PHYSICAL REVIEW A VOLUME 31, NUMBER 3 MARCH 1985

Double-diffusive convection and λ bifurcation

Jerry F. Magnan* and Edward L. Reiss
Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, Illinois 60201

(Received 16 May 1983; revised manuscript received 23 January 1984)

We analyze convection in a rectangular box where two "substances," such as temperature and a solute, are diffusing. The solutions of the Boussinesq theory depend on the thermal and solute Rayleigh numbers R_T and R_s, respectively, in addition to other geometrical and fluid parameters. As R_T is increased, the conduction state becomes linearly unstable with respect to steady (periodic) convection states if $R_s < \bar{R}_s$ ($> \bar{R}_s$). The critical value $\bar{R}_s$ is characterized by the frequency $\omega=0$ appearing as a root of algebraic multiplicity two and geometrical multiplicity one of the linearized stability theory. Asymptotic approximations of the solutions of the nonlinear theory are obtained for R_s near $\bar{R}_s$ by the Poincaré-Lindstedt method. It is found that a periodic (steady-state) solution bifurcates supercritically (subcritically) from the conduction state at $R_T=R_c^p$ (R_T^s), where $R_c^p < R_T^s$. The periodic branch joins the steady-state branch with an "infinite-period bifurcation" at $R_T=R_b$, where $R_c^p < R_b < R_T^s$. The shape of the resulting bifurcation diagram suggests the term, λ bifurcation. The infinite-period bifurcation corresponds to a heteroclinic orbit in the appropriate amplitude-phase plane. The stabilities of the bifurcation states are determined by solving the convection initial-value problem using the multiscale method.

I. INTRODUCTION

Double-diffusive convection differs from ordinary thermal convection because of the presence of a dissolved solute, such as salt, in the convecting fluid. The resulting solute diffusion may alter the stability characteristics of the fluid motion. For example, the transition from the conduction to the convection states may be delayed by this diffusion. More general double-diffusive convection, such as occurs in binary fluids, involves the diffusion of two general scalar fields.

The precise qualitative features of the transition from the conduction to the convection states as the thermal Rayleigh number R_T increases, depends on the magnitude of the solute Rayleigh number R_s. For $R_s>0$ (negative solute gradient) some of the features of the transitions to two-dimensional convection in an infinite layer with the Rayleigh boundary conditions, as known from analytical and numerical studies,[1–5] are summarized in Figs. 1 and 2. The quantity $\bar{R}_s$ in these figures can be further characterized as the value of R_s at which the convection states which branch from the conduction state at the lowest critical value of R_T change from steady to periodic. Extensive numerical studies[1,6] suggest more complex responses at larger amplitude motions.

In this paper we study double-diffusive convection in a rectangular box. We employ the Boussinesq theory and assume the Rayleigh or "slippery" conditions on the walls of the box. The side walls of the box are insulated from heat and solute flux. The temperature and solute concentrations are specified on the horizontal walls. The problem is formulated in dimensionless variables in Sec. II. The solutions depend on the six parameters R_T, R_s, σ, D, a, and b, where σ and D are the Prandtl and Schmidt numbers, respectively, and a and b are the aspect ratios of the convection box.

The linearized theory for the stability of the conduction state is summarized in Sec. III and Figs. 1 and 2. We consider values of the aspect ratios a and b for which the convection states branching from the critical value of R_T

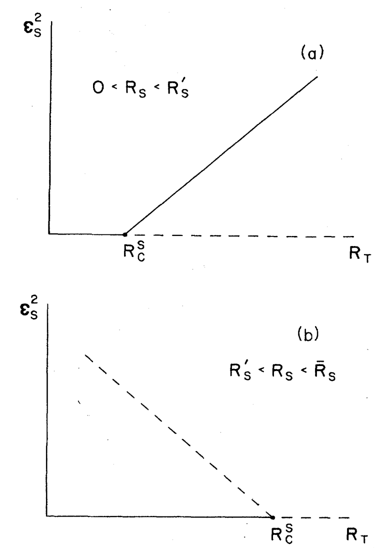

FIG. 1. Bifurcation diagrams for steady-state solution branches, where R_s' is defined in (4.5) and $\bar{R}_s$ is defined in (3.12). Small parameter ϵ_s is an amplitude of the steady convection states. Branches bifurcate, supercritically in (a) (as in ordinary thermal convection), and subcritically in (b) (which does not occur in ordinary thermal convection), from the conduction branch ($\epsilon_s=0$) at $R_T=R_c^s$. Thus the critical value R_s' separates supercritical from subcritical steady-state bifurcation. Dashed lines indicate the linearly unstable solution branches.

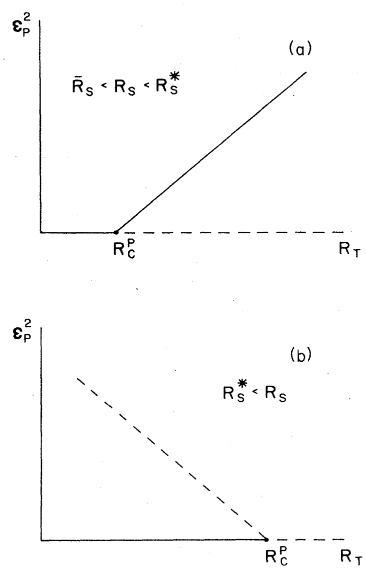

FIG. 2. Bifurcation diagrams for periodic solution branches, where R_s^* is a critical value of R_s that is defined in Sec. V. It separates supercritical from subcritical bifurcation of periodic solutions. Small parameter ϵ_p is an amplitude of the periodic convection states. Bifurcation of periodic solutions from the conduction state, which is a Hopf bifurcation, is not possible in ordinary thermal convection. Dashed lines indicate linearly unstable solution branches. Thus for $R_s > R_s^*$ the critical bifurcation point is a Hopf bifurcation point and it corresponds to unstable periodic solutions branching from the conduction state.

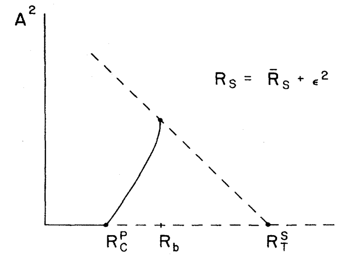

FIG. 3. Bifurcation diagram for R_s slightly greater than $\bar{R}_s$. Amplitudes of the periodic (steady-state) branches are $A = \pm A_0$ $(=\pm\sqrt{\alpha/\beta})$. Periodic (steady-state) branch bifurcates supercritically (subcritically) at $R_T = R_c^p$ $(=R_T^s)$ from the conduction branch ($A=0$). At $R_T = R_b$ the periodic and steady states are "joined" by an infinite-period bifurcation. Dashed lines indicate the linearly unstable solution branches. The periodic branch is orbitally stable as we have discussed in Sec. VIII.

are spatially two dimensional. The critical thermal Rayleigh number R_c is the smallest bifurcation point of the conduction state. We denote this critical value by R_c^p (R_c^s) if the corresponding convection state is periodic (steady). For $R_s > \bar{R}_s$ $(<\bar{R}_s)$ we have $R_c = R_c^p$ (R_c^s) such that $R_c^p \to R_c^s$ as $R_s \to \bar{R}_s$; i.e., the value $\bar{R}_s$ corresponds to a coalescence point of a steady and a periodic bifurcation point. In addition, there are no periodic bifurcation points of the conduction state for $R_s < \bar{R}_s$. In Sec. IV and V and Figs. 1 and 2 we summarize the nonlinear perturbation analysis of the convection states that branch from R_c^p and R_c^s. We find a singularity in the frequency of the periodic solutions that branch from R_c^p as $R_s \to \bar{R}_s$. Thus, the perturbation method (Poincaré-Lindstedt) is invalid as $R_s \to \bar{R}_s$. In addition, the frequency $\omega = 0$ is a double eigenvalue of the linearized theory for $R_s = \bar{R}_s$. The nonlinear interaction of a periodic and a steady-state mode near a double-zero eigenvalue was discussed in Ref. 7 for ordinary differential equations and the possible bifurcations were classified.

In Secs. VI and VII we analyze the convection states branching from the conduction state for R_s near $\bar{R}_s$ by a systematic perturbation method. The response is summarized in the bifurcation diagram in Fig. 3, where A^2 is the square of an amplitude of the fluid motion. We refer to this response as λ bifurcation because of the shape of the figure. A periodic solution branches supercritically from R_c^p and a steady-state solution branches subcritically from R_T^s, where $R_c^p < R_T^s$. The periodic branch terminates at $R_T = R_b$, as shown. The period of the solution varies with R_T. For R_T near R_c the period is "large." However, as $R_T \to R_b$ the period becomes infinite, so that the solutions on the periodic branch as $R_T \to R_b$ approach a heteroclinic orbit, see Fig. 4. Thus, R_b is not a secondary bifurcation point, in the usual sense, of the steady states branching from R_T^s. The linearized stability of these solutions to three-dimensional disturbances is analyzed in Sec. VIII by solving the convection initial-value problem by a multitime method. It is found that the periodic states have asymptotic orbital stability for $R_T^p < R < R_b$, in the sense discussed in Sec. VIII. The steady states are unstable for $R_T < R_T^s$.

The perturbation analysis in Secs. VI and VII, which is essentially the Poincaré-Lindstedt method in the small parameter ϵ, defined by $\epsilon^2 \equiv R_s - \bar{R}_s$, contains a novel feature. It is found that the amplitude A satisfies a Duffing equation, which has a one-parameter family of periodic solutions, as shown in Fig. 4. This is unusual because the Duffing equation corresponds to a conservative system and the convection problem is a dissipative system. However, the initial conditions and the coefficients in the

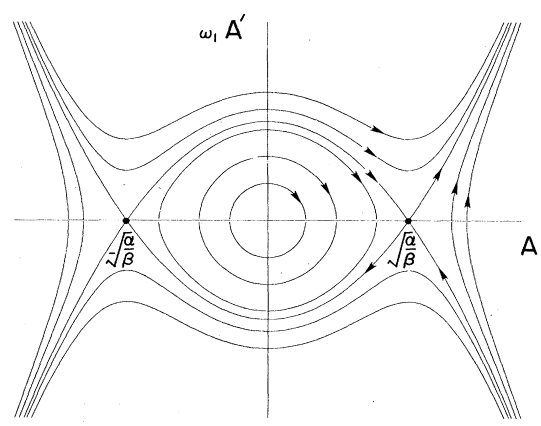

FIG. 4. Phase plane diagram for the Duffing equation (6.11a) for a particular value of r_2 in the interval $r_2(0) \le r_2 \le r_2(1)$. For each r_2 in this interval only one of the closed orbits in the figure, i.e., the one which satisfies (7.3b) and (7.6), is a periodic solution of the convection problem.

Duffing equation depend on R_T. This dependence is determined by analyzing higher-order terms in the Poincaré-Lindstedt expansion. Thus, our analysis determines the amplitude as a unique function of R_T, or equivalently selects for each R_T a unique member of the one-parameter family of periodic solutions. This suggests another mathematical mechanism to determine isolated periodic solutions, other than the conventional limit cycles.

Double-diffusive convection is an infinite layer with R_s near $\overline{R}_s$ has been analyzed previously using different methods. Approximate solutions of the Boussinesq theory were obtained in Ref. 8 by an amplitude expansion in terms of specified spatial modes. Then by considering a five-mode truncation of this system the time-dependent amplitudes of these modes were found to satisfy a system of first-order, ordinary differential equations which depend on the parameter ϵ. These equations, which were originally derived in Ref. 2 (see also Refs. 3 and 9), are then reduced in Ref. 8 by an iteration procedure to a single third-order equation representing a singular perturbation problem. An approximation is then made by setting $\epsilon=0$ in this equation which eliminates the third derivative term. A Duffing equation is thus obtained for the amplitude which is related to the Duffing equation obtained in the present work. The evolution equation for the first constant of integration in the solution of the Duffing equation is then obtained by an averaging procedure on the third-order equation. From this equation the bifurcation diagram and its stabilities with respect to special two-dimensional disturbances are obtained. The second integration constant is omitted, however, since its slow time dependence is not considered in Ref. 8. Although the analysis in Ref. 8 is somewhat nonuniform (some of these nonuniformities are discussed in Ref. 8), some of the main qualitative features are correctly obtained, as we shall show. We demonstrate in this paper that a direct application of the standard Poincaré-Lindstedt method to the partial differential equation Boussinesq problem systematically yields asymptotic approximations of the solutions. Furthermore, our multitime analysis establishes stability with respect to three-dimensional disturbances and suggests a possible mechanism for "irregular" response in the solution of the nonlinear initial-value problem, as we discuss in Sec. VIII.

Mode amplitude equations similar to the ones employed in Ref. 8 were derived in Ref. 9 from a simplified physical model of convection. They were analyzed in Refs. 9 and 10 by an asymptotic method, but λ bifurcation was not obtained. More recently, the method of normal forms was employed[11] to obtain approximate solutions of the initial-value problem for the Boussinesq theory with a restricted class of initial data. Other convection systems, such as rotating, magnetic, and binary convection[12,13] can exhibit λ bifurcation. They were analyzed in Refs. 14, 8, and 15 by using different methods. Energy methods were used in Ref. 16 to determine the global stability of thermohaline convection. These give a maximum value of $R_T=R_T(R_s)$ below which all disturbances of the conduction state decay. This result supplements the local stability analyses mentioned above.

II. FORMULATION

In dimensionless variables, the Boussinesq theory for double-diffusive convection in a rectangular box is to solve the differential equations

$$\begin{aligned}
&\vec{u}_t+(\vec{u}\cdot\vec{\nabla})\vec{u}=\sigma[-\nabla p+(R_T T-R_s S)\vec{k}+\triangle\vec{u}\,]\,,\\
&\vec{\nabla}\cdot\vec{u}=0\,,\\
&T_t+(\vec{u}\cdot\vec{\nabla}T)-\vec{u}\cdot\vec{k}=\triangle T\,,\\
&S_t+(\vec{u}\cdot\vec{\nabla}S)-\vec{u}\cdot\vec{k}=D\,\Delta S\,,
\end{aligned} \tag{2.1a}$$

for the velocity $\vec{u}=(u,v,w)$, the pressure p, the reduced temperature, and the reduced solute concentrations T and S, respectively, subject to the Rayleigh boundary conditions,

$$u=v_x=w_x=T_x=S_x=0,\quad \text{for } x=0,\pi a \tag{2.1b}$$

$$u_y=v=w_y=T_y=S_y=0,\quad \text{for } y=0,\pi b \tag{2.1c}$$

$$u_z=v_z=w=T=S=0,\quad \text{for } z=0,\pi\ . \tag{2.1d}$$

In (2.1), Δ is the three-dimensional Laplacian with respect to the dimensionless variables $\vec{x}=(x,y,z)$, $\vec{k}$ is the unit vector in the z (vertical) direction, and a and b are the aspect ratios of the box. The spatial coordinates are scaled by the height d of the box, and time by d^2/κ_T, where κ_T is the thermometric conductivity of the fluid. The parameters σ, D, R_T, and R_s are defined by

$$\begin{aligned}
&\sigma\equiv\frac{\nu}{\kappa_T}>0,\quad D\equiv\frac{\kappa_s}{\kappa_T}>0\,,\\
&R_T\equiv\frac{g\alpha(T_l-T_u)d^3}{\nu\kappa_T}>0\,,\\
&R_s\equiv\frac{g\beta(S_l-S_u)d^3}{\nu\kappa_T}>0\,,
\end{aligned} \tag{2.2}$$

where κ_s is the thermometric and solute "conductivity," ν is the kinematic viscosity of the fluid, g is the gravitational constant, T_l, T_u, S_l, and S_u are the specified values of the temperature and solute concentration on the lower and upper surfaces of the box, and, finally, α and β are the expansion coefficients in the equation of state of the fluid,

$$\rho=\rho_0(1-\alpha T+\beta S) \tag{2.3}$$

for the density ρ.

In deriving (2.1) we have assumed that the Dufour and Sorret effects are negligible. That is, we have assumed that the heat and mass fluxes are proportional to the temperature and solute gradients, respectively. The quantities T and S are the deviations of the temperature and solute concentration from their values for the conduction state. Thus, the conduction state is given by

$$\vec{u}\equiv\vec{0}\ \text{ and }\ T=S\equiv0,\ \ p=0\ . \tag{2.4}$$

It is a solution of (2.1) for all values of the parameters (2.2). The aspect ratios a and b are the ratios of the x and y lengths of the box with respect to its vertical height.

The boundary conditions (2.1b) and (2.1c) imply that the vertical sides of the box are impervious to heat and solute flux. In addition, (2.1c)—(2.1d) imply that the side walls of the box are rigid with respect to fluid motions normal to the wall, but they are slippery with respect to tangential fluid motions. These fluid conditions, which are called the Rayleigh boundary conditions, are usually unrealistic for laboratory experiments performed in rigid boxes. Realistically, the fluid sticks to the walls of the box and the proper fluid boundary conditions are $\vec{u}=0$ on the walls. However, the Rayleigh conditions may be reasonable for convection occuring in natural circumstances such as in the ocean and in the atmosphere. The principal mathematical virtue of the Rayleigh conditions is that the linearized convection theory can be solved explicitly, as we show in Sec. III, whereas explicit solutions for a box with sticky wall boundary conditions have not been obtained.

To study solutions of (2.1) which are periodic in t with period $2\pi/\omega$, the formulation (2.1) is modified by adding the periodicity conditions

$$\begin{aligned}
&\vec{u}(\vec{x},\ t+2\pi/\omega)=\vec{u}(x,t)\ ,\\
&T(\vec{x},\ t+2\pi/\omega)=T(\vec{x},t)\ ,\\
&S(\vec{x},\ t+2\pi/\omega)=S(\vec{x},t)\ ,
\end{aligned}\tag{2.5}$$

for all t. Thus the periodic problem, which we denote as problem $\mathscr{P}$, consists of solving (2.1) subject to the periodicity conditions (2.5). We wish to study the solutions of problem $\mathscr{P}$ and their variations with the Rayleigh number R_T for fixed values of the remaining parameters. Thus, we refer to a, b, σ, D, and R_s as the system parameters.

For two-dimensional convection we have $v\equiv 0$ and the solutions of (2.1) are independent of y. Then, by defining a stream function ψ by

$$u=\psi_z,\quad w=-\psi_x\ , \tag{2.6}$$

problem $\mathscr{P}$ can be reduced to solving

$$\begin{aligned}
&\Delta\psi_t-\sigma\,\Delta^2\psi+\sigma(R_TT_x-R_sS_x)=J(\psi,\Lambda\psi)\ ,\\
&T_t-\Delta T+\psi_x=J(\psi,T)\ ,\\
&S_t-D\,\Delta S+\psi_x=J(\psi,S)\ ,
\end{aligned}\tag{2.7a}$$

on the rectangle $0\le x\le \pi a$, $0\le z\le\pi$ and subject to the boundary and periodicity conditions

$$\begin{aligned}
&\psi=\psi_{xx}=T_x=S_x=0 \quad\text{on } x=0,\pi a\\
&\psi=\psi_{zz}=T=S=0 \quad\text{on } z=0,\pi
\end{aligned}\tag{2.7b}$$

$$\begin{aligned}
&\psi(x,\ z,\ t+2\pi/\omega)=\psi(x,z,t)\ ,\\
&T(x,\ z,\ t+2\pi/\omega)=T(x,z,t)\ ,\\
&S(x,\ z,\ t+2\pi/\omega)=S(x,z,t)\ .
\end{aligned}\tag{2.7c}$$

In (2.7), Δ and Δ^2 are the two-dimensional Laplacian and biharmonic operators, respectively. In addition, the nonlinear operator $J(f,g)$ is defined for any two smooth functions f and g by

$$J(f,g)\equiv f_xg_z-f_zg_x\ . \tag{2.8}$$

The solutions of (2.7) are translationally invariant in time. Hence, periodic solutions can be determined only within an arbitrary phase shift. To fix this shift we impose the following normalization condition:

$$\psi_t(x_0,z_0,0)=0 \tag{2.9}$$

for some appropriate choice of x_0 and z_0. For example, we can choose x_0 and z_0 to correspond to a local spatial maximum or minimum of ψ.

III. THE LINEAR STABILITY THEORY

The linear theory of convection is obtained by linearizing (2.1) about the conduction state (2.4). The solutions of the resulting problem are given by

$$(u,v,w,T,s,p)=e^{qt}(A_1\phi^{scc},A_2\phi^{csc},A_3\phi^{ccs},A_4\phi^{ccs},A_5\phi^{ccs},A_6\phi^{ccc})\ , \tag{3.1}$$

where the functions ϕ^{scc}, etc., are defined by

$$\phi^{scc}\equiv\sin(mx/a)\cos(ny/b)\cos(sz),\ \text{etc.}\ ,\qquad m,n,s=1,2,\ldots \tag{3.2}$$

and the amplitudes A_j satisfy a linear system of homogeneous, algebraic equations. The matrix of this system is a function of the exponential factor q in (3.1), the indices m, n, and s, and the system parameters. From the condition that the determinant of this matrix must vanish, we obtain, for each fixed triple (m,n,s), a quartic equation for q. One root is

$$q^{(1)}=-\sigma Q_{mns}<0\ , \tag{3.3}$$

where Q_{mns} is defined, for $m,n,s=1,2,\ldots$, by

$$Q_{mns}\equiv Q^0_{mn}+s^2,\quad Q^0_{mn}\equiv\left[\frac{m}{a}\right]^2+\left[\frac{n}{b}\right]^2\ . \tag{3.4}$$

Since $q^{(1)}<0$, the solution (3.1) corresponding to this root decays exponentially in t. The remaining three roots of the quartic are the roots of

$$q^3+a_1q^2+a_2q+a_3=0\ . \tag{3.5}$$

The coefficients in (3.5) are defined by

$$a_1 \equiv (1+\sigma+D)Q_{mns} ,$$
$$a_2 \equiv \frac{\sigma Q^0_{mn}}{Q_{mns}}(R^0 - R_T) , \tag{3.6}$$
$$a_3 \equiv \sigma D Q^0_{mn}(R^s_T - R_T) .$$

The quantities R^0, R^s_T, and R^p_T in (3.6) are defined by

$$R^0 \equiv R_s + \left[\frac{\sigma+D(1+\sigma)}{\sigma}\right]\lambda_{mns}, \quad \lambda_{mns} \equiv Q^3_{mns}/Q^0_{mn} , \tag{3.7}$$

$$R^s_T \equiv \frac{R_s}{D} + \lambda_{mns} ,$$
$$R^p_T \equiv \left[\frac{D+\sigma}{1+\sigma}\right]R_s + \frac{(1+D)(\sigma+D)}{\sigma}\lambda_{mns} . \tag{3.8}$$

It follows from (3.5), (3.6), and the Routh-Hurwitz criteria that the conduction state is stable if $R_T < R_c$, and it is unstable if $R_T > R_c$. Here, the critical thermal Rayleigh number R_c is the minimum of R^s_T and R^p_T over all the positive integers m, n, and s. Since R^s_T and R^p_T are linear functions of R_s and λ_{mns}, it follows from the definition of λ_{mns} in (3.7) that

$$R_c = \begin{cases} R^s_c & \text{for } R_s \le \bar{R}_s \\ R^p_c & \text{for } R_s \ge \bar{R}_s \end{cases} \tag{3.9}$$

where R^s_c and R^p_c are defined by

$$R^s_c \equiv \frac{R_s}{D} + \underline{\lambda}, \quad R^p_c \equiv \left[\frac{D+\sigma}{1+\sigma}\right]R_s + \frac{(1+D)(\sigma+D)}{\sigma}\underline{\lambda} , \tag{3.10}$$

and the minimum value $\underline{\lambda}$ is defined by

$$\underline{\lambda} \equiv \min_{m,n,s} \lambda_{mns} = \min_{m,n} \lambda_{mn1} = \lambda_{MN1} . \tag{3.11}$$

Here, M and N are the values of m and n for which the minimum is achieved. Thus R^s_c and R^p_c occur for the same values of m, n, and s, and hence of $\underline{\lambda}_{mns}$. In addition, the critical solute Rayleigh number $\bar{R}_s$, determined from the condition that $R^s_c = R^p_c$, is given by

$$\bar{R}_s \equiv \frac{D^2(1+\sigma)}{(1-D)\sigma}\underline{\lambda} . \tag{3.12}$$

It follows from (3.5) that at $R_T = R^s_c$ for $R_s < \bar{R}_s$ two roots have negative real parts and one root vanishes. This suggests that $R_T = R^s_c$ is a bifurcation point of steady convection states, as we demonstrate in Sec. IV. Similarly, $R_T = R^p_c$ for $R_s > \bar{R}_s$ is a bifurcation point of periodic convection states (see Sec. V) because two roots of (3.5) are complex with negative (positive) real parts for $R_T < R^p_c$ ($> R^p_c$), and they are imaginary at $R_T = R^p_c$ giving the frequency

$$\omega_0^2 = a_2(R^p_c) = \frac{D^2 Q^2_{mn1}}{\bar{R}_s}(R_s - \bar{R}_s) . \tag{3.13}$$

Consequently, instability of the conduction state occurs by the bifurcation of steady (periodic) convection states for $R_s < \bar{R}_s$ ($> \bar{R}_s$). At $R_s = \bar{R}_s$ the steady and periodic bifurcation points coincide and $q = i\omega_0 = 0$ is a double root of (3.5) and thus a double eigenvalue of the linearized theory. This leads to λ bifurcation for R_s near $\bar{R}_s$, as we demonstrate in Sec. VI.

The precise values of M and N in (3.11) depend on the aspect ratios a and b only and not on the other system parameters. The critical Rayleigh number R_c may correspond to either a simple or a multiple eigenvalue of the linearized theory, depending on the values of a and b. In addition, the eigenfunctions (3.1) for $R_T = R_c$ may be either two dimensional with $M \ne 0$, $N = 0$ (y rolls), or $M = 0$, $N \ne 0$ (x rolls), or three dimensional with $M \ne 0$, $N \ne 0$, depending on the values of a and b. In the analysis in Secs. IV and V, we assume that a and b are in a range such that R_c is a simple eigenvalue of the linearized theory with a two-dimensional eigenfunction. Thus, we analyze the bifurcation of two-dimensional convection states which are independent of y, using the formulation (2.7) of problem $\mathscr{P}$.

IV. BIFURCATION OF STEADY CONVECTION SOLUTIONS ($R_s < \bar{R}_s$)

We obtain asymptotic expansions of steady two-dimensional solutions of (2.7a) and (2.7b) that bifurcate from the conduction state at $R_T = R^s_c$ using the modified perturbation method. The small amplitude parameter ϵ_s is defined by

$$\epsilon_s^2 \equiv \int_0^{\pi a} \int_0^{\pi} (\Delta\psi)^2 dx\, dz . \tag{4.1}$$

Since the analysis is standard, we omit all details. The results are

$$\psi = -\left[BD\frac{Q}{(Q^0)^{1/2}}\sin\left[\frac{Mx}{a}\right]\sin z\right]\epsilon_s + O(\epsilon_s^3) ,$$
$$T = \left[BD\cos\left[\frac{Mx}{a}\right]\sin z\right]\epsilon_s - \left[\frac{B^2 Q}{8}D^2\sin(2z)\right]\epsilon_s^2 + O(\epsilon_s^3) , \tag{4.2}$$
$$S = \left[B\cos\left[\frac{Mx}{a}\right]\sin z\right]\epsilon_s - \left[\frac{B^2 Q}{8}\sin(2z)\right]\epsilon_s^2 + O(\epsilon_s^3) ,$$

$$R_T = R^s_c + \frac{Q}{8D}B^2(D^3R^s_c - R_s)\epsilon_s^2 + O(\epsilon_s^3) , \tag{4.3}$$

where we have employed the notation

$$Q \equiv Q_{M01} = (M/a)^2 + 1 ,$$
$$Q^0 \equiv Q^0_{M0} = (M/a)^2, \quad B \equiv \pm\frac{2(M/a)}{\pi a^{1/2} D Q^2} . \tag{4.4}$$

From (4.3) we deduce that the bifurcation is supercritical (subcritical) if $R_s < R'_s$ ($> R'_s$) where $R'_s < \bar{R}_s$ is defined by

$$R_s' \equiv \frac{D^3\lambda}{1-D^2} . \tag{4.5}$$

We have used the definition of R_c^s in (3.10) in deriving (4.5). If $R_s = R_s'$, it is necessary to obtain additional terms in the expansions (4.2) and (4.3) to determine the direction of bifurcation.

From a linear stability analysis of the steady convection states (4.2) and (4.3), which we do not present, it can be shown that the supercritical (subcritical) states are stable (unstable). Furthermore, we observe from (4.2)–(4.4) that as the Schmidt number $D \to 0$, the solute concentration S and the critical thermal Rayleigh number R_c^s become unbounded. Thus, the perturbation analysis is valid only for D bounded away from zero. The bifurcation diagram and its stability are shown in Fig. 1. If $D=1$ and $R_s=0$ then the results of this section also describe the bifurcation of steady solutions in ordinary Bénard convection.

V. BIFURCATION OF PERIODIC CONVECTION SOLUTIONS ($R_s > \bar{R}_s'$)

The periodic convection states that bifurcate from the conduction state at $R_T = R_c^p$ for $R_s > \bar{R}_s$ are obtained by employing the Poincaré-Lindstedt method. Thus, we define a scaled time τ by

$$\tau \equiv \omega t , \tag{5.1}$$

where the frequency ω of the periodic motion is to be determined, and a new small amplitude parameter ϵ_p is defined by

$$\epsilon_p^2 \equiv \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi a} (\Delta\psi)^2 dx\, dz\, d\tau . \tag{5.2}$$

We then seek asymptotic expansions of the solutions (2.7) in the form

$$\begin{aligned} \vec{\psi}(x,z,\tau,\epsilon_p) &= \sum_{j=1}^{\infty} \vec{\psi}_p^{(j)}(x,z,\tau)\epsilon_p^j , \\ R &= R_c^p + \sum_{j=1}^{\infty} \rho_j \epsilon_p^j, \quad \omega = \omega_0 + \sum_{j=1}^{\infty} \Omega_j \epsilon_p^j , \end{aligned} \tag{5.3}$$

where $\vec{\psi}$ is the vector with components (ψ, T, S). Because of (5.1), the solutions $\vec{\psi}$ are periodic in τ of period 2π. The coefficients in these expansions are determined in the usual way by substituting (5.1)–(5.3) into (2.7)–(2.9). This leads to systems of linear equations for these coefficients. The lowest-order system $[O(\epsilon_p)]$ corresponds to the linearized convection theory at $R = R_c^p$ with the normalization conditions

$$\begin{aligned} &\int_0^{2\pi} \int_0^{\pi} \int_0^{\pi a} (\Delta\psi_p^{(1)})^2 dx\, dz\, d\tau = 1 , \\ &\left. \frac{\partial \psi_p^{(1)}}{\partial \tau} \right|_{x=x_0, z=z_0, \tau=0} = 0 , \end{aligned} \tag{5.4}$$

that are obtained from (5.2) and (2.9), respectively. Thus, we find

$$\begin{aligned} \psi_p^{(1)} &= B_p \cos\tau \sin(Mx/a) \sin z , \\ T_p^{(1)} &= -B_p \frac{M}{a}(Q^2+\omega_0^2)^{-1} \times \cos(\tau - p_1) \cos\left[\frac{Mx}{a}\right] \sin z , \\ S_p^{(1)} &= -B_p \frac{M}{a}(D^2Q^2+\omega_0^2)^{-1} \times \cos(\tau - p_2) \cos\left[\frac{Mx}{a}\right] \sin z , \end{aligned} \tag{5.5}$$

where ω_0 is given by (3.13) and the amplitude B_p and phases p_1 and p_2 are defined by

$$\begin{aligned} B_p &\equiv 2(\pi^3 a Q^2)^{-1/2} , \\ \tan p_1 &\equiv \omega_0/Q = D[(R_s - \bar{R}_s)/\bar{R}_s]^{1/2} , \\ \tan p_2 &\equiv \omega_0/DQ = [(R_s - \bar{R}_s)/\bar{R}_s]^{1/2} . \end{aligned} \tag{5.6}$$

The state described by (5.5) corresponds to a standing wave that oscillates with a frequency of one on the τ scale and hence a frequency of $\omega_0 + O(\epsilon_p)$ on the t scale. In addition, the fluid velocity, temperature, and salinity are phase shifted. We observe from (3.13) and (5.6) that the frequency and the phase shifts vanish as $R_s \to \bar{R}_s$. The fact that $D<1$, and hence the solute phase shift exceeds the thermal phase shift, explains some of the effects of double-diffusive convection.

The solvability conditions for the $O(\epsilon_p^2)$ problem imply that Ω_1 and ρ_1 satisfy a system of two homogeneous, linear algebraic equations defined by

$$Z \begin{bmatrix} \Omega_1 \\ \rho_1 \end{bmatrix} = 0 . \tag{5.7}$$

The elements of the matrix Z are listed in Appendix A. Since $\det Z \neq 0$ if $R_s > \bar{R}_s$, (5.7) implies that $\Omega_1 = \rho_1 = 0$ for $R_s > \bar{R}_s$. If $R_s = \bar{R}_s$ then $\det Z = 0$ and it follows from (5.7) that $\rho_1 = 0$, but Ω_1 is arbitrary. Similarly, the solvability conditions for the $O(\epsilon_p^3)$ problem give

$$Z \begin{bmatrix} \Omega_2 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} , \tag{5.8}$$

where the quantities g_1 and g_2 are defined in Appendix A. The solution $\vec{\psi}^{(2)}$ of the $O(\epsilon_p^2)$ problem is also given in Appendix A since it is used in obtaining (5.8). Since $\det Z \neq 0$ for $R_s > \bar{R}_s$, we can solve (5.8) uniquely for Ω_2 and ρ_2 when R_s exceeds $\bar{R}_s$. Furthermore, Ω_2 becomes unbounded as $R_s \to \bar{R}_s$ because $\det Z = 0$ for $R_s = \bar{R}_s$. This implies that the Poincaré-Lindstedt method and hence the expansions (5.3) are not uniformly valid in R_s as $R_s \to \bar{R}_s$. Other methods, which have been applied to the thermohaline[17] and binary fluid[18] convection problems, have produced results which also break down at the coalescence point of the steady and periodic solutions. In Sec. V we study the solutions of (2.7) at and near the singularity $R_s = \bar{R}_s$ by modifying the Poincaré-Lindstedt method.

Graphs of the variation of Ω_2 and ρ_2 with R_s are shown in Ref. 1 for typical values of the system parameters. Furthermore, we observe from our own graphs of these functions that ρ_2 is a monotonic function of R_s. However, Ω_2 may be either monotonic or have a positive maximum, depending on the value of the system parameters. Moreover, we find that $\Omega_2 \to -\infty$ and ρ_2 attains a constant, positive value as $R_s \to \bar{R}_s$. In addition, $\rho_2(R_s)$ and $\Omega_2(R_s)$ have the unique zeros $R_s = R_s^*$ and $R_s = R_s^+$, respectively. Thus, for $R_s < R_s^*$ ($> R_s^*$), $\rho_2 > 0$ (<0) and hence the period convection states bifurcate supercritically (subcritically) from R_c^p. Since $\Omega_2 < 0$ (>0) for $R_s < R_s^+$ ($> R_s^+$), the system responds as a "soft spring" ("hard spring") for $R_s < R_s^+$ ($> R_s^+$). Thus, there is a smooth transition from the conduction state to the periodic convection states for $\bar{R}_s < R_s < R_s^*$ since the bifurcation is supercritical, and, as can be demonstrated, these convection states are stable. Since the periodic convection states bifurcate subcritically for $R_s > R_s^*$ and the subcritical states are unstable, there must be a jump transition as R_T increases past R_c^p. This transition from the conduction state to a resulting large-amplitude state of the system cannot be described by the present local analysis. The bifurcation diagram and its stability are shown in Fig. 2. We observe from (5.3), (5.5), (5.8), and (A3) that as $D \to 0$, the perturbation analysis remains valid. Thus, when $D=0$ and $R_s > 0$ the resulting convection state is periodic.

VI. λ BIFURCATION

The asymptotic expansions (5.3) for the bifurcating periodic convection states of (2.7) are not uniformly valid as $R_s \to \bar{R}_s$, as we have described in Sec. VI. To analyze the solutions of (2.7) that bifurcate from the conduction state for R_s near $\bar{R}_s$, we first define a new small parameter ϵ by

$$R_s = \bar{R}_s + \epsilon^2 \ . \tag{6.1}$$

Then we introduce the time scale τ from (5.1) and seek asymptotic expansions of the solutions of (2.7) in the form

$$\vec{\psi}(x,z,\tau,\epsilon) = \sum_{j=1}^{\infty} \vec{\psi}^{(j)}(x,z,\tau)\epsilon^j \ ,$$
$$R_T = \bar{R}_c + \sum_{j=1}^{\infty} r_j \epsilon^j, \quad \omega = \sum_{j=1}^{\infty} \omega_j \epsilon^j \ . \tag{6.2}$$

In (6.2), $\vec{\psi}^{(j)}$ is the vector $(\psi^{(j)}, T^{(j)}, S^{(j)})$ and $\bar{R}_c$ is the critical thermal Rayleigh number at $R_s = \bar{R}_s$, i.e.,

$$\bar{R}_c = R_c^s(\bar{R}_s) = R_c^p(\bar{R}_s) = \frac{\sigma + D}{\sigma(1-D)} \lambda \ . \tag{6.3}$$

We have assumed in the expansion (6.2) that $\omega \to 0$ as $R_s \to \bar{R}_s$, because of expression (3.13) for the frequency ω_0.

The coefficients in (6.2) are determined by inserting (5.1) and (6.1)–(6.3) into (2.7). This leads to a sequence of linear problems, the first four of which are given below:

$$\begin{bmatrix} -\Delta^2\psi^{(j)} + \bar{R}_c T_x^{(j)} - \bar{R}_s S_x^{(j)} \\ -\Delta T^{(j)} + \psi_x^{(j)} \\ -D\Delta S^{(j)} + \psi_x^{(j)} \end{bmatrix} = \vec{G}^{(j)} \ ,$$
$$\psi^{(j)} = \psi_{zz}^{(j)} = T^{(j)} = S^{(j)} = 0 \quad \text{for } z = 0, \pi$$
$$\psi^{(j)} = \psi_{xx}^{(j)} = T_x^{(j)} = S_x^{(j)} = 0 \quad \text{for } x = 0, \pi a \tag{6.4}$$
$$\vec{\psi}^{(j)}(x,z,\tau+2\pi) = \vec{\psi}^{(j)}(x,z,\tau) \ ,$$
$$\psi_\tau^{(j)}(x_0,y_0,0) = 0 \quad \text{for } j = 1,2,\ldots \ .$$

The column vectors $\vec{G}^{(j)}$, which have the components $G_1^{(j)}$, $G_2^{(j)}$, and $G_3^{(j)}$, are given for $j = 1,2,3,4$ by

$$\vec{G}^{(1)} \equiv 0 \ , \tag{6.5a}$$

$$\vec{G}^{(2)} \equiv \begin{bmatrix} -\sigma^{-1}\omega_1\Delta\psi_\tau^{(1)} - r_1 T_x^{(1)} + \sigma^{-1}J(\psi^{(1)},\Delta\psi^{(1)}) \\ -\omega_1 T_\tau^{(1)} + J(\psi^{(1)},T^{(1)}) \\ -\omega_1 S_\tau^{(1)} + J(\psi^{(1)},S^{(1)}) \end{bmatrix} , \tag{6.5b}$$

$$\vec{G}^{(3)} \equiv \begin{bmatrix} -\sigma^{-1}(\omega_1\Delta\psi_\tau^{(2)} + \omega_2\Delta\psi^{(1)}) - r_2 T_x^{(1)} + S_x^{(1)} + \sigma^{-1}[J(\psi^{(1)},\Delta\psi^{(2)}) + J(\psi^{(2)},\Delta\psi^{(1)})] \\ -(\omega_1 T_\tau^{(2)} + \omega_2 T_\tau^{(1)}) + J(\psi^{(1)},T^{(2)}) + J(\psi^{(2)},T^{(1)}) \\ -(\omega_1 S_\tau^{(2)} + \omega_2 S_\tau^{(1)}) + J(\psi^{(1)},S^{(2)}) + J(\psi^{(2)},S^{(1)}) \end{bmatrix} , \tag{6.5c}$$

$$\vec{G}^{(4)} \equiv \begin{bmatrix} -\sigma^{-1}(\omega_1\Delta\psi_\tau^{(3)} + \omega_2\Delta\psi_\tau^{(2)} + \omega_3\Delta\psi_\tau^{(1)}) - r_2 T_x^{(2)} - r_3 T_x^{(1)} \\ \quad + S_x^{(2)} + \sigma^{-1}[J(\psi^{(1)},\Delta\psi^{(3)}) + J(\psi^{(2)},\Delta\psi^{(2)}) + J(\psi^{(3)},\Delta\psi^{(1)})] \\ -(\omega_1 T_\tau^{(3)} + \omega_2 T_\tau^{(2)} + \omega_3 T_\tau^{(1)}) + J(\psi^{(1)},T^{(3)}) + J(\psi^{(2)},T^{(2)}) + J(\psi^{(3)},T^{(1)}) \\ -(\omega_1 S_\tau^{(3)} + \omega_2 S_\tau^{(2)} + \omega_3 S_\tau^{(1)}) + J(\psi^{(1)},S^{(3)}) + J(\psi^{(2)},S^{(2)}) + J(\psi^{(3)},S^{(1)}) \end{bmatrix} . \tag{6.5d}$$

Problem (6.4) with $j=1$ is the linearized two-dimensional stability theory at $R_s = \bar{R}_s$, $R = \bar{R}_c$ with $\omega = 0$. The solution of this problem, which must be periodic and satisfy the translational invariance normalization condition, is given by

$$\vec{\psi}^{(1)} = A(\tau)\vec{\psi}_0 \equiv A(\tau) \begin{bmatrix} -D[Q/(Q^0)^{1/2}]\sin(Mx/a) \\ D\cos(Mx/a) \\ \cos(Mx/a) \end{bmatrix} \sin z \ , \tag{6.6a}$$

where the amplitude $A(\tau)$ is a periodic function of period 2π to be determined, and

$$A'(0) = 0 \ . \tag{6.6b}$$

In (6.6b) the prime denotes differentiation with respect to

τ. Since the homogeneous problems corresponding to (6.4) with $j>1$ have nontrivial solutions, the inhomogeneous terms $G^{(j)}$ must satisfy the solvability conditions

$$\langle \vec{G}^{(j)}, \vec{\psi}^* \rangle \equiv \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi a} (\vec{G}^{(j)} \cdot \vec{\psi}^*) dx\, dz\, d\tau = 0 , \quad j=2,3,\ldots \tag{6.7}$$

where $\vec{G}^{(j)} \cdot \vec{\psi}^*$ is the usual dot product of two vectors, and $\vec{\psi}^*$ is the eigenvector of the adjoint linear problem. It is given by

$$\vec{\psi}^* \equiv \bar{R}_s^{-1} \begin{bmatrix} D[Q/(Q^0)^{1/2}]\sin(Mx/a) \\ D\bar{R}_c \cos(Mx/a) \\ -\bar{R}_s \cos(Mx/a) \end{bmatrix} \sin z , \tag{6.8}$$

By employing (6.5b) and (6.6a), the solvability condition (6.7) with $j=2$ yields

$$r_1 = 0 . \tag{6.9}$$

Then by substituting (6.9) into (6.4) with $j=2$ and solving the resulting problem for $\vec{\psi}^{(2)}$ we obtain

$$\vec{\psi}^{(2)} = B(\tau)\vec{\psi}_0 + A'(\tau) \begin{bmatrix} -[\omega_1/(Q^0)^{1/2}]\sin(Mx/a) \\ \dfrac{(1-D)\omega_1}{Q}\cos(Mx/a) \\ 0 \end{bmatrix} \sin z$$

$$- \frac{QA^2}{8} \begin{bmatrix} 0 \\ D^2 \\ 1 \end{bmatrix} \sin(2z) , \tag{6.10}$$

where $B(\tau)$ is a periodic function of period 2π to be determined. The solvability condition (6.7) with $j=3$ then gives the following equation for $A(\tau)$:

$$A'' + \alpha A - \beta A^3 = 0 , \tag{6.11a}$$

where α and β are defined by

$$\alpha \equiv \frac{(M/a)^2 \sigma(1-r_2 D)}{(1+\sigma+D)Q\omega_1^2}, \quad \beta \equiv \frac{D^2 Q^3}{8\omega_1^2} > 0 , \tag{6.11b}$$

In addition, A satisfies the initial and periodicity conditions,

$$A'(0)=0, \quad A(\tau+2\pi)=A(\tau) \text{ for all } \tau . \tag{6.11c}$$

The coefficients corresponding to α and β of the Duffing equation obtained in Ref. 8 are the same as the expressions (6.11), modulo the different scales used in Ref. 8.

Since α and β are functions of r_2 and ω_1 and since we are studying the variation of the solution with R_T, or equivalently with r_2, we need only determine how ω_1 varies with r_2 to evaluate the amplitude $A(\tau)$. The frequency $\omega_1(r_2)$ is deduced from the solvability condition (6.7) with $j=4$. To apply this condition, we require the solution of (6.4) with $j=3$. By employing the results (6.6), (6.9), and (6.10) to evaluate $\vec{G}^{(3)}$, we obtain, finally,

$$\vec{\psi}^{(3)} = C(\tau)\vec{\psi}_0 + \begin{bmatrix} 0 \\ b_1 \\ c_1 \end{bmatrix} \cos(Mx/a)\sin z$$

$$+ \begin{bmatrix} 0 \\ b_3 \\ c_3 \end{bmatrix} \sin(2z) + \begin{bmatrix} a_2 \sin(Mx/a) \\ b_2 \cos(Mx/a) \\ c_2 \cos(Mx/a) \end{bmatrix} \sin(3z) , \tag{6.12}$$

where $C(\tau)$ is a periodic function of period 2π to be determined and the coefficients a_j, b_j, and c_j are defined by

$$b_1 \equiv \frac{1}{Q}\left\{ -\omega_1[DB' + (1-D)Q^{-1}A''] - \omega_2 DA' - \frac{D^3}{8}Q^2 A^3 \right\} ,$$

$$c_1 \equiv \frac{1}{DQ}\left\{ -(\omega_1 B' + \omega_2 A') - \frac{DQ^2}{8}A^3 \right\} ,$$

$$a_2 \equiv -\frac{D^2}{8} H \frac{Q^5}{(M/a)} (Q^3 - P^3)^{-1} A^3 ,$$

$$b_2 \equiv \frac{D^2 Q^2}{8P}[D + HQ^3(Q^3-P^3)^{-1}]A^3 , \tag{6.13a}$$

$$c_2 \equiv \frac{Q^2}{8P}[1 + HDQ^3(Q^3-P^3)^{-1}]A^3 ,$$

$$b_3 \equiv \frac{1}{4}\left[\left[\frac{D^2}{4}(2+Q) - D \right] \omega_1 AA' - D^2 Q^2 AB \right] ,$$

$$c_3 \equiv \frac{1}{4D}[\tfrac{1}{4}(Q-2)\omega_1 AA' - DQAB] ,$$

where P and H are defined by

$$P = (M/a)^2 + 9, \quad H \equiv (1+\sigma+D)/\sigma . \tag{6.13b}$$

The solvability condition with $j=4$ gives a linear, nonhomogeneous Hill's equation for the amplitude $B(\tau)$ in (6.10) and (6.12):

$$B'' + (\alpha - 3\beta A^2)B = F(A, \omega_1, r_2, r_3) , \tag{6.14a}$$

where F is defined by

$$F \equiv \frac{1}{\omega_1(\bar{R}_s - D^3\bar{R}_c)} \left\{ D^2(1-D)Qr_2 A' + \frac{D^2(1-D)}{Q}\bar{R}_c \omega_1^2 A''' + D^3 Q^2 \omega_1 r_3 A \right.$$

$$\left. + \frac{DQ^2}{2}\left[\left[1 + \frac{Q}{8} \right](D^4\bar{R}_c - \bar{R}_s) + \tfrac{3}{4}(\bar{R}_s - D^3\bar{R}_c) \right] A^2 A' \right\} . \tag{6.14b}$$

In addition, $B(\tau)$ satisfies the periodicity and initial conditions

$$B(\tau+2\pi)=B(\tau) \quad \text{for all } \tau$$
$$B'(0)=-\frac{\omega_1}{DQ}(-\alpha A_0+\beta A_0^3)\ . \tag{6.14c}$$

The coefficient ω_1 will be obtained in Sec. VII by applying a solvability condition to the nonhomogeneous problem (6.14).

VII. THE AMPLITUDE PROBLEMS

It follows from (6.11) and (6.14) that

$$A=B\equiv 0 \tag{7.1}$$

are solutions of the amplitude problems for all values of r_2. This implies that $\psi^{(1)}=\psi^{(2)}\equiv 0$ so that (7.1) corresponds to the conduction state. If $\alpha<0$ ($r_2>D^{-1}$), then (7.1) is the only steady solution of the amplitude problems. However, if $\alpha>0$ ($r_2<D^{-1}$), then there are additional steady solutions that are given by

$$A=\pm\left[\frac{\alpha}{\beta}\right]^{1/2},\quad B\equiv 0\ . \tag{7.2}$$

Since $\alpha\rightarrow 0$ as $r_2\rightarrow D^{-1}$, the solutions (7.2) correspond to a branch of steady convection states that bifurcate subcritically from $R_T^s=\bar{R}_c+\epsilon^2/D+O(\epsilon^3)$, as we illustrate in Fig. 3. The phase plane diagram for (6.11a) is sketched in Fig. 4. Thus, the steady convection states are unstable since they correspond to a saddle point.

It is easy to demonstrate, e.g., from a phase plane analysis, that there are no periodic solutions of amplitude problem (6.11) for $\alpha<0$ ($r_2>D^{-1}$). Periodic solutions of (6.11) exist for $\alpha>0$ ($r_2<D^{-1}$) but only for sufficiently small values of $A_0\equiv A(0)$, i.e., for an initial point within the separatrix in the phase plane. The periodic solutions are given by

$$A(\tau)\equiv A_0\,\mathrm{sn}\left[\frac{2K(k)}{\pi}\tau+K(k)\right], \tag{7.3a}$$

where A_0 is defined by

$$A_0^2\equiv 2(\alpha/\beta)\frac{k^2}{1+k^2}\ , \tag{7.3b}$$

$K(k)$ is the complete elliptic integral of the first kind with modulus k, and r_2, ω_1, and k satisfy

$$\omega_1^2=\left[\frac{\pi^2(M/a)^2\sigma}{4(1+\sigma+D)Q}\right]\frac{(1-r_2D)}{(1+k^2)[K(k)]^2}\ . \tag{7.4}$$

In (7.3a), $\mathrm{sn}(y)$ is the Jacobian elliptic function. It is a periodic function of period $4K$ and is defined for k in the interval $0\le k\le 1$. Since α/β is independent of ω_1, (7.3b) expresses A_0 as a function of r_2 and k. Equation (7.4) expresses ω_1 as a function of r_2 and k. The required third relation between these quantities is obtained from an analysis of the amplitude problem (6.14) for $B(\tau)$.

If $F\equiv 0$ then the homogeneous problem (6.14) for B has a 2π-periodic solution that is proportional to $A'(\tau)$ since the homogeneous equation is the variational equation for (6.11). There are no other 2π-periodic solutions of the homogeneous problem for B as we can demonstrate. Thus the nonhomogeneous term F must satisfy the solvability condition that it is orthogonal to A':

$$\int_0^{2\pi}FA'd\tau=0\ . \tag{7.5}$$

When this condition is satisfied it guarantees that B is periodic. After some tedious calculations this yields the following equation relating r_2 and k:

$$r_2=r_2(k)\equiv\frac{\xi_1X-\xi_3Y}{(\xi_1X-\xi_3Y)D+\xi_2Z}\ . \tag{7.6}$$

Here,

$$\xi_1\equiv\frac{\sigma D^2(1-D)\bar{R}_c(M/a)^2}{5(1+\sigma+D)Q^2},\quad \xi_2\equiv D^2(1-D)Q\ ,$$
$$\xi_3\equiv\frac{8\sigma(M/a)^2}{5D(1+\sigma+D)Q^2}\times\left[\left[1+\frac{Q}{8}\right](D^4\bar{R}_c-\bar{R}_s)+\tfrac{3}{4}(\bar{R}_s-D^3\bar{R}_c)\right],$$
$$X(k)\equiv(-7k^4+22k^2-7)E(k)+(11k^4-18k^2+7)K(k)\ , \tag{7.7}$$
$$Y(k)\equiv 2(k^4-k^2+1)E(k)+(-k^4+3k^2-2)K(k)\ ,$$
$$Z(k)\equiv(1+k^2)[(1+k^2)E(k)+(k^2-1)K(k)]\ ,$$

and $E(k)$ is the complete elliptic integral of the second kind. Then (7.3b), (7.4), and (7.6) express $A_0(k)$, $\omega_1(k)$, and $r_2(k)$ in terms of the parameter k. Specifically, we deduce from these equations the following limiting values:

$$r_2(0)=\frac{\sigma+D}{\sigma+1},\quad A_0(0)=0\ ,$$
$$\omega_1^2(0)=\frac{\sigma(1-D)}{(\sigma+1)}\frac{(M/a)^2}{Q}\ , \tag{7.8a}$$
$$r_2(1)=(D+\mu)^{-1}\ ,$$
$$A_0^2(1)=\frac{8\sigma(M/a)^2}{(1+\sigma+D)D^2Q^4}\frac{\mu}{D+\mu},\quad \omega_1(1)=0\ , \tag{7.8b}$$

where μ is defined by

$$\mu=\frac{2\xi_2}{4\xi_1-\xi_3}\ . \tag{7.9}$$

Notice that $\omega_1(0)\epsilon$ is just the linearized frequency given by (3.13). In addition, we obtain the limiting slopes

$$\lim_{k\to 0}\frac{dA_0^2}{dr_2}>0\ ,\quad \lim_{k\to 1}\frac{dA_0^2}{dr_2}=\infty\ . \tag{7.10}$$

For each fixed k in the interval $0\le k\le 1$, (7.6) gives a unique value of r_2, and (7.3b) and (7.4) then give unique values for A_0^2 and ω_1. Because of the complicated transcendental expression (7.6) for r_2 it is difficult to deduce analytically the qualitative features of the dependence of

A_0 and ω_1 on r_2. Thus, we have numerically evaluated these equations for typical values of the system parameters. This yields response curves such as the one sketched in Fig. 3. The periodic solutions branch from the conduction state $A_0 \equiv 0$ at the bifurcation point, which corresponds to $k=0$ and $r_2 = r_2(0)$. The period at this point, in the t scale, is given by

$$T(0) = \frac{2\pi}{\omega_1(0)\epsilon} + O(1) , \tag{7.11}$$

where $\omega_1(0)$ is given in (7.8a). Thus, the response curve branches with positive slope as indicated in (7.10). The numerical evaluations suggest that this curve is monotonic. It terminates at $k=1$ with the values given in (7.8b) and with an infinite slope, as indicated in (7.10). We observe that $A_0(1)$ is equal to the value of the steady solution (7.2) evaluated at $r_2 = r_2(1)$. Furthermore, since $\omega_1(1)=0$, the period is infinite. Thus, when $k=1$, (7.3) corresponds to the two steady states on the heteroclinic orbit in the phase plane. The periodic solution (7.3) connects to the stationary solution (7.2) at an infinite-period bifurcation. Therefore, in this generalized sense $r_2(1), A_0(1)$ is a "secondary bifurcation" point of both the steady convection state that bifurcates from R_T^s and of the periodic convection state that bifurcates from R_c^p. We refer to this type of one-sided branching of states as λ bifurcation.

Stable (unstable) solution branches are indicated in Fig. 3 by solid (dashed) curves. The stability analysis is outlined in Sec. VIII. The results suggest the following sequence of events as r_2, or equivalently as R_T increases, for R_s near $\overline{R}_s$. For small r_2 the fluid is in the stable conduction state. At $r_2 = r_2(0)$, i.e., at $R_T = R_c^p$, the conduction state destabilizes and the fluid commences a state of periodic convection as given to lowest order by $\vec{\psi} = A(\tau)\vec{\psi}_0 + O(\epsilon^2)$. The amplitude of this motion increases as r_2 increases and the period increases to infinity as r_2 approaches $r_2(1)$, i.e., as R_T approaches R_b. When r_2 exceeds $r_2(1)$ the solutions must jump since there are no nearby stable convection states. The solutions might jump to another periodic or steady convection state or to some other, possibly aperiodic convection state. The final state after the jump cannot be determined by our perturbation analysis. Thus, careful numerical and/or physical experiments are required to determine the ultimate fate of the system as r_2 exceeds $r_2(1)$. Of course, imperfections which are always present in physical systems, and which have appropriate frequency content, may substantially alter the qualitative features of the solution for r_2 near $r_2(1)$.

Since $A(\tau)$ has zeros at $\tau = \tau_n \equiv (2n+1)\pi/2$, $n = 0, 1, \ldots$, the leading-order approximations $\vec{\psi}^{(1)}\epsilon$ of the solutions vanish at $\tau = \tau_n$, and hence they are small near τ_n. Therefore, the approximation $\vec{\psi} = \vec{\psi}^{(1)}\epsilon$ is not uniformly valid in τ; see the discussion in Ref. 19. To obtain a uniformly valid first approximation we must retain the next term in the asymptotic expansion (6.2) to get the leading-order asymptotic approximation as

$$\vec{\psi} = \vec{\psi}^{(1)}\epsilon + \vec{\psi}^{(2)}\epsilon^2/2 + O(\epsilon^3) . \tag{7.12}$$

We solve the amplitude equation (6.14) to determine the uniform expansion (7.12). The solution for B that satisfies the condition $\vec{\psi}_\tau^{(2)}(x_0, z_0, 0) = 0$ is

$$B(\tau) = A'(\tau) \left[-\frac{\omega_1}{DQ} + \int_0^\tau \int_0^s \frac{F(r)A'(r)}{[A'(s)]^2} dr\, ds \right] , \tag{7.13}$$

where the integral represents the particular solution of (6.14). It is possible to explicitly evaluate the integral in (7.13) in terms of elliptic functions and integrals and the θ function. However, because of its excessive length we do not present this result here.

We observe that, for $D \to 0$, both the steady and periodic solutions of the λ bifurcation become unbounded. Thus, the perturbation analysis is valid only for D bounded away from zero. The problem has to be rescaled to consider the case of $D \to 0$.

VIII. STABILITY

To determine the stability of the two-dimensional convection states that were derived in Secs. VI and VII, we solve the initial-value problem for the three-dimensional Boussinesq theory. Thus, we consider (2.1) and impose the small-amplitude initial conditions

$$\begin{aligned} &\vec{u}(\vec{x},0) = \epsilon^2 \vec{u}^{\,0}(\vec{x}), \quad T(\vec{x},0) = \epsilon^2 T^0(\vec{x}) , \\ &S(\vec{x},0) = \epsilon^2 S^0(\vec{x}) , \end{aligned} \tag{8.1}$$

where $\overset{0}{\vec{u}}(\vec{x})$, $T^0(\vec{x})$, and $S^0(\vec{x})$ are prescribed initial data, and ϵ is the small parameter defined in (6.1). Specifically, we analyze this initial-value problem for R_s and R_c near $\overline{R}_s$ and $\overline{R}_c$, respectively. Then, to test the stability of the periodic convection states we choose initial data in (8.1) which are "close" to these convection states. We show that the solution of the resulting linearized initial-value problem decays to the periodic orbit as $t \to \infty$.

The multitime method is employed in the analysis. To apply this method, we first define two slow times τ_1 and τ_2 by

$$\frac{d\tau_1}{dt} = \Omega(\tau_2)\epsilon , \quad \tau_2 \equiv \epsilon^2 t . \tag{8.2}$$

The time τ_1 corresponds to the oscillation time since $\omega = O(\epsilon)$ for $(R_s - \overline{R}_s) = O(\epsilon^2)$ and τ_2 is the decay time to the periodic states as is suggested by (3.5) with $R_T - \overline{R}_c = O(\epsilon^2)$. Then we seek an asymptotic expansion of the solution of (2.1) and (8.1) in the form

$$\begin{aligned} &\vec{\Phi}(\vec{x},t,\epsilon) = \sum_{j=1}^{\infty} \vec{\Phi}^{(j)}(\vec{x},t,\tau_1,\tau_2)\epsilon^j , \\ &R_t = \overline{R}_c + \sum_{j=1}^{\infty} r_j \epsilon^j . \end{aligned} \tag{8.3}$$

Here, $\vec{\Phi}$ is the vector with components (u,v,w,T,S,p) and we assume that the expansion coefficients $\vec{\Phi}^{(j)}$ are bounded functions of each of their arguments.

The coefficients in (8.3) are determined by inserting (6.1), (8.2), and (8.3) into (2.1) and (8.1). This leads to a sequence of linear initial-value problems for $j = 1, 2, \ldots$ given by

$$L\vec{\Phi}^{(j)} \equiv \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & \partial_x \\ 0 & L_0 & 0 & 0 & 0 & \partial_y \\ 0 & 0 & L_0 & 0 & 0 & \partial_z \\ 0 & 0 & -1 & L_0 & 0 & 0 \\ 0 & 0 & -1 & 0 & L_0 & 0 \\ \partial_x & \partial_y & \partial_z & 0 & 0 & 0 \end{bmatrix} \vec{\Phi}^{(j)} = \vec{H}_{(j)} , \tag{8.4a}$$

$$L_0 \equiv \frac{1}{\sigma}\partial_t - \Delta ;$$

$$\begin{aligned} &u^{(j)} = v_x^{(j)} = w_x^{(j)} = T_x^{(j)} = S_x^{(j)} = 0 \quad \text{for } x = 0, \pi a \\ &u_y^{(j)} = v^{(j)} = w_y^{(j)} = T_y^{(j)} = S_y^{(j)} = 0 \quad \text{for } y = 0, \pi b \\ &u_z^{(j)} = v_z^{(j)} = w^{(j)} = T^{(j)} = S^{(j)} = 0 \quad \text{for } z = 0, \pi ; \end{aligned} \tag{8.4b}$$

$$\begin{aligned} \vec{u}^{(j)}(\vec{x},0,0,0) &= T^{(j)}(\vec{x},0,0,0) \\ &= S^{(j)}(\vec{x},0,0,0) = 0, \quad j \neq 2 \\ \vec{u}^{(2)}(\vec{x},0,0,0) &= \vec{u}^{0}(\vec{x}) , \\ T^{(2)}(\vec{x},0,0,0) &= T^{0}(\vec{x}) , \\ S^{(2)}(\vec{x},0,0,0) &= S^{0}(\vec{x}) . \end{aligned} \tag{8.4c}$$

The nonhomogeneous vector $\vec{H}^{(j)}$ depends on $\vec{\Phi}^{(i)}$ and r_i for $i < j$ and contains derivatives with respect to τ_1 and τ_2. Since these expressions are lengthy, we do not present them here.

Since $\vec{H}^{(1)} \equiv 0$, problem (8.4) with $j=1$ is the linearized, three-dimensional stability theory at $R_T = \overline{R}_c$ and $R_s = \overline{R}_s$ (see Sec. III). The solutions of the linearized theory are determined for each m, n, and s by the root (3.3) and the three roots $q = q_{mnsl}$, $l = 1,2,3$, of the cubic (3.5). For the critical values of m, n, and s $(M,0,1)$ we have $a_2 = a_3 = 0$ in (3.5), as we can show from (3.6) and (3.7). This implies that (3.5) has the double root $q=0$. However, there is only one linearly independent eigenvector $\vec{\phi}$ corresponding to this root. Thus $q=0$ is a root of algebraic multiplicity two and geometric multiplicity one. This is an important feature of the underlying mathematical structure of the present problem which is shared by other bifurcation problems. To obtain a complete set of eigenvectors, we must add the generalized eigenvector corresponding to the double root. The roots (3.3) and the roots of (3.5) for $(m,n,s) \neq (M,0,1)$ are negative and correspond to exponentially decaying modes. We assume that all the eigenvectors corresponding to the negative roots, and the eigenvector $\vec{\phi}$ and generalized eigenvector $t\vec{\phi} + \vec{\phi}_0$ corresponding to the double-zero root, form a complete set. The solution of (8.4) with $j=1$ can then be written concisely as

$$\vec{\Phi}^{(1)} = \mathscr{A}(\tau_1,\tau_2)\vec{\phi} + \mathscr{B}(\tau_1,\tau_2)(t\vec{\phi} + \vec{\phi}_0) + \mathscr{L}_{\text{EDT}} . \tag{8.5}$$

Here, $\mathscr{A}$ and $\mathscr{B}$ are functions of the slow times to be determined so that $\vec{\Phi}^{(1)}$ satisfies the initial conditions (8.4c) with $j=1$, $\mathscr{L}_{\text{EDT}}$ is the notation for the sum of all the exponentially decaying terms, and the expressions for the vectors $\vec{\phi}$ and $\vec{\phi}_0$ are given in Appendix B. By assumption $\vec{\Phi}^{(1)}$ is a bounded function of t and hence we deduce from (8.5) that

$$\mathscr{B}(\tau_1,\tau_2) \equiv 0 . \tag{8.6}$$

This condition is formally obtained from (8.8) with $j=2$. By substituting (8.5) into the initial conditions we get

$$\mathscr{A}(0,0) = 0 . \tag{8.7}$$

We have assumed that $\vec{\phi}$ and the eigenvectors corresponding to the negative roots form an orthogonal set of functions.

In order to have bounded solutions of (8.4) the nonhomogeneous vectors $\vec{H}^{(j)}$ in (8.4) must satisfy the following conditions for every vector $\vec{\phi}^*$ that spans the null space of the adjoint linear problems:

$$\lim_{T\to\infty} \left[\frac{1}{T} \int_0^T \int_0^\pi \int_0^{\pi b} \int_0^{\pi a} (\vec{H}^{(j)} \cdot \vec{\Phi}^*) dx\, dy\, dz\, dt \right] = 0 \quad \text{for } j = 2,3,\ldots . \tag{8.8}$$

In Appendix B we list the adjoint eigenvectors corresponding to $\vec{\phi}$ and $t\vec{\phi} + \vec{\phi}_0$. All other adjoint eigenvectors are exponentially decaying in t and do not contribute to the conditions (8.8).

The solvability condition (8.8) with $j=2$ and $\vec{\Phi}^* = \vec{\phi}^*$ yields

$$r_1 = 0 . \tag{8.9}$$

We obtain the bounded solution

$$\vec{\Phi}^{(2)} = \mathscr{A}^{(2)}(\tau_1,\tau_2)\vec{\phi} - \mathscr{A}_{\tau_1}\vec{\phi}_0 + \mathscr{A}^2\vec{\phi}_1 + \mathscr{L}_{\text{EDT}} \tag{8.10a}$$

by substituting (8.9) into (8.4) with $j=2$ and solving the resulting problem. In (8.10a) $\mathscr{A}^{(2)}$ is to be determined, and the vector $\vec{\phi}_1$ is defined by

$$\vec{\phi}_1 \equiv \left[0,\ 0,\ 0\ -\frac{D^2 Q}{8}\sin(2z),\ -\frac{Q}{8}\sin(2z),\ \frac{D^2 Q^2}{4\sigma Q^0}\left[\cos\left[\frac{2Mx}{a}\right] + (Q^0 - Q^2)\cos(2z) \right] \right] . \tag{8.10b}$$

By inserting (8.10) into the initial conditions for $j=2$ in (8.4c) we find that

$$\mathscr{A}_{\tau_1}(0,0) = a^* , \tag{8.11}$$

where a^* is a linear functional of the initial data $\vec{u}^0$, T^0, and S^0 which, because of its length, we do not present here. The initial conditions for the amplitude $\mathscr{A}$ are given by (8.7) and (8.11).

We substitute (8.5), (8.6), (8.9), and (8.10) in the solvability condition (8.8) with $j=3$ and deduce that $\mathscr{A}(\tau_1,\tau_2)$ satisfies a Duffing equation given by

$$\mathscr{A}_{\tau_1\tau_1} + \overline{\alpha}\mathscr{A} - \overline{\beta}\mathscr{A}^3 = 0 . \tag{8.12}$$

where $\overline{\alpha} = \omega_1^2\alpha/\Omega^2$, $\overline{\beta} = \omega_1^2\beta/\Omega^2$. This equation corresponds to (6.11a). Thus the variation of $\mathscr{A}$ with the slow time τ_1 is determined by the initial-value problem (8.12), (8.7), and (8.11). The problem of determining the bounded solutions of this initial value problem is referred to as problem $\mathscr{A}$.

The steady states of (8.12), given by

$$\mathscr{A}^2=\frac{\alpha}{\beta} \quad \text{for } r_2 \le D^{-1} \tag{8.13}$$

correspond to saddle points in the phase plane and therefore are unstable to two-dimensional disturbances. It is easily verified from the first integral of (8.12) or from a phase plane analysis of problem $\mathscr{A}$ that bounded solutions in τ_1 exist for $r_2 < D^{-1}$ only if the initial data is confined within the separatrix. Then (8.7) and (8.11) imply that a^* satisfies

$$(a^*)^2 \le \alpha^2/(2\beta) . \tag{8.14}$$

If a^* satisfies (8.14) and $r_2 < D^{-1}$ then the solution of problem $\mathscr{A}$, periodic in τ_1, is given by

$$\mathscr{A}(\tau_1,\tau_2)=\mathscr{A}_0(\tau_2)\mathrm{sn}[\gamma(\tau_2)\tau_1+\theta(\tau_2)] , \tag{8.15a}$$

where the slower time-dependent amplitude $\mathscr{A}_0(\tau_2)$, and "frequency" $\gamma(\tau_2)$ are given in terms of the function $k(\tau_2)$ by

$$\mathscr{A}_0^2(\tau_2)=\frac{2(\alpha/\beta)k^2(\tau_2)}{1+k^2(\tau_2)} ,$$
$$\gamma^2(\tau_2)=\bar{\alpha}/[1+k^2(\tau_2)] . \tag{8.15b}$$

The second "constant" of integration $\theta(\tau_2)$ and the modulus of the elliptic function $k(\tau_2)$ are to be determined. Thus, if $\theta(\tau_2)$ and $k(\tau_2)$ are obtained, the leading-order term in the asymptotic expansion of the solution of the three-dimensional Boussinesq initial-value problem is evaluated. The values of $\theta(0)$ and $k(0)$, which are determined from (8.15a), the initial conditions (8.7) and (8.11), and the first integral of (8.12) are given by

$$\theta(0)=\begin{cases} 0 & \text{if } a^*\mathscr{A}_0(0)>0 \\ 2K & \text{if } a^*\mathscr{A}_0(0)<0 \end{cases} \tag{8.16a}$$

and

$$\frac{k^2(0)}{[1+k^2(0)]^2}=\frac{\beta(a^*)^2}{2\alpha^2} , \tag{8.16b}$$

where the sign of $\mathscr{A}_0(0)$ is arbitrary. We omit the details of obtaining (8.16).

The bounded solution of the initial-value problem (8.4) with $j=3$ is

$$\vec{\Phi}^{(3)}=\mathscr{A}^{(3)}(\tau_1,\tau_2)\vec{\phi} + \vec{J}(\mathscr{A},\mathscr{A}_{\tau_1},\mathscr{A}_{\tau_1\tau_1},\mathscr{A}_{\tau_2},\mathscr{A}^{(2)},\mathscr{A}^{(2)}_{\tau_1},x,z)+\mathscr{Z}_{\mathrm{EDT}} . \tag{8.17}$$

The known vector $\vec{J}$ is not listed explicitly because it is a lengthy function of the indicated arguments.

Finally, the solvability condition (8.8) with $j=4$ yields the following Hill's equation for the amplitude $\mathscr{A}^{(2)}(\tau_1,\tau_2)$:

$$\mathscr{A}^{(2)}_{\tau_1\tau_1}+[\bar{\alpha}-3\bar{\beta}\mathscr{A}^2(\tau_1,\tau_2)]\mathscr{A}^{(2)}=\Gamma(\mathscr{A},r_2,r_3,\Omega) , \tag{8.18a}$$

where Γ is defined by

$$\Gamma\equiv\frac{\Omega^{-1}}{\bar{R}_s-D^3\bar{R}_c}\left\{-2(\bar{R}_s-D^3\bar{R}_c)\mathscr{A}_{\tau_1\tau_2}+D^2(1-D)Qr_2\mathscr{A}_{\tau_1}+\Omega^2\frac{D^2(1-D)}{Q}\bar{R}_c\mathscr{A}_{\tau_1\tau_1\tau_1}+\Omega^{-1}D^3Q^2r_3\mathscr{A}\right.$$
$$\left.+\frac{DQ^2}{2}\left[\left[1+\frac{Q}{8}\right](D^4\bar{R}_c-\bar{R}_s)+\tfrac{3}{4}(\bar{R}_s-D^3\bar{R}_c)\right]\mathscr{A}^2\mathscr{A}_{\tau_1}\right\} . \tag{8.18b}$$

This equation corresponds to (6.14). The values of $\mathscr{A}^{(2)}(0,0)$ and $\mathscr{A}^{(2)}_{\tau_1}(0,0)$ are obtained by employing the initial conditions (8.4c), but we do not list them here. We observe that $\mathscr{A}^{(2)}=\mathscr{A}_{\tau_1}$ is a solution of the homogeneous equation corresponding to (8.18a). Therefore, a necessary condition for the existence of a bounded solution of (8.18) on the τ_1 time scale is

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T \Gamma\mathscr{A}_{\tau_1}d\tau_1\right]=0 . \tag{8.19a}$$

This condition implies that

$$\lim_{n\to\infty}\frac{\gamma}{4nK}\int_0^{4nK/\gamma}\Gamma\mathscr{A}_{\tau_1}d\tau_1=0 , \tag{8.19b}$$

where n is an integer and $4K/\gamma=2\pi$ is the period of $\mathscr{A}(\tau_1,\tau_2)$ in the time τ_1. Since $\mathscr{A}$ and $\mathscr{A}_{\tau_1}$ are periodic functions of τ_1, (8.19b) can be replaced by the equivalent solvability condition

$$\int_0^{2\pi}\Gamma\mathscr{A}_{\tau_1}d\tau_1=0 . \tag{8.20}$$

By employing (8.14), (8.15), and (8.18b), we obtain from (8.20), after a lengthy calculation, the following nonlinear differential equation for $k(\tau_2)$:

$$\mathscr{P}_{\tau_2}(k(\tau_2))=\mathscr{Q}(k(\tau_2),r_2) . \tag{8.21}$$

Here the nonlinear functions $\mathscr{P}$ and $\mathscr{Q}$ are defined by

$$\mathscr{P}\equiv\frac{K(k(\tau_2))}{1+k^2(\tau_2)}\{[1+k^2(\tau_2)]E(k(\tau_2)) + [k^2(\tau_2)-1]K(k(\tau_2))\} , \tag{8.22a}$$

$$\mathscr{Q}\equiv\frac{K(k(\tau_2))}{[1+k^2(\tau_2)]^2\bar{R}_s-D^3\bar{R}_c}[-\bar{\xi}_1(\bar{\alpha}/\bar{\beta})X(k(\tau_2)) +\bar{\xi}_2(\bar{\alpha}/\bar{\beta})Y(k(\tau_2)) +\bar{\xi}_3r_2Z(k(\tau_2))] , \tag{8.22b}$$

where X, Y, and Z are defined in (7.7) and $\bar{\xi}_1$, $\bar{\xi}_2$, and $\bar{\xi}_3$ are defined by

$$\bar{\xi}_1=\tfrac{1}{40}D^4(1-D)Q^2\bar{R}_c\,,\quad \bar{\xi}_3=D^2(1-D)Q\,,$$
$$\bar{\xi}_2=\tfrac{1}{5}DQ^2\left[\left[1+\frac{Q}{8}\right](D^4\bar{R}_c-\bar{R}_s)+\tfrac{3}{4}(\bar{R}_s-D^3\bar{R}_c)\right]. \tag{8.22c}$$

The initial condition $k(0)$ for (8.21) is obtained from (8.16b) with a^* satisfying (8.14). When this initial-value problem is solved for $k(\tau_2)$, the quantities $\mathscr{A}_0(\tau_2)$ and $\gamma(\tau_2)$ are given by (8.15b) and consequently $\mathscr{A}(\tau_1,\tau_2)$ is determined within a phase shift $\theta(\tau_2)$ by (8.15a).

To test the stability of the periodic solutions obtained in Secs. VI and VII, we first observe that $r_2=r_2(k)$ given by (7.6) is a steady solution of (8.21), i.e., it satisfies $\mathscr{D}(k,r_2)=0$. Denoting the inverse of this function by $k=k_0(r_2)$, we then linearize (8.21) about k_0 to obtain

$$k_{\tau_2}=\mathscr{S}(k_0)k\,, \tag{8.23}$$

where $\mathscr{S}(k_0)$ is a lengthy transcendental expression which we do not present here. However, we can show that it has the following properties:

$$\mathscr{S}(0)=\mathscr{S}(1)=0 \text{ and } \mathscr{S}(k_0)<0 \text{ for } 0<k_0<1\,. \tag{8.24}$$

Thus (8.23) and (8.24) imply that if k is initially close to k_0 it decays exponentially to k_0. Therefore, according to (8.15), (7.3), and (7.4), the amplitude $\mathscr{A}_0$ and frequency γ decay, respectively, to the amplitude A_0 and to the frequency $(2K/\pi)\omega_1$ of the periodic solutions obtained in Sec. VII. Consequently, for fixed r_2 in the interval $(\sigma+D)/(\sigma+1)<r_2<(D+\mu)^{-1}$, if the initial data (8.1) is sufficiently close to the periodic orbit determined asymptotically in Secs. VI and VII, the asymptotic approximation of the solution of the initial-value problem obtained in this section converges as $\tau_2\to\infty$ to this orbit modulo a phase shift depending on the slow time τ_2. Thus the periodic solutions of Secs. VI and VII are asymptotically ($\tau_2\to\infty$), orbitally stable in the sense we have just described, provided that $0<k_0<1$. This stability interval for k_0 is, from (7.6) and (8.3), equivalent to R_T in the interval

$$R_c^p<R_T<R_b\,, \tag{8.25}$$

as shown in Fig. 3.

To completely determine $\mathscr{A}(\tau_1,\tau_2)$, and hence the leading term in the asymptotic expansion (8.3) of the solution of the initial-value problem, we must find $\theta(\tau_2)$. An appropriate equation for this quantity can be deduced. Although we have not obtained this equation because of the unwieldy calculations involved, its solution could yield a phase-modulated amplitude $\mathscr{A}(\tau_1,\tau_2)$ which suggests a more complicated solution of the initial-value problem. However, the evolution equation for θ may imply that $\theta\to$const as $\tau_2\to\infty$, which would suggest that the periodic solutions found in Secs. VI and VII are in fact linearly and asymptotically stable.

ACKNOWLEDGMENTS

The research reported in this paper was supported by the U.S. Air-Force Office of Scientific Research under Grant No. AFOSR-80-0016 and the U.S. Office of Naval Research under Contract No. N00014-80-C-0196. The authors wish to thank Professor Thomas Erneux, Professor Thomas Mahar, and Professor William Kath for helpful discussions.

APPENDIX A

The matrix Z which appears in (5.7) and (5.8) is given by

$$Z=\begin{bmatrix} z_1 & z_3\\ z_2 & z_4\end{bmatrix}, \tag{A1a}$$

where

$$z_1=\omega_0\left[\frac{R_c^p}{(Q^2+\omega_0^2)^2}-\frac{DR_s}{(D^2Q^2+\omega_0^2)^2}\right], \tag{A1b}$$

$$z_2=\frac{\left[\dfrac{M}{a}\right]^2(\omega_0^2-Q^2)}{(Q^2+\omega_0^2)^2}R_c^p-\frac{\left[\dfrac{M}{a}\right]^2(\omega_0^2-D^2Q^2)}{(D^2Q^2+\omega_0^2)^2}R_s-\frac{Q}{\sigma}\,, \tag{A1c}$$

$$z_3=-\frac{1}{2(Q^2+\omega_0^2)}\,, \tag{A1d}$$

and

$$z_4=-\frac{\left[\dfrac{M}{a}\right]^2\omega_0}{(Q^2+\omega_0^2)}\,. \tag{A1e}$$

The quantities g_1 and g_2 which appear in (5.8), are given by

$$g_1=-\frac{\left[\dfrac{M}{a}\right]^2B_p^2}{32Q}\left[\frac{R_c^p}{(Q^2+\omega_0^2)^2}\left[Q^2+\frac{Q(2Q-\omega_0^2)}{\omega_0^2+4}-\frac{\omega_0^2(Q+2)}{\omega_0^2+4}\right]\right.$$
$$\left.-\frac{R_s}{(D^2Q^2+\omega_0^2)^2}\left[DQ^2+\frac{DQ(2D^2Q-\omega_0^2)}{\omega_0^2+4D^2}-\frac{D\omega_0^2(Q+2)}{\omega_0^2+4D^2}\right]\right], \tag{A2a}$$

$$g_2 = -\frac{\left[\frac{M}{a}\right]^4 B_p^2}{16}\left[\frac{R_c^p}{(Q^2+\omega_0^2)^2}\left[Q+\frac{Q(Q+2)}{\omega_0^2+4}+\frac{2Q-\omega_0^2}{\omega_0^2+4}\right]\right.$$

$$\left. -\frac{R_s}{(D^2Q^2+\omega_0^2)^2}\left[Q+\frac{D^2Q(Q+2)}{\omega_0^2+4D^2}+\frac{2D^2Q-\omega_0^2}{\omega_0^2+4D^2}\right]\right] . \tag{A2b}$$

The solution $\vec{\psi}_p^{(2)}$ of the $O(\epsilon_p^2)$ problem is given by

$$\psi_p^{(2)}=0 , \tag{A3a}$$

$$T_p^{(2)}=-\tfrac{1}{8}B_p^2Q^0[(Q^2+\omega_0^2)(4+\omega_0^2)]^{-1}[\tfrac{1}{2}Q(4+\omega_0^2)+\cos(2\tau-p_3)]\sin(2z) , \tag{A3b}$$

$$S_p^{(2)}=-\tfrac{1}{8}B_p^2Q^0[(D^2Q^2+\omega_0^2)(4D^2+\omega_0^2)]^{-1}[\tfrac{1}{2}Q(4D^2+\omega_0^2)+\cos(2\tau-p_4)]\sin(2z) , \tag{A3c}$$

where the phases p_3 and p_4 are defined by

$$\tan p_3 \equiv \frac{\omega_0(2+Q)}{2Q-\omega_0^2} , \tag{A4a}$$

$$\tan p_4 \equiv \frac{D\omega_0(2+Q)}{2D^2Q-\omega_0^2} . \tag{A4b}$$

APPENDIX B

The vectors $\vec{\phi}$ and $\vec{\phi}_0$ and the corresponding adjoint vectors $\vec{\phi}^{\,*}$ and $\vec{\phi}_0^{\,*}$ which appear in Sec. VIII are given by the following expressions:

$$\vec{\phi}\equiv[-DQ(a/M)\phi^{sc},0,DQ\phi^{cs},D\phi^{cs},\phi^{cs},-(DQ^2/Q^0)\phi^{cc}] , \tag{B1}$$

$$\vec{\phi}^{\,*}\equiv\{DQ(a/M)\phi^{sc},0,(DQ/\bar{R}_s)\phi^{cs},(D\bar{R}_T/\bar{R}_s)\phi^{cs},-\phi^{cs},[DQ^2/(Q^0\bar{R}_s)]\phi^{cc}\} , \tag{B2}$$

$$\vec{\phi}_0\equiv\{(a/m)\phi^{sc},0,-\phi^{cs},[(D-1)/Q]\phi^{cs},0,[(1+D\sigma^{-1})Q/Q^0]\phi^{cc}\} , \tag{B3}$$

and

$$\vec{\phi}_0^{\,*}\equiv\{[a/(M\bar{R}_s)]\phi^{sc},0,-(1/\bar{R}_s)\phi^{cs},[\bar{R}_T(D-1)/(R_sQ)]\phi^{cs},0,-[(1+\sigma^{-1}D)Q/(Q^0\bar{R}_s)]\phi^{cc}\} , \tag{B4}$$

where the trigonometric functions ϕ^{sc}, etc., are defined by $\phi^{sc}\equiv\sin[(M/a)x]\cos z$, etc.

---

*Present address: Department of Mathematics, The Florida State University, Tallahassee, FL 32306.

[1]H. E. Huppert and D. R. Moore, J. Fluid Mech. **78**, 825 (1976).
[2]G. Veronis, J. Marine Res. **23**, 1 (1965).
[3]G. Veronis, J. Fluid Mech. **34**, 315 (1968).
[4]J. S. Turner, *Buoyancy Effects in Fluids* (Cambridge University, London, 1973).
[5]P. G. Baines and A. E. Gill, J. Fluid Mech. **37**, 289 (1969).
[6]L. N. Da Costa, E. Knobloch, and N. O. Weiss, J. Fluid Mech. **109**, 25 (1981).
[7]W. F. Langford , SIAM J. Appl. Math. **37**, 22 (1979).
[8]E. Knobloch and M. R. E. Proctor, J. Fluid Mech. **108**, 291 (1981).
[9]W. L. Siegmann and L. A. Rubenfeld, SIAM J. Appl. Math. **29**, 540 (1975).
[10]L. A. Rubenfeld and W. L. Siegmann, SIAM J. Appl. Math **32**, 871 (1977).
[11]P. H. Coullet and E. A. Spiegel, SIAM J. Appl. Math. **43**, 776 (1983).
[12]S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Clarendon, Oxford, 1961), Chaps. 3 and 4.
[13]R. S. Schechter, M. G. Velarde, and J. K. Platten, Adv. Chem. Phys. **26**, 265 (1974).
[14]J. Guckenheimer and E. Knobloch, Geophys. Astrophys. Fluid Dyn. **23**, 247 (1983).
[15]H. Brand, P. Hohenberg, and V. Steinberg, Phys. Rev. A **27**, 591 (1983).
[16]D. D. Joseph, Arch. Rational Mech. Anal. **36**, 285 (1970).
[17]R. L. Sani, AIChE J. **11**, 971 (1965).
[18]D. Gutkowicz-Krusin, M. A. Collins, and J. Ross, Phys. Fluids **22**, 1451 (1979).
[19]E. L. Reiss, SIAM J. Appl. Math. **38**, 249 (1980).