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Fractal dimension function for energy levels

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To characterize fractals, a function is introduced which is a natural extension of the fractal dimension. This fractal dimension function is easily evaluated numerically and is amenable to a theoretical description. In particular, an intimate connection between fractal dimension and statistics is uncovered. As illustrative examples the concept is applied to sequences of random numbers and to the energy levels of coupled harmonic oscillators.

In recent years much attention has been paid to fractals. The investigations of fractal properties cover by now a long list of different subjects ranging from simple geography to the study of intricate mathematical models. A few examples are landscapes,[1] critical fluctuations in phase transitions,[2] percolation on fractal lattices,[3] states of quasiperiodic potentials,[4] turbulence,[5] and strange attractors.[6]

Dimension is perhaps the most basic property of a fractal. The notion of dimension is not unique and there are several relevant definitions of dimension available in the literature (see, e.g., Refs. 1 and 6). Here, we concentrate on metric properties of the fractal and, hence, the relevant quantity is the fractal dimension.[1] The fractal is viewed as a set which is a bounded subset of a d-dimensional Euclidean space and its fractal dimension D derives from[1,7]

$$N(\delta)\sim\delta^{-D} \tag{1}$$

as $\delta\rightarrow 0$. The quantity $N(\delta)$ is the minimum number of d-dimensional cubes of side δ needed to cover the fractal. In many examples of interest the validity of relation (1) persists for a large range of values of δ. This fact simplifies the numerical search for D and may explain why it is useful to retain the concept of a fractal dimension also for sets with an inner cutoff,[1] i.e., a length scale below which the fractal behavior subsides. Clearly, δ should not be chosen smaller than the inner cutoff.

Of course, the validity of relation (1) is at best approximate for finite δ and the fractal dimension obtained from it will depend on the choice of δ, i.e., $D=D(\delta)$. As will become clear below, this behavior is rather an advantage than a disadvantage. It is one goal of this contribution to illustrate that the function $D(\delta)$ is characteristic of the specific fractal under consideration and that its knowledge provides us with additional information on the system. To be more general, we consider a set with n elements. n can be either the true number of elements of a "fractal" with an inner cutoff or an intermediate number of elements obtained after some step in the construction of a fractal. An example for the latter is the well-known classic example of a Cantor set obtained by the limiting process of deleting middle thirds (as illustrated, for instance, in Fig. 80 of Ref. 1). After p steps in the construction of this set, we have $n=2^p$.

To proceed we introduce coarse graining of length δ into the description of the set. For the sake of illustration let us consider a set of n points in the interval $[0,1]$. Each point is covered by a bar of length δ with $\delta/2$ on each side of the point. Once δ exceeds the smallest distance between two points, the bars begin to overlap and the coarse graining is done by the union of the overlapping bars being now of lengths l_i, $i=1,\ldots,m$. For the number of bars, m, we obviously have $m\leq n$, and when δ reaches the size of the largest distance between two points the whole set is covered by a single bar, i.e., $m=1$. This concept of coarse-graining allows for a useful and unambiguous definition of a fractal dimension function $D=D(n,\delta)$ which is obtained from the equation[8]

$$\sum_{i=1}^{m} l_i^D=1 , \tag{2}$$

where, for a given set, m and the l_i solely depend on δ. Current evidence[1,9] supports the conjecture that for $n\rightarrow\infty$ and small enough δ, $D(n,\delta)$ takes on the same value as the fractal dimension defined by relation (1). For finite n and δ, definition (2) is superior to definition (1) both from the conceptual and practical points of view. For finite n and δ the proportionality constant in (1) also depends on n and δ and, consequently, $D(n,\delta)$ is ill defined. If $-d\ln N(\delta)/d\ln(\delta)$ is taken to be the approximate dimension, a smoothing of $N(\delta)$ is required, since $N(\delta)$ is a staircase function. Moreover, computation of $N(\delta)$ in general requires optimization of cube positions, whereas the evaluation of D from Eq. (2) is very simple. The determination of the l_i is straightforward and D is obtained by searching the zeros of the function $F(D)\equiv\sum l_i^D-1$ using standard methods.

The definition (2) of a fractal dimension function definitively has the advantage of being amenable to a theoretical description. We again consider a set of n points on the real axis. Let $P(S)$ be the distribution of the next-neighbor spacings S in our set of points. Two auxiliary functions $I_\nu(\delta)$, $\nu=0,1$, are introduced

$$I_\nu(\delta)=\int_\delta^\infty S^\nu P(S)dS \tag{3}$$

which fulfill the normalization conditions $I_0(0)=n$ and $I_1(0)=n\bar{S}$, where $\bar{S}$ is the mean next-neighbor distance. $I_0(\delta)$ gives the mean number of next-neighbor spacings larger than δ and hence the number $\bar{N}$ of bars available.

The mean next-neighbor gap $\bar{g}$ between bars is deduced from $\bar{g}=I_1(\delta)/I_0(\delta)-\delta$. Now, the identity $n\bar{S}=\bar{N}(\bar{g}+\bar{l})$ for the total size of the original interval comprising the set of points determines the mean length $\bar{l}$ of the bars. Putting [see Eq. (2)]

$$\bar{N}(\bar{l}/n\bar{S})^D=1 , \tag{4}$$

we obtain our final result for the fractal dimension function

$$D(n,\delta)=\frac{\ln I_0(\delta)}{\ln I_0(\delta)-\ln[I_1(0)-I_1(\delta)+\delta I_0(\delta)]+\ln I_1(0)} . \tag{5}$$

The denominator in Eq. (5) vanishes at some irrelevantly large value of δ at which the mean length of bars equals the total length of the interval, i.e., $\bar{l}=n\bar{S}$, for which D cannot be determined from Eq. (4).

To elucidate Eq. (5) we discuss an example. We consider a set of *n random* points in some bounded interval on the real axis. Using a standard procedure to generate uniformly distributed random numbers, the fractal dimension function has been calculated via Eq. (2) for $n=1000$, 5000, and 10 000. The results of this very simple computation are shown as circles in Fig. (1). To compare with the theoretical prediction of Eq. (5) we make use of the fact that the distribution of the next-neighbor spacings in a set of random numbers is the well-known Poisson distribution[10]

$$P(S)=(n/\bar{S})\exp(-S/\bar{S}) . \tag{6a}$$

The fractal dimension function which now reads

$$D(n,\delta)=1+\ln X/[\ln(n)-\delta/\bar{S}-\ln X] , \tag{6b}$$

where $X=1-\exp(-\delta/\bar{S})$, is also shown in Fig. 1 and compares enjoyably well with the numerical result. With growing n the random points gradually cover the corresponding interval which is fully covered for $n\rightarrow\infty$ leading to a fractal dimension $D=1$. Indeed, $D(n,\delta)$ in Eq. (6) monotonously approaches 1 as n is increased at fixed δ. For fixed n the function $D(n,\delta)$ is closer to 1 the larger the values of δ are, reflecting the higher degree of coarse graining introduced into the set.

It has been demonstrated above that fractal and statistical properties of sets are closely related to each other. Statistical methods have been extensively used to discuss the behavior of nuclear energy levels[11] and more recently also of atomic[12] and molecular levels.[13] As an important result of these investigations it has emerged that once the energy levels correspond to "complicated" states, they are amenable to statistical analysis using random matrix methods. In particular, the distribution of the next-neighbor spacings of these energy levels closely resembles the Wigner distribution[10]

$$P(S)=(\pi nS/2\bar{S}^2)\exp(-\pi S^2/4\bar{S}^2) . \tag{7a}$$

On the other hand, this distribution leads, via Eq. (5), to the following fractal dimension function

$$D(n,\delta)=1+\ln Y/[\ln(n)-\pi\delta^2/(4\bar{S}^2)-\ln Y] , \tag{7b}$$

where $Y=\Phi(\sqrt{\pi}\delta/2\bar{S})$ and Φ is the error function.[14] In analogy to our first example, we have computed the fractal dimension function using definition (2) and sequences of 1000, 5000, and 10 000 random numbers obeying the constraint of a Wigner distribution. These numerical results, shown as crosses in Fig. 1, are again in good agreement with the theoretical results [Eq. (7b)] drawn as solid

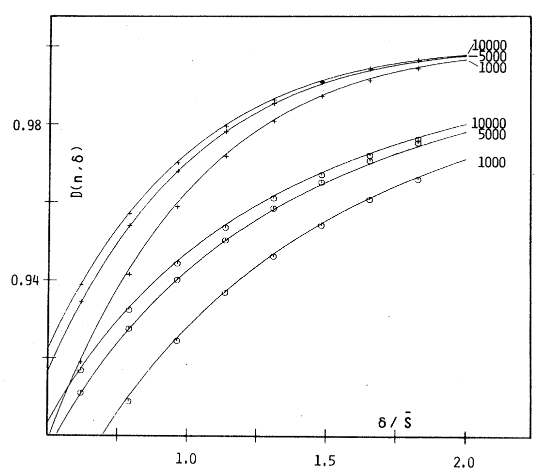

FIG. 1. Fractal dimension functions for sets of random numbers: $n=1000$, 5000, and 10 000. The circles and crosses denote numerically obtained results for uniformly distributed and Wigner-type random numbers, respectively. The corresponding theoretical results, Eqs. (6) and (7), respectively, are shown as solid curves.

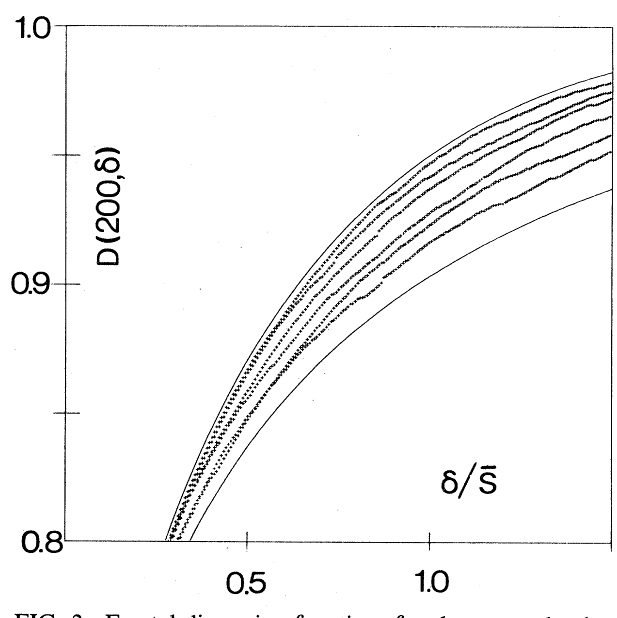

FIG. 2. Fractal dimension functions for the energy levels of the system of coupled oscillators (see the text). Each series of crosses has been computed using a piece of the spectrum with 200 energy levels. The "curves" move upwards with growing energy. The limiting functions $D(200,\delta)$ of purely regular and irregular sequences given by Eqs. (6) and (7), respectively, are shown as solid curves.

lines in the same figure. $D(n,\delta)$ behaves differently for the two types of sets, the "Wigner set" and "Poisson set," as a function of δ as well as of n although the fractal dimension D is equal in both cases ($D=1$). This underlines the usefulness of the concept of a fractal dimension function for the characterization of fractals with or without an inner cutoff.

Classical nonintegrable systems may exhibit a transition from quasiperiodic to chaotic motion (see, e.g., Refs. 15 and 16 for a review). In the case of quantum systems it seems more appropriate to adopt the terms "regular" and "irregular" motion[17] which have been subject to many investigations (see, e.g., Refs. 18 and 19). The distribution of nearest-neighbor spacings is used as a tool to distinguish between regular and irregular spectral sequences. Interestingly, the Poisson and Wigner distributions discussed above are the appropriate distributions associated with regular and irregular spectral sequences, respectively.[18,20]

The classical model system of two harmonic oscillators with equal frequencies coupled by a quartic term $\sim q_1^2 q_2^2$, where q_1 and q_2 are the coordinates of the oscillators, shows a transition from quasiperiodicity to chaoticity.[21] Recently it has been shown that the analogous quantum system exhibits a transition from regularity to irregularity.[22] The fractal dimension function may be used to demonstrate and characterize this transition. Starting at some energy, the spectrum of the system[23] is cut into pieces of 200 levels each. For each piece the fractal dimension function has been computed via Eq. (2) and depicted in Fig. 2. With growing energy the fractal dimension function withdraws from the one associated with the Poisson set and smoothly approaches the function associated with the Wigner set, thereby uncovering a transition from regularity to irregularity.

In conclusion we may say that the concept of a fractal dimension function leads to a powerful method to analyze fractals. This function provides additional information on the system, is easily evaluated numerically, and interrelates the notion of fractal dimension to statistics. The applications to subsets of one-dimensional Euclidean space have been chosen because of simplicity. Extensions to more general fractals should be straightforward.

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