambda)=-\frac{m}{\hbar^2}\frac{e^{iK(x-x')}}{\alpha+iK}, \quad x'<0<x \tag{3.10}$$

$$G_1(x;x';\lambda)=\frac{mi}{\hbar^2 K}\left[e^{-iK(x-x')}-\frac{\alpha}{\alpha+iK}e^{iK(x+x')}\right], \quad 0<x<x'$$

$$G_1(x;x';\lambda)=\frac{mi}{\hbar^2 K}\left[e^{iK(x-x')}-\frac{\alpha}{\alpha+iK}e^{iK(x+x')}\right], \quad 0<x'<x\,.$$

An alternative to Eq. (3.1c) is

$$G_i(x,y;x',y';\lambda)=\sum_l \chi_l(y)\chi_l^*(y')G_1(x;x';\lambda-E_l)\,, \tag{3.11}$$

where χ_l are a complete orthonormal set of eigenfunctions of $-(\hbar^2/2m)\partial^2/\partial y^2$ [compare Eqs. (2.9b) and (3.3a)] having energy E_l, and the summation sign again includes integration over the continuous spectrum. It is readily verified that Eq. (3.11) does satisfy Eq. (3.2) when $\sum_l \chi_l\chi_l^*$ is a resolution of the identity in the one-dimensional y space. Now, however, in contrast to ψ_n of Eq. (3.1c), there can be no doubt about completeness, because the eigenfunctions of $-(\hbar^2/2m)\partial^2/\partial y^2$ are the plane waves

$$\chi_l \equiv \chi(y;k_y)=\frac{1}{\sqrt{2\pi}}e^{ik_y y} \tag{3.12a}$$

having energy

$$E_l=\frac{\hbar^2 k_y^2}{2m}\,. \tag{3.12b}$$

Thus we find, using Eq. (3.10), that for $x<x'<0$

$$G_i(x,y;x',y';\lambda)=\frac{1}{2\pi}\frac{mi}{\hbar^2}\int_{-\infty}^{\infty}dk_y\frac{e^{ik_y(y-y')}}{q}\times\left[e^{-iq(x-x')}-\frac{\alpha}{\alpha+iq}e^{-iq(x+x')}\right], \tag{3.13a}$$

where

$$q=\left[\frac{2m\lambda}{\hbar^2}-k_y^2\right]^{1/2}\,. \tag{3.13b}$$

The corresponding expressions for G_i in the other domains of x and x', e.g., in $x<0<x'$, are obtained similarly from Eqs. (3.10) and (3.11); we shall not take the space to write down these expressions. In Eq. (3.13a), and in the corresponding expressions for G_i valid in other domains of x and x', the values of y and y' are unrestricted.

In the k_y plane, q of Eq. (3.13b) has branch points at $k_y=\pm(2m\lambda/\hbar^2)^{1/2}$. Correspondingly, cuts through $k_y=\pm(2m\lambda/\hbar^2)^{1/2}$ are required; because k_y runs over all real values in the integral (3.13a), these cuts should not intersect the real k_y axis. Furthermore, it is implicit that q, which by virtue of Eq. (3.11) replaces K appearing in Eq. (3.10), is constrained in phase as was K in the quadratically integrable G_1. In other words, from Eq. (3.5b), we must have

$$0<\arg q<\pi \tag{3.14}$$

at all real values of k_y.

From Eq. (3.13b),

$$\arg q=\tfrac{1}{2}\arg\left[\left[\frac{2m\lambda}{\hbar^2}\right]^{1/2}-k_y\right]+\tfrac{1}{2}\arg\left[\left[\frac{2m\lambda}{\hbar^2}\right]^{1/2}+k_y\right]. \tag{3.15}$$

Suppose λ is such that $\sqrt{\lambda}$ lies in the first quadrant. It then can be seen that the requirements of the preceding paragraph can be met by drawing the cuts as shown in Fig. 2, with the understanding that at any k_y the values of the arguments in Eq. (3.15) are to be found by analytic continuation from $k_y=0$, where each of the arguments in Eq. (3.15) is equal to $\arg\sqrt{\lambda}$. For example, Fig. 2 shows the values of these arguments when k_y lies at the point A

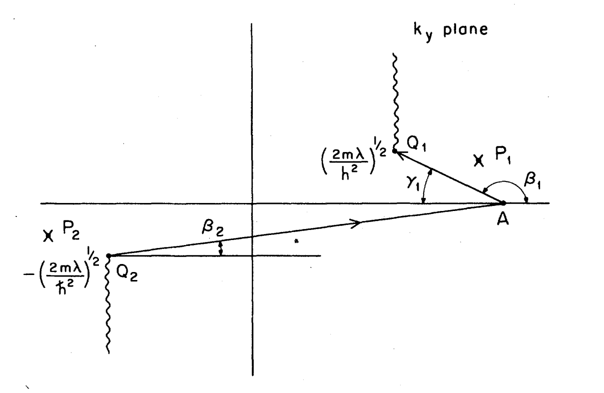

FIG. 2. Diagram showing the cuts and poles in the k_y plane for the integral of Eq. (3.13a). The cuts start at $Q_1=(2m\lambda/\hbar^2)^{1/2}$ and $Q_2=-(2m\lambda/\hbar^2)^{1/2}$. The poles P_1 and P_2 (located at the crosses) denote the values of k_y at which $\alpha+iq=0$. The diagram also shows the values of the arguments of $[(2m\lambda/\hbar^2)^{1/2}-k_y]$ and $[k_y+(2m\lambda/\hbar^2)^{1/2}]$; these values are β_1 and β_2, respectively, with $\beta_1+\gamma_1=\pi$. The figure has been drawn for $\sqrt{\lambda}$ lying in the first quadrant.

on the real axis. If AQ_1 is understood to be directed from A to Q_1, and Q_2A is directed from Q_2 to A, then

$$\arg\left[\left[\frac{2m\lambda}{\hbar^2}\right]^{1/2}-k_y\right]=\arg AQ_1=\beta_1\,,$$
$$\arg\left[\left[\frac{2m\lambda}{\hbar^2}\right]^{1/2}+k_y\right]=\arg\left[k_y--\left[\frac{2m\lambda}{\hbar^2}\right]^{1/2}\right]=\arg Q_2A=\beta_2\,, \tag{3.16a}$$

where it is obvious from the construction that

$$0<\beta_1<\pi\,,\qquad 0<\beta_2<\pi\,. \tag{3.16b}$$

Therefore, using Eq. (3.15), the inequality (3.14) necessarily holds. As a matter of fact, since $\gamma_1>\beta_2$ in Fig. 2 because $AQ_1<Q_2A$, and since $\beta_1=\pi-\gamma_1$, we actually have

$$0<\arg q<\frac{\pi}{2} \tag{3.16c}$$

when k_y is at A; indeed it can be seen that Eq. (3.16c) holds for any k_y on the real axis when the cuts are as shown in Fig. 2. However, though the inequality (3.14) continues to hold, $\arg q$ will exceed $\pi/2$ for real k_y when $\sqrt{\lambda}$ lies in the second quadrant.

In addition to the branch points at Q_1 and Q_2, G_i of Eq. (3.13a) has poles at

$$k_y=\pm\left[\frac{2m\lambda}{\hbar^2}+\alpha^2\right]^{1/2}\,, \tag{3.17a}$$

where

$$\alpha+iq=0\,. \tag{3.17b}$$

The locations of these poles, for the case that $\sqrt{\lambda}$ lies in the first quadrant, are given by the crosses at P_1 and P_2 in Fig. 2.

The foregoing completes the task of obtaining $G_i(x,y;x',y';\lambda)$. Incidentally, the result (3.13a) agrees with the result we would have obtained from Eq. (3.1c), and similarly for other (than $x<x'<0$) ranges of x,x', showing that the orthonormal set of eigenfunctions ψ_n of H_i indeed are complete. Next, recalling Eq. (3.1), we want the limit of Eq. (3.13a) as $\lambda\to E$ from values of λ having a positive imaginary part. Suppose for specificity, and as is surely sufficient for the purposes of this paper, that $E>0$, i.e., that $k^2>\alpha^2$ according to Eq. (2.9a). Then as $\lambda\to E$ the termini Q_1,Q_2 of the cuts in Fig. 2 will approach the real k_y axis; the poles P_1,P_2 given by Eq. (3.17a) similarly approach the real k_y axis. Correspondingly, as values of λ in the first quadrant approach $E>0$, the integration contour over k_y in Eq. (3.13a) must be deformed below the real axis near P_1 and Q_1, and above the real axis near P_2 and Q_2. Hence we find that

$$\begin{aligned}
G_i^{(+)}(x,y;x',y';E)&=\frac{1}{2\pi}\frac{mi}{\hbar^2}\int_\Gamma dk_y\frac{e^{ik_y(y-y')}}{p}\left[e^{-ip(x-x')}-\frac{\alpha}{\alpha+ip}e^{-ip(x+x')}\right],\quad x<x'<0\\
&=\frac{1}{2\pi}\frac{mi}{\hbar^2}\int_\Gamma dk_y\frac{e^{ik_y(y-y')}}{p}\left[e^{ip(x-x')}-\frac{\alpha}{\alpha+ip}e^{+ip(x+x')}\right],\quad x'<x<0\\
&=\frac{-1}{2\pi}\frac{m}{\hbar^2}\int_\Gamma dk_y e^{ik_y(y-y')}\frac{e^{-ip(x-x')}}{\alpha+ip},\quad x<0<x'\\
&=-\frac{1}{2\pi}\frac{m}{\hbar^2}\int_\Gamma dk_y e^{ik_y(y-y')}\frac{e^{ip(x-x')}}{\alpha+ip},\quad x'<0<x\\
&=\frac{1}{2\pi}\frac{mi}{\hbar^2}\int_\Gamma dk_y\frac{e^{ik_y(y-y')}}{p}\left[e^{-ip(x-x')}-\frac{\alpha}{\alpha+ip}e^{ip(x+x')}\right],\quad 0<x<x'\\
&=\frac{1}{2\pi}\frac{mi}{\hbar^2}\int_\Gamma dk_y\frac{e^{ik_y(y-y')}}{p}\left[e^{ip(x-x')}-\frac{\alpha}{\alpha+ip}e^{ip(x+x')}\right],\quad 0<x'<x
\end{aligned} \tag{3.18}$$

where

$$p=(k^2-\alpha^2-k_y^2)^{1/2}\,, \tag{3.19a}$$

the contour of integration Γ is as shown in Fig. 3, and the phase of p from (3.19a) is specified everywhere in the k_y plane by the understanding that near $k_y=0$

$$\arg[(k^2-\alpha^2)^{1/2}-k_y]\cong\arg[k_y+(k^2-\alpha^2)^{1/2}]\cong 0\,. \tag{3.19b}$$

With Eq. (3.19b) and the branch cuts as drawn, we have the following for real k_y,

when $k_y>(k^2-\alpha^2)^{1/2}$:

$$\arg[(k^2-\alpha^2)^{1/2}-k_y]=\pi\,,\qquad \arg[(k^2-\alpha^2)^{1/2}+k_y]=0\,,\qquad \arg p=\frac{\pi}{2}\,; \tag{3.20a}$$

when $-(k^2-\alpha^2)^{1/2}<k_y<(k^2-\alpha^2)^{1/2}$:

$$\arg[(k^2-\alpha^2)^{1/2}-k_y]=\arg[(k^2+\alpha^2)^{1/2}+k_y]=\arg p=0\,; \tag{3.20b}$$

when $k_y < -(k^2-\alpha^2)^{1/2}$:

$$\arg[(k^2-\alpha^2)^{1/2}-k_y]=0 ,$$
$$\arg[(k^2-\alpha^2)^{1/2}+k_y]=\pi , \quad (3.20c)$$
$$\arg p=\frac{\pi}{2} .$$

The phases of $\arg p$ in Eqs. (3.20a) and (3.20c) show that the points $k_y=\pm k$ indeed are poles of the integrals in Eq. (3.18), i.e., that p does indeed equal $i\alpha$, not $-i\alpha$, at P_1,P_2 of Fig. 3. We are not able to perform the integrals over k_y in Eq. (3.18), but we will be able to use this integral representation of $G_i^{(+)}(x,y;x',y',E)$, as will be seen.

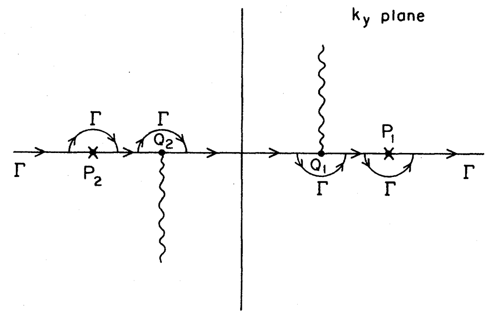

FIG. 3. The contour of integration Γ for the integrals of Eq. (3.18). The termini Q_1,Q_2 of the branch cuts are at $k_y=\pm(k^2-\alpha^2)^{1/2}$. The poles P_1,P_2 lie at $k_y=\pm k$. Figure is drawn on the assumption that $k^2>\alpha^2$; by definition, k is >0.

IV. NONUNIQUENESS OF SOLUTIONS TO THE LS EQUATION

A. Demonstration that Ψ_i satisfies the inhomogeneous LS equation

We will verify that the LS equation (1.1) holds for $G_i^{(+)}$ of Eq. (3.18) and Ψ_i of Eqs. (2.10) and (2.11). From comparison of Eqs. (2.6b) and (2.9b)

$$V_i \equiv H-H_i=-g\delta(u)-g\delta(r) . \quad (4.1)$$

Also, Eqs. (2.5) imply that the Jacobians of the transformations from x,y to u,v or from x,y to r,s, are equal to unity. Thus Eq. (1.1) for $\Psi\equiv\Psi_i$ is

$$\begin{aligned}\Psi_i(x,y)&=\psi_i(x,y)+g\int dx'dy'G_i^{(+)}(x,y;x',y';E)[\delta(u')+\delta(r')]\Psi_i(x',y')\\&=\psi_i(x,y)+g\int du'dv'G_i^{(+)}(x,y;x',y';E)\delta(u')\Psi_i(x',y')+g\int dr'ds'G_i^{(+)}(x,y;x',y';E)\delta(r')\Psi_i(x',y')\\&=\psi_i(x,y)+g\int_{-\infty}^{\infty}dv'[G_i^{(+)}(x,y;x',y';E)\Psi_i(x',y')]_{u'=0}+g\int_{-\infty}^{\infty}ds'[G_i^{(+)}(x,y;x',y';E)\Psi_i(x',y')]_{r'=0}\end{aligned} \quad (4.2)$$

with ψ_i given by Eq. (2.8a). Referring to Fig. 1, we see that on the line $u'=0$, $\Psi_i(x',y')$ in Eq. (4.2) has the form $\Psi_{i\mathrm{I}}$ or $\Psi_{i\mathrm{III}}$ for $v'<0$, but has the form $\Psi_{i\mathrm{IV}}$ or $\Psi_{i\mathrm{VI}}$ for $v'>0$; here it is indifferent whether we use $\Psi_{i\mathrm{I}}$ or $\Psi_{i\mathrm{III}}$, because these forms are equal on $u'=0$ by Eq. (2.7a), and similarly for the pair $\Psi_{i\mathrm{IV}}$ and $\Psi_{i\mathrm{VI}}$. Hence Eq. (4.2) becomes

$$\begin{aligned}\Psi_i(x,y)=\sqrt{\alpha}e^{iky}e^{-\alpha|x|}&+g\int_{-\infty}^{0}dv'(G_i^{(+)}\Psi_{i\mathrm{I}})_{u'=0}+g\int_0^{\infty}dv'(G_i^{(+)}\Psi_{i\mathrm{IV}})_{u'=0}\\&+g\int_{-\infty}^{0}ds'(G_i^{(+)}\Psi_{i\mathrm{II}})_{r'=0}+g\int_0^{\infty}ds'(G_i^{(+)}\Psi_{i\mathrm{III}})_{r'=0} .\end{aligned} \quad (4.3)$$

In Eq. (4.3) $G_i^{(+)}\equiv G_i^{(+)}(x,y;x',y';E)$ and $\Psi_i\equiv\Psi_i(x',y')$, of course.

Equation (2.5c) tells us that $x'=\frac{1}{2}\sqrt{3}v'$, $y'=-\frac{1}{2}v'$ on $u'=0$; similarly, $x'=-\frac{1}{2}\sqrt{3}s'$, $y'=-\frac{1}{2}s'$ on $r'=0$. Therefore, using Eq. (2.10), Eq. (4.3) can be rewritten as

$$\begin{aligned}\Psi_i(x,y)=\sqrt{\alpha}e^{iky}e^{-\alpha|x|}&+g\int_{-\infty}^{0}dv'G_i^{(+)}(x,y;\tfrac{1}{2}\sqrt{3}v';-\tfrac{1}{2}v';E)Ae^{(-ik+\sqrt{3}\alpha)v'/2}\\&+g\int_0^{\infty}dv'G_i^{(+)}(x,y;\tfrac{1}{2}\sqrt{3}v',-\tfrac{1}{2}v';E)(Be^{(-ik-\alpha\sqrt{3})v'/2}+Ce^{ikv'})\\&+g\int_{-\infty}^{0}ds'G_i^{(+)}(x,y;-\tfrac{1}{2}\sqrt{3}s',-\tfrac{1}{2}s';E)Ae^{(-ik+\alpha\sqrt{3})s'/2}\\&+g\int_0^{\infty}ds'G_i^{(+)}(x,y;-\tfrac{1}{2}\sqrt{3}s',-\tfrac{1}{2}s';E)(Be^{(-ik-\alpha\sqrt{3})s'/2}+Ce^{iks'}) .\end{aligned} \quad (4.4)$$

So far, the values of x and y in Eq. (4.4) have been unrestricted. To proceed further, we must choose the sign of x, so that we can determine which of the six possible formulas (3.18) must be used in the various terms of Eq. (4.4); there still are no restrictions on y. So suppose for the present that $x<0$. Then in the first integral on the right-hand side of Eq. (4.4), we must use the formula for $G_i^{(+)}$ in the range $x'<x<0$ when $\frac{1}{2}\sqrt{3}v'<x$, but must use the $x<x'<0$ formula for $G_i^{(+)}$ when $x<\frac{1}{2}\sqrt{3}v'$. In the second integral on the right-hand side of Eq. (4.4), we use only the $x<0<x'$ formula for $G_i^{(+)}$. Proceeding in this fashion, and actually substituting the appropriate formulas from Eq. (3.18), we find that for $x<0$ Eq. (4.4) reduces to

$$
\begin{aligned}
\Psi_i(x,y)-\sqrt{\alpha}e^{iky}e^{\alpha x}
&=\frac{i\alpha}{2\pi}\int_{-\infty}^{2x/\sqrt{3}}dv' Ae^{(-ik+\sqrt{3}\alpha)v'/2}\int_\Gamma dk_y\frac{e^{ik_y(y+v'/2)}}{p}\left[e^{ip(x-\sqrt{3}v'/2)}-\frac{\alpha}{\alpha+ip}e^{-ip(x+\sqrt{3}v'/2)}\right]\\
&+\frac{i\alpha}{2\pi}\int_{2x/\sqrt{3}}^{0}dv' Ae^{(-ik+\sqrt{3}\alpha)v'/2}\int_\Gamma dk_y\frac{e^{ik_y(y+v'/2)}}{p}\left[e^{-ip(x-\sqrt{3}v'/2)}-\frac{\alpha}{\alpha+ip}e^{-ip(x+3v'/2)}\right]\\
&-\frac{\alpha}{2\pi}\int_0^\infty dv'(Be^{-(ik+\alpha\sqrt{3})v'/2}+Ce^{ikv'})\int_\Gamma dk_y e^{ik_y(y+v'/2)}\frac{e^{-ip(x-\sqrt{3}v'/2)}}{\alpha+ip}\\
&-\frac{\alpha}{2\pi}\int_{-\infty}^{0}ds' Ae^{(-ik+\alpha\sqrt{3})s'/2}\int_\Gamma dk_y e^{ik_y(y+s'/2)}\frac{e^{-ip(x+\sqrt{3}s'/2)}}{\alpha+ip}\\
&+\frac{i\alpha}{2\pi}\int_0^{-2x/\sqrt{3}}ds'(Be^{-(ik+\alpha\sqrt{3})s'/2}+Ce^{iks'})\int_\Gamma dk_y\frac{e^{ik_y(y+s'/2)}}{p}\left[e^{-ip(x+\sqrt{3}s'/2)}-\frac{\alpha}{\alpha+ip}e^{-ip(x-\sqrt{3}s'/2)}\right]\\
&+\frac{i\alpha}{2\pi}\int_{-2x/\sqrt{3}}^{\infty}ds'(Be^{-(ik+\alpha\sqrt{3})s'/2}+Ce^{iks'})\int_\Gamma dk_y\frac{e^{ik_y(y+s'/2)}}{p}\left[e^{ip(x+\sqrt{3}s'/2)}-\frac{\alpha}{\alpha+ip}e^{-ip(x-\sqrt{3}s'/2)}\right]. \quad (4.5)
\end{aligned}
$$

We stress that in Eq. (4.5) the limit as $\epsilon\to 0$, which appeared in the defining equation (3.1a) for $G_i^{(+)}(E)$, has disappeared; the limit at $\epsilon\to 0$ has been taken while deriving Eq. (3.18). Correspondingly, all arguments in the literature[8–12,14–16]—concerning the validity or invalidity of taking the limit as $\epsilon\to 0$ in expressions such as the LS equation's key term $G_i^{(+)}V_i\Psi$—are totally irrelevant to what follows.

If inversion of the order of integration in Eq. (4.5) is permissible, the integrals over v' and s' can be performed. Consider, e.g., the third repeated integral on the right-hand of Eq. (4.5), wherein $v'\to\infty$, and let us concentrate on that portion of the contour Γ where $k_y\to\infty$. In these domains of v' and k_y, conditions for inversion of the order of integration of the pertinent repeated infinite integral in Eq. (4.5) [namely the third repeated integral on the right-hand side of Eq. (4.5)] can be stated. Specifically, sufficient conditions for the equality

$$\int_a^\infty dv'\int_b^\infty dk_y f(v',k_y)=\int_b^\infty dk_y\int_a^\infty dv' f(v',k_y) \quad (4.6a)$$

are essentially[22] (i) that $\int_a^\infty dv' f(v',k_y)$ converges uniformly in any fixed interval $b<k_y<K$, however large K; (ii) that $\int_b^\infty dk_y f(v',k_y)$ converges uniformly in any fixed interval $a<v'<V$, however large V; and (iii) that one of the infinite integrals

$$\int_a^\infty dv'\int_b^K dk_y f(v',k_y) \quad\text{or}\quad \int_b^\infty dk_y\int_a^v dv' f(v',k_y) \quad (4.6b)$$

converges uniformly in the respective unbounded interval $K<\infty$ or $V<\infty$.

The modifications of Eqs. (4.6) appropriate to the portion of Γ where $k_y\to-\infty$, or to the domain $v'\to-\infty$, or the integrals involving s' in Eq. (4.5), are completely obvious and need not be written down. Hence it can be seen that when the integrals over k_y are performed first in Eq. (4.5), those integrals do converge uniformly, because Eqs. (3.20) and the signs of x, v', and s' guarantee that the integrands in those integrals are not exponentially increasing as $k_y\to\pm\infty$ along Γ. For example, in the domain $-\infty<v'\le 2x/\sqrt{3}$, remembering we have specified $x<0$, we have

$$
\begin{aligned}
&x-\tfrac{1}{2}\sqrt{3}v'\ge 0\,,\\
&x+\tfrac{1}{2}\sqrt{3}v'<0\,.
\end{aligned} \quad (4.7)
$$

Thus for the double integral in Eq. (4.5) involving the range $-\infty<v'\le 2x/\sqrt{3}$, the exponents $ip(x-\frac{1}{2}\sqrt{3}v')$ and $-ip(x+\frac{1}{2}\sqrt{3}v')$ appearing in that double integral go to $-\infty$ as $k_y\to\pm\infty$, which is enough to guarantee uniform convergence of the k_y integration, except perhaps near $v'=2x/\sqrt{3}$, where $ip(x-\frac{1}{2}\sqrt{3}v')$ does not go to $-\infty$ because $x-\frac{1}{2}\sqrt{3}v'=0$; in the small interval near $v'=2x/\sqrt{3}$, however, the uniform convergence of the k_y integration over the term involving $\exp[ip(x-\frac{1}{2}\sqrt{3}v')]$ (in the $-\infty<v'\le 2x/\sqrt{3}$ domain) is guaranteed by the factor $p^{-1}\exp[ik_y(y+\frac{1}{2}v')]$, which essentially behaves like $(\sin k_y)/k_y$ at infinite k_y. The uniform convergence of the integrals over k_y for the other domains of v' in Eq. (4.5), and for the various domains of s', similarly are seen to be guaranteed by the appropriate generalizations of the inequalities (4.7) to those domains.

When the integration over v' in $-\infty<v'\le 2x/\sqrt{3}$ is performed before the integration over k_y in Eq. (4.5), the same inequalities (4.7) will guarantee uniform convergence of the v' integration when $\arg p=\pi/2$, i.e., when k_y is real and $|k|>|(k^2-\alpha^2)|^{1/2}$, recalling Eqs. (3.20). For other values of k_y on the contour Γ, especially for $-(k^2-\alpha^2)^{1/2}<k_y<(k^2-\alpha^2)^{1/2}$, where p is purely real according to Eq. (3.20b), the exponential factors involving p (relied on in the preceding paragraph) may not guarantee uniform convergence as the v' integration is performed. Uniform convergence of the v' integration over $-\infty<v'\le 2x/\sqrt{3}$ will be guaranteed, however, by the factor $\exp(\sqrt{3}\alpha v'/2)$ appearing in the corresponding integral of Eq. (4.5). Other integrals over v' and s' in Eq. (4.5) are similarly kept uniformly convergent by similar exponential factors; for instance, the factor $\exp(-\sqrt{3}\alpha s'/2)$ guarantees the uniform convergence of

integration over s' in the range $-2x/\sqrt{3} \le s' < \infty$.

On the other hand, not all the terms in Eq. (4.5) have integrals containing the appropriate factors $\exp(\pm\sqrt{3}\alpha v'/2)$ or $\exp(\pm\sqrt{3}\alpha s'/2)$; such factors specifically are missing from the integrals in Eq. (4.5) involving the numerical factor C defined in Eq. (2.11). To enable inversion of the order of integration for the term involving $Ce^{ikv'}$ in Eq. (4.5), an additional factor $e^{-\sigma v'}$ $(\sigma > 0)$ can be included in the integrand with the limit as $\sigma \to 0$ to be taken when the integration has been completed. This procedure is legitimate because the convergence of the integrals in Eq. (4.4) is not in question here; what is in question is only their uniform convergence when $G_i^{(+)}$ in Eq. (4.4) is replaced by its integral representation (3.18). In fact, Hobson[23] proves that when $\int_a^b dx\,\varphi(x)$ exists and is a continuous function of the upper limit b for all $b > a$, the limit of $\int_a^\infty dx\,\varphi(x)e^{-\sigma x}$ as $\sigma \to 0$ exists and equals $\int_a^\infty dx\,\varphi(x)$. Thus the extra factor $e^{-\sigma v'}$, with $\sigma \to 0$, can be introduced into the term having the factor $Ce^{ikv'}$ in Eq. (4.4) *before* $G_i^{(+)}(x,y;\frac{1}{2}\sqrt{3}v',-\frac{1}{2}v')$ is replaced by its representation from Eq. (3.18).

Adopting the procedure just described, for any fixed σ the integration over v'—in the range $0 < v' < \infty$, for the term in Eq. (4.5) having the factor $Ce^{ikv'}$ modified by the extra factor $e^{-\sigma v'}$—now will be uniformly convergent for values of k_y on Γ, provided Γ remains exactly on or infinitesimally near the real k_y axis, as we presently assume Γ does remain. Similar remarks pertain to the integrals involving $Ce^{iks'}$ in Eq. (4.5). Consequently, assuming provisionally that condition (iii) stated in connection with Eqs. (4.6) is satisfied, Eq. (4.5) can be replaced by

$$
\begin{aligned}
\Psi_i(x,y) - \sqrt{\alpha}e^{iky}e^{\alpha x}
&= \frac{i\alpha}{2\pi}\int_\Gamma dk_y A\frac{e^{ik_y y}}{p}\int_{-\infty}^{2x/\sqrt{3}} dv' e^{(-ik+\sqrt{3}\alpha)v'/2}e^{ik_y v'/2}\left[e^{ipx}e^{-ip\sqrt{3}v'/2} - \frac{\alpha}{\alpha+ip}e^{-ipx}e^{-ip\sqrt{3}v'/2}\right] \\
&+ \frac{i\alpha}{2\pi}\int_\Gamma dk_y A\frac{e^{ik_y y}}{p}\int_{2x/\sqrt{3}}^{0} dv' e^{(-ik+\sqrt{3}\alpha)v'/2}e^{ik_y v'/2}e^{-ipx}\left[e^{ip\sqrt{3}v'/2} - \frac{\alpha}{\alpha+ip}e^{-ip\sqrt{3}v'/2}\right] \\
&- \frac{\alpha}{2\pi}\int_\Gamma dk_y \frac{e^{ik_y y}}{\alpha+ip}B\int_0^\infty dv' e^{-(ik+\alpha\sqrt{3})v'/2}e^{ik_y v'/2}e^{-ipx}e^{ip\sqrt{3}v'/2} \\
&- \frac{\alpha}{2\pi}\int_\Gamma dk_y A\frac{e^{ik_y y}}{\alpha+ip}\int_{-\infty}^{0} ds' e^{(-ik+\alpha\sqrt{3})s'/2}e^{ik_y s'/2}e^{-ipx}e^{-ip\sqrt{3}s'/2} \\
&+ \frac{i\alpha}{2\pi}\int_\Gamma dk_y \frac{e^{ik_y y}}{p}B\int_0^{-2x/\sqrt{3}} ds' e^{-(ik+\alpha\sqrt{3})s'/2}e^{ik_y s'/2}e^{-ipx}\left[e^{-ip\sqrt{3}s'/2} - \frac{\alpha}{\alpha+ip}e^{ip\sqrt{3}s'/2}\right] \\
&+ \frac{i\alpha}{2\pi}\int_\Gamma dk_y \frac{e^{ik_y y}}{p}B\int_{-2x/\sqrt{3}}^{\infty} ds' e^{-(ik+\alpha\sqrt{3})s'/2}e^{ik_y s'/2}e^{ip\sqrt{3}s'/2}\left[e^{ipx} - \frac{\alpha}{\alpha+ip}e^{-ipx}\right] \\
&- \frac{\alpha}{2\pi}\lim_{\sigma\to 0}\int_\Gamma dk_y C\frac{e^{ik_y y}}{\alpha+ip}\int_0^\infty dv' e^{-\sigma v'}e^{ikv'}e^{ik_y v'/2}e^{-ipx}e^{ip\sqrt{3}v'/2} \\
&+ \frac{i\alpha}{2\pi}\lim_{\sigma\to 0}\int_\Gamma dk_y C\frac{e^{ik_y y}}{p}\int_0^{-2x/\sqrt{3}} ds' e^{-\sigma s'}e^{iks'}e^{ik_y s'/2}e^{-ipx}\left[e^{-ip\sqrt{3}s'/2} - \frac{\alpha}{\alpha+ip}e^{ip\sqrt{3}s'/2}\right] \\
&+ \frac{i\alpha}{2\pi}\lim_{\sigma\to 0}\int_\Gamma dk_y C\frac{e^{ik_y y}}{p}\int_{-2x/\sqrt{3}}^{\infty} ds' e^{-\sigma s'+iks'}e^{ik_y s'/2}e^{ip\sqrt{3}s'/2}\left[e^{ipx} - \frac{\alpha}{\alpha+ip}e^{-ipx}\right]. \qquad (4.8)
\end{aligned}
$$

Introduction of the factor $e^{-\sigma s'}$ is not really necessary in the next to the last double integral in Eq. (4.8), involving integration over the finite s' interval $0 \le s' \le -2x/\sqrt{3}$, but does no harm. Note that the signs of the exponents in all terms in Eq. (4.8), whether proportional to $e^{\pm ip\sqrt{3}v'/2}$ or $e^{\pm ip\sqrt{3}s'/2}$, always are such that—by virtue of Eqs. (3.20)—convergence of the integrals over v' and s' as v' or $s' \to \pm\infty$ is guaranteed, for large $|k_y|$ on the real k_y axis.

Performing the integrals over v' and s' in Eq. (4.8) (for $x < 0$ only, we remember) yields, after considerable but quite straightforward algebra,

$$\Psi_i(x,y)-\sqrt{\alpha}e^{ik_y y}e^{\alpha x}$$

$$=\frac{i\alpha}{\pi}\int_\Gamma dk_y\frac{e^{ik_y y}}{p}\left[Ae^{[\alpha+i(k_y-k)/\sqrt{3}]x}\left[\frac{1}{i(k_y-k)+\sqrt{3}(\alpha-ip)}-\frac{1}{i(k_y-k)+\sqrt{3}(\alpha+ip)}\right]\right.$$
$$\left.+Be^{[\alpha-i(k_y-k)/\sqrt{3}]x}\left[\frac{1}{i(k_y-k)-\sqrt{3}(\alpha+ip)}-\frac{1}{i(k_y-k)-\sqrt{3}(\alpha-ip)}\right]\right]$$
$$+\frac{i\alpha}{\pi}\lim_{\sigma\to 0}\int_\Gamma dk_y\frac{e^{ik_y y}}{p}Ce^{[\sigma-i(k+k_y/2)]2x/\sqrt{3}}\left[\frac{1}{-2\sigma+i(2k+k_y-p\sqrt{3})}-\frac{1}{-2\sigma+i(2k+k_y+p\sqrt{3})}\right]$$
$$+\frac{i\alpha}{\pi}\int_\Gamma dk_y\frac{e^{ik_y y}}{p}e^{-ipx}\left[A\left[\frac{1}{i(k_y-k)+\sqrt{3}(\alpha+ip)}-\frac{\alpha-ip}{\alpha+ip}\frac{1}{i(k_y-k)+\sqrt{3}(\alpha-ip)}\right]\right.$$
$$\left.+B\left[\frac{-1}{i(k_y-k)-\sqrt{3}(\alpha+ip)}+\frac{\alpha-ip}{\alpha+ip}\frac{1}{i(k_y-k)-\sqrt{3}(\alpha-ip)}\right]\right]$$
$$+\frac{i\alpha}{\pi}\lim_{\sigma\to 0}\int_\Gamma dk_y\frac{e^{ik_y y}}{p}e^{-ipx}C\left[\frac{-1}{-2\sigma+i(2k+k_y-p\sqrt{3})}+\frac{\alpha-ip}{\alpha+ip}\frac{1}{-2\sigma+i(2k+k_y+p\sqrt{3})}\right]. \quad (4.9)$$

We now return to condition (iii) associated with Eqs. (4.6). As we have explained, the written form of Eqs. (4.6) is pertinent to inversion of the order of integration in the third repeated integral on the right-hand side of Eq. (4.5). The term proportional to B in that particular repeated integral, after inversion of the order of integration, became the third repeated integral on the right-hand side of Eq. (4.8). If, e.g., the v' integration in this third integral on the right-hand side of Eq. (4.8) had run to an upper limit V rather than to ∞ [as in the double integral on the right-hand side of Eq. (4.6b)], we would have found an additional term in the integrand of Eq. (4.9), proportional to $B\exp[V(-ik-\alpha\sqrt{3}+ik_y+ip\sqrt{3})/2]$. But this additional term in Eq. (4.9) would be exponentially decreasing for large $|k_y|$ as $V\to\infty$, because of Eqs. (3.20). Therefore there is no doubt that for this extra term just described, $\int dk_y$ in Eq. (4.9) would converge uniformly for values of V in the unbounded interval $V<\infty$, as condition (iii) associated with Eqs. (4.6) demands. Similar remarks obviously pertain to other extra terms which would appear in Eq. (4.9) if the $\pm\infty$ integration limits of v' or s' in Eq. (4.8) were replaced by $\pm V$ or $\pm S$, respectively, with V and S large and >0. Moreover, the infinite integrals over k_y already present in Eq. (4.9) are convergent because their integrands all are of order k_y^{-2} at infinite k_y, even when those integrands do not have an exponentially decreasing factor e^{-ipx} at $|k_y|\to\infty$ ($x<0$, remember).

It follows from the preceding paragraph that, as applied to the integrals in Eq. (4.5), condition (iii) of Eqs. (4.6)—or of the appropriate obvious modification of Eqs. (4.6)—is satisfied when the factors $e^{-\sigma v'}$ and $e^{-\sigma s'}$ are properly included in Eq. (4.5), as discussed above. We already have explained that conditions (i) and (ii) of Eqs. (4.6) (or of their appropriate obvious modifications) are satisfied for every fixed $\sigma>0$ in Eq. (4.9). Thus we now can conclude that the manipulations—introducing the limits as $\sigma\to 0$ and then inverting the order of integration—which led from Eq. (4.5) to Eq. (4.8), and thence ultimately to Eq. (4.9), indeed were legitimate. In Eq. (4.9) however, the limit as $\sigma\to 0$ can be taken under the integral sign by virtue of the following theorem, also proved by Hobson:[24] If $\int_a^\infty dk_y\,|\varphi(k_y)|$ exists, and if $|f(k_y,\sigma)|$ is bounded for all $a\le k<\infty$ in a domain $0<\sigma\le\epsilon$ ($\epsilon>0$), then

$$\lim_{\sigma\to 0}\int_a^\infty dk_y\varphi(k_y)f(k_y,\sigma)=\int_a^\infty dk_y\varphi(k_y)f(k_y,0)\,. \quad (4.10)$$

For example, in the first term involving σ on the right-hand side of Eq. (4.9), write

$$\frac{1}{p}\frac{1}{-2\sigma+i(2k+k_y-p\sqrt{3})}=\frac{1}{p[i(2k+k_y-p\sqrt{3})]}\frac{1}{1+\dfrac{2i\sigma}{2k+k_y-p\sqrt{3}}}\,. \quad (4.11\text{a})$$

Then if we denote

$$\varphi(k_y)=\frac{1}{p(2k+k_y-p\sqrt{3})}\,, \quad (4.11\text{b})$$

the requirements for the relation (4.10) are seen to be satisfied at $|k_y|\to\infty$, the only domain of k_y in Eq. (4.9) where taking the limit $\sigma\to 0$ under the integral sign might be questioned; in particular, $|\varphi(k_y)|$ of Eq. (4.11b), being of order k_y^{-2}, is integrable as $k_y\to\infty$. Similar arguments pertain to the limit as $k_y\to-\infty$ in the first term involving σ on the right-hand side of Eq. (4.9), as well as to the other terms involving σ in Eq. (4.9).

Setting $\sigma=0$ under the integral sign in Eq. (4.9) we find—though it hardly seems possible—that in the integrand of Eq. (4.9) the combination of terms proportional to e^{-ipx} vanishes identically. The algebra needed to verify this assertion is tedious, but straightforward; it is necessary to make explicit use of Eqs. (2.11). The remaining terms in Eq. (4.9) (the terms whose integrands do not involve e^{-ipx}) also simplify very considerably, with the result that Eq. (4.9) becomes

$$\Psi_i(x,y)-\sqrt{\alpha}e^{iky}e^{\alpha x}=\frac{\alpha\sqrt{3}}{2\pi}\int_\Gamma dk_y\left[Ae^{(\alpha-ik/\sqrt{3})x}e^{ik_y(y+x/\sqrt{3})}\frac{1}{(k_y-k)(k_y-k_1)}+Be^{(\alpha+ik/\sqrt{3})x}e^{ik_y(y-x/\sqrt{3})}\frac{1}{(k_y-k)(k_y-k_2)}+Ce^{-2ikx/\sqrt{3}}e^{ik_y(y-x/\sqrt{3})}\frac{1}{(k_y-k_1)(k_y-k_2)}\right], \tag{4.12a}$$

where

$$k_1=\tfrac{1}{2}(-k+i\alpha\sqrt{3}),\quad k_2=\tfrac{1}{2}(-k-i\alpha\sqrt{3}). \tag{4.12b}$$

The contour Γ in Eq. (4.12a) still is the contour shown in Fig. 3. However, because the integrand in Eq. (4.12a) no longer contains the double-valued quantity p defined by Eq. (3.19a), the branch cuts drawn in Fig. 3 can be discarded. It also will be noted that in Eq. (4.12a) there no longer is a pole at $k_y=-k$ (we recall that k has been specified >0). Thus, in Eq. (4.12a) the contour Γ of Fig. 3 can be replaced by the contour Γ of Fig. 4; Fig. 4 also shows the poles k_1,k_2 of the integrand (4.12a).

We further recall that, starting with Eq. (4.5), our results have been restricted to the domain $x<0$. Thus our present task, which is to verify that Eq. (4.12a) indeed is true, is confined to verification for the three sectors I, III, and V shown in Fig. 1. Referring to Eqs. (2.5), we see that the interiors of these sectors are defined by the inequalities

$$\text{I: } y>-\frac{x}{\sqrt{3}},\quad \text{III: } \frac{x}{\sqrt{3}}<y<-\frac{x}{\sqrt{3}},\quad \text{V: } y<\frac{x}{\sqrt{3}}, \tag{4.13}$$

with the understanding that $x<0$. With $x<0$, the inequality $y+x/\sqrt{3}>0$ implies $y-x/\sqrt{3}>0$; similarly, $y-x/\sqrt{3}<0$ implies $y+x/\sqrt{3}<0$. Moreover, when $y+x/\sqrt{3}>0$, the integration contour for the term in (4.12a) containing the factor $\exp[ik_y(y+x/\sqrt{3})]$ can be closed in the upper half-plane; when $y+x/\sqrt{3}<0$, the integration contour for that same term can be closed in the lower half-plane. The contours for the terms in (4.12a) involving $\exp[ik_y(y-x/\sqrt{3})]$ can be closed similarly.

Therewith, the integrals in Eq. (4.12a) can be evaluated by residues in the three sectors I, III, and V. We then find, without difficulty, that Eq. (4.12a) *is* consistent with Eq. (2.10). For instance, in sector I ($y+x/\sqrt{3}>0$, $y-x/\sqrt{3}>0$), evaluating (4.12a) by residues yields

$$\Psi_i(x,y)-\sqrt{\alpha}e^{iky}e^{\alpha x}=i\alpha\sqrt{3}\left[Ae^{(\alpha-ik/\sqrt{3})x}\left[e^{ik(y+x/\sqrt{3})}\frac{1}{(k-k_1)}+e^{ik_1(y+x/\sqrt{3})}\frac{1}{k_1-k}\right]+Be^{(\alpha+ik/\sqrt{3})x}e^{ik(y-x/\sqrt{3})}\frac{1}{(k-k_2)}+Ce^{-2ikx/\sqrt{3}}e^{ik_1(y-x/\sqrt{3})}\frac{1}{(k_1-k_2)}\right]. \tag{4.14}$$

Using Eqs. (2.5) we observe that

$$\left[\alpha-\frac{ik}{\sqrt{3}}\right]x+ik_1\left[y+\frac{x}{\sqrt{3}}\right]=-\alpha r+iks,\quad -\frac{2ikx}{\sqrt{3}}+ik_1\left[y-\frac{x}{\sqrt{3}}\right]=-\alpha r+iks. \tag{4.15}$$

Straightforward algebra now shows that Eq. (4.14) indeed is consistent with Eqs. (2.10); in particular, we find that the expression (4.14) reduces to $(A-\sqrt{\alpha})e^{iky}e^{\alpha x}$. The consistency of Eq. (4.12a) with Eqs. (2.10) in the sectors III and V defined by the inequalities (4.13) is shown similarly.

This completes our objective of verifying that Ψ_i of Eqs. (2.10) satisfies the LS equation (1.1) for $x<0$. It then is obvious by symmetry that Ψ_i of Eqs. (2.10) also will satisfy the LS equation for $x>0$; in any event explicit demonstration that Ψ_i of Eq. (2.10) also satisfies Eq. (1.1) for $x>0$ proceeds along the same lines as for $x<0$ and is equally successful, as the reader now should have no difficulty in ascertaining.

B. Demonstration that Ψ_f satisfies the homogeneous LS equation

We now shall verify that Ψ_f of Eqs. (2.13) satisfies Eq. (1.3). The equation to be verified now, corresponding to Eq. (4.3), is

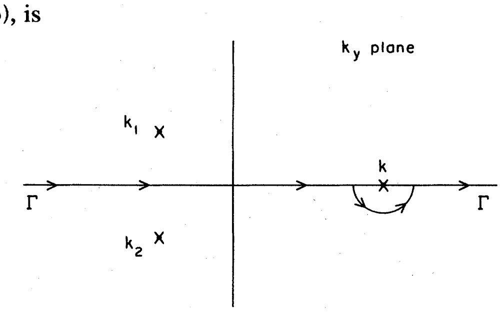

FIG. 4. Contour of integration Γ for Eq. (4.12a). Crosses denote poles. The values of k_1 and k_2 are given by Eq. (4.12b).

$$\Psi_f(x,y)=g\int_{-\infty}^{0}dv'(G_i^{(+)}\Psi_{f\mathrm{I}})_{u'=0}+g\int_{0}^{\infty}dv'(G_i^{(+)}\Psi_{f\mathrm{IV}})_{u'=0}+g\int_{-\infty}^{0}ds'(G_i^{(+)}\Psi_{f\mathrm{II}})_{r'=0}+g\int_{0}^{\infty}ds'(G_i^{(+)}\Psi_{f\mathrm{III}})_{r'=0}\,. \tag{4.16}$$

Again choosing $x<0$, the right-hand side of Eq. (4.16) becomes the obvious analog of the right-hand side of Eq. (4.5); we merely replace $\Psi_i(u'=0)$ by $\Psi_f(u'=0)$. Thus the analog of Eq. (4.5) is

$$\Psi_f(x,y)=\frac{i\alpha}{2\pi}\int_{-\infty}^{2x/\sqrt{3}}dv'(\sqrt{\alpha}e^{ikv'}+Ce^{(-ik+\sqrt{3}\alpha)v'/2}+De^{(-ik+\sqrt{3}\alpha)v'/2})$$
$$\times\int_\Gamma dk_y\frac{e^{ik_y(y+v'/2)}}{p}\left[e^{ip(x-\sqrt{3}v'/2)}-\frac{\alpha}{\alpha+ip}e^{-ip(x+\sqrt{3}v'/2)}\right]+\cdots\,, \tag{4.17}$$

where, to save space, we have written explicitly only the first integral on the right-hand side of Eq. (4.17); this integral has replaced the first integral on the right-hand side of Eq. (4.5). The contour Γ in Eq. (4.17) is the same as in Fig. 3, with the same poles and branch cuts.

It now is apparent from the untruncated version of Eq. (4.17) that inversion of the order of integration in Eq. (4.17) can be justified as in Eq. (4.5), namely by introducing the appropriate convergence factors $e^{\pm\sigma v'}$ or $e^{\pm\sigma s'}$ under the integral sign, in the limit $\sigma\to 0$ ($\sigma>0$); for example, in the very first term on the right-hand side of Eq. (4.17), we obviously must introduce the factor $e^{\sigma v'}$ to guarantee the necessary uniform convergence of the integration over v' in the range $-\infty<v'\le 2x/\sqrt{3}$. It also is quite apparent that at the very end, after the order of integration has been interchanged, and the integration over v' performed, the procedure of Eqs. (4.11), together with the theorem (4.10), will permit setting $\sigma=0$ under the remaining integral sign (for integration over k_y).

In this fashion, proceeding as previously, but after even more tedious (though still straightforward) algebra than before, we arrive at the following analog of Eq. (4.12a):

$$\Psi_f(x,y)=\frac{\alpha\sqrt{3}}{2\pi}\int_\Gamma dk_y\left[(C+D)e^{(\alpha-ik/\sqrt{3})x}e^{ik_y(y+x/\sqrt{3})}\frac{1}{(k_y-k)(k_y-k_1)}\right.$$
$$+(D+\sqrt{\alpha})e^{(\alpha+ik/\sqrt{3})x}e^{ik_y(y-x/\sqrt{3})}\frac{1}{(k_y-k)(k_y-k_2)}$$
$$\left.+\frac{\sqrt{\alpha}e^{2ikx/\sqrt{3}}e^{i(y+x/\sqrt{3})k_y}+Ce^{-2ikx/\sqrt{3}}e^{i(y-x/\sqrt{3})k_y}}{(k_y-k_1)(k_y-k_2)}\right]\,, \tag{4.18}$$

where Γ again is the contour of Fig. 4, with k_1,k_2 as in Eq. (4.12b). As Eq. (4.18) shows, once again there has occurred a miraculous cancellation of terms in the integral proportional to e^{-ipx}.

Evaluating the integrals in Eq. (4.18) by residues, we readily verify that Eq. (4.18) is consistent with Eqs. (2.13). For example, remembering $x<0$, evaluating (4.18) by residues in sector III ($y-x/\sqrt{3}>0$, $y+x/\sqrt{3}<0$) yields

$$\Psi_f(x,y)$$
$$=i\alpha\sqrt{3}\left[(D+\sqrt{\alpha})e^{(\alpha+ik/\sqrt{3})x}e^{ik(y-x/\sqrt{3})}\frac{1}{k-k_2}\right.$$
$$-\sqrt{\alpha}e^{2ikx/\sqrt{3}}e^{i(y+x/\sqrt{3})k_2}\frac{1}{k_2-k_1}$$
$$\left.+Ce^{-2ikx/\sqrt{3}}e^{ik_1(y-x/\sqrt{3})}\frac{1}{k_1-k_2}\right]\,. \tag{4.19}$$

Now supplementing Eqs. (4.15) by

$$\frac{2ikx}{\sqrt{3}}+ik_2\left[y+\frac{x}{\sqrt{3}}\right]=-\alpha u+ikv\,, \tag{4.20}$$

one readily sees that the right-hand side of Eq. (4.19) indeed does reduce to $\Psi_{f\mathrm{III}}$ of Eqs. (2.13).

Now, it is not possible to infer immediately that our having shown Ψ_f of Eq. (2.13) satisfies the homogeneous LS equation (1.3) for $x<0$ means Ψ_f also will satisfy Eq. (1.3) for $x>0$; Ψ_f (unlike Ψ_i) is not symmetric about the y axis. Nevertheless, Ψ_f does satisfy Eq. (1.3) for $x>0$; after the results already obtained, a contrary result for $x>0$ scarcely would be credible. We shall not detail the verification that Ψ_f satisfies the homogeneous LS equation when $x>0$.

V. CONCLUSION

The immediately preceding result indicates that the homogeneous LS equation has been correctly derived, and has solutions as previously predicted.[3] In addition, our explicit verifications that Ψ_i of (2.10) satisfies Eq. (1.1), and that Ψ_f of (2.13) satisfies Eq. (1.3), immediately provide an explicit demonstration that any

$$\Psi=\Psi_i+c\Psi_f \tag{5.1}$$

(c any number) satisfies (1.1). In other words, we have explicitly demonstrated that in our model the LS equation (1.1) does not have a unique solution.

Moreover, the nonuniqueness is broader than indicated by Eq. (5.1). In Eqs. (2.13), the final f channel corresponded to propagation with particles 2 and 3 bound, as explained in connection with Eqs. (2.12) and (2.13). However, because of the obvious symmetry of the problem, we just as readily could have let the final channel correspond to propagation with particles 1 and 3 bound [it will be recalled that in the "incident" i channel of Eq. (2.8a), particles 1 and 2 are bound]. Thus we just as readily could have constructed an analog $\hat{\Psi}_f$ of Eqs. (2.13) by making the following changes in Eqs. (2.10): $(x,y)\to(r,s)$, $(r,s)\to(u,v)$, $(u,v)\to(x,y)$, I→V, II→III, III→VI, IV→I, V→IV, VI→II. By symmetry, the scattering solution $\hat{\Psi}_f$ constructed in this fashion must satisfy Eq. (1.3) just as well as Ψ_f itself does. Therefore any

$$\Psi=\Psi_i+c\Psi_f+d\hat{\Psi}_f \tag{5.2}$$

also satisfies the LS equation (1.1), where c and d are any numbers.

The function Ψ_f satisfying Eq. (1.3) was constructed to be a scattering solution of the Schrödinger equation corresponding to particle 1 incident on the bound pair 2,3. This incoming part of Ψ_f is the term $\sqrt{\alpha}e^{ikv}e^{\alpha u}$ in $\Psi_{f\mathrm{I}}$ of Eqs. (2.13), or equivalently is the term $\sqrt{\alpha}e^{ikv}e^{-\alpha u}$ in $\Psi_{f\mathrm{III}}$. At $u=0$, these are finite-amplitude waves proceeding along increasing v. From Fig. 1 ones see that the boundary between sectors I and III corresponds to $v<0$, so that on this boundary a wave proceeding along increasing v is coming *in* from $v=-\infty$. Correspondingly, the scattered part $\Phi=\Psi-\psi_i$ [recall Eq. (1.4)] of any Ψ from Eq. (5.1) with $c\neq0$ is not everywhere outgoing at infinity, even though this Ψ satisfies the LS equation (1.1) with identical incoming wave ψ_i together with a kernel containing the outgoing Green's function $G_i^{(+)}$ for the same i channel. Of course, the solution to the LS equation (1.1) can be specified uniquely with the aid of the additional boundary condition requirement that the scattered part be everywhere outgoing, but our results illustrate that in three-particle systems—unlike two-particle systems—there can be no assurance that any given solution of Eq. (1.1) will satisfy this boundary condition, i.e., will have a purely outgoing Φ.

In sum, the results we have obtained in this paper are completely consistent with the earlier results[1-3] which have been criticized;[8-12] the results herein obtained also serve as counterexamples to those criticisms. Our present results also confirm that—in the inhomogeneous and homogeneous LS equations (1.1) and (1.3)—the Green's function can and should be computed at real energy E; as suggested earlier, the literature seems to be in disagreement on this point.[8-12,14-16,25]

It is true that our actual calculations have been limited to McGuire's very simple one-dimensional model of a three-particle system. Nevertheless, this simple model is sufficiently realistic to encompass scattering solutions manifesting particle rearrangement; as the preceding paragraphs have made manifest, the possibility of particle rearrangement—which permits Ψ_f to propagate inward from infinity in channels wherein $G_i^{(+)}$ does not propagate—is the key factor permitting the construction of solutions satisfying the homogeneous LS equation (1.3).

We conclude finally that any claims that the LS equation (1.1) has unique scattering solutions in three-particle systems first must explain why our results cannot be extrapolated to actual three-dimensional three-particle systems, as well as why "proofs" that the solutions are unique fail in one dimension but not in three dimensions. An abbreviated and much less detailed account of the foregoing results has appeared recently.[26]

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[21]Compare P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), p. 826.

[22]E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series* (Dover, New York, 1957), Vol. II, p. 350.

[23]E. W. Hobson, Ref. 22, p. 329, Ex. 3.

[24]E. W. Hobson, Ref. 22, p. 329, Ex. 2.

[25]S. Mukherjee, Phys. Lett. **100A**, 325 (1984).

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