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# Privman-Fisher hypothesis on finite systems: Verification in the case of a relativistic Bose gas with pair production

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The Privman-Fisher hypothesis on the singular part of the free-energy density of a finite system, near the bulk critical point $T=T_c$, is examined in the context of an ideal relativistic Bose gas confined to a cuboidal enclosure $(L_1 \times L_2 \times L_3)$ under periodic boundary conditions. Taking into account the possibility of particle-antiparticle pair production in the system, explicit expressions are derived for the free energy, the specific heat, and the condensate density at temperatures close to $T_c$, and the special cases of a cube, a square channel and a film are investigated at length. The various predictions of the Privman-Fisher hypothesis are fully borne out and the scaling functions governing the critical behavior of the system are found to be universal—irrespective of the severity of the relativistic effects. The influence of the latter enters only through the nonuniversal scale factors, $C_1$ and $C_2$, which depend on the particle mass $m$ and density $\rho$ as well.

## I. INTRODUCTION

In a recent paper Privman and Fisher[1] have argued that the "singular" part of the free-energy density of a finite hypercubical system $(L \times L \times \cdots \times L = L^d$, $d$ being less than the upper critical dimension $d_>$), near the bulk critical point $T=T_c$, may be expressed in the form[2]

$$f^{(s)}(t,h;L) \equiv \frac{F^{(s)}}{Vk_BT} \approx L^{-d}Y(C_1tL^{1/\nu}, C_2hL^{\Delta/\nu}) , \tag{1}$$

where $t$ and $h$ are the (reduced) temperature and field variables,

$$t=\frac{T-T_c}{T_c}, \quad h=\frac{\mu_{\rm eff}H}{k_BT} , \tag{2}$$

$x_1$ $(=C_1tL^{1/\nu})$ and $x_2$ $(=C_2hL^{\Delta/\nu})$ are the appropriate scaled variables, $\nu$ and $\Delta$ being the familiar bulk indices, while $C_1$ and $C_2$ are certain *nonuniversal*, system-dependent scale factors. The function $Y(x_1,x_2)$ is then a *universal* function, common to all systems in the same universality class as the given system. The more significant features of expression (1) are that (i) no nonuniversal metric factor, $C_0$, appears in front of the function $Y(x_1,x_2)$ and (ii) the variable $L$ here denotes the actual physical dimension of the system (and not one scaled in terms of any elementary length appropriate to the situation). As indicated by Privman and Fisher, the above formulation is valid for a cylindrical system $(L^{d-1}\times\infty)$ as well; in our investigation it seems to hold for a film $(L\times\infty^{d-1})$ too. Of course, the precise nature of the scaling function $Y(x_1,x_2)$ varies significantly as we move from one geometry to another; the same is true if we alter the set of boundary conditions to which the system is subjected.

Of pivotal importance in expression (1) are the scale factors, $C_1$ and $C_2$, whose determination may seem to require an explicit evaluation of the function $f^{(s)}(t,h;L)$ for the given finite system. In reality, such an evaluation is necessary only if one is interested in determining the exact form of the scaling function $Y(x_1,x_2)$; insofar as the scale factors are concerned, they can by determined from a study of the corresponding bulk system instead. To see this, we rewrite (1) in the form

$$f^{(s)}(t,h;L) \approx C_1^{d\nu}\,|\,t\,|^{d\nu}Z\left[C_1tL^{1/\nu}, \frac{C_2h}{C_1^{\Delta}\,|\,t\,|^{\Delta}}\right] , \tag{3}$$

where $Z(x_1,x_2')$ is some other universal function, and let $L\rightarrow\infty$. We obtain

$$f^{(s)}(t,h;\infty) \approx C_1^{d\nu}\,|\,t\,|^{d\nu}W^{\pm}\left[\frac{C_2h}{C_1^{\Delta}\,|\,t\,|^{\Delta}}\right] , \tag{4}$$

where $W^{\pm}(x_2')$ denote the limiting forms of the function $Z(x_1,x_2')$ as $x_1\rightarrow\pm\infty$. Now, remembering the hyperscaling relation $d\nu=2-\alpha$, we readily recover the standard bulk result

$$f^{(s)}(t,h;\infty) \approx A_1\,|\,t\,|^{2-\alpha}W^{\pm}(A_2h/|\,t\,|^{\Delta}) , \tag{5}$$

with the proviso that

$$C_1 \propto A_1^{1/(2-\alpha)}, \quad C_2 \propto A_1^{\Delta/(2-\alpha)}A_2 , \tag{6}$$

the constants of proportionality being universal. The scale factors $C_1$ and $C_2$ are, therefore, determinable from the bulk parameters $A_1$ and $A_2$ of the free-energy function $f^{(s)}(t,h;\infty)$. In actual practice, however, one need not invoke the free-energy function for this purpose; any bulk function, or functions, containing two *independent* bits of information on the singularity of the problem should do the job.

In the present paper we propose to test the scaling hypothesis (1) in the case of an ideal, relativistic Bose gas confined to restricted geometries, taking into account the possibility that particle-antiparticle pairs may be produced in the system.[3,4] Using methods developed in earlier papers,[5,6] hereafter referred to as I and II, explicit expressions are derived for various thermodynamic functions of the field-free system ($x_2=0$) confined to a cuboidal enclosure ($L_1 \times L_2 \times L_3$) under periodic boundary conditions. The special cases of a cube, a square channel, and a film are examined in detail, and a comparison is made with the corresponding results emerging from (1). Quite generally, the two sets of results are found to be in complete agreement. Additionally, while the scale factors $C_1$ and $C_2$, and certain asymptotic forms of the scaling function $Y(x_1,x_2)$ and its derivatives, can be determined from the appropriate bulk results,[4,7] our analysis of the finite system enables us to derive the complete mathematical form of these functions valid for all values of $x_1$. In view of the fact that these functions are characteristic of the geometry of the enclosure, finite-size corrections to the various thermodynamic properties of the system are also geometry dependent.

In Secs. II and III we carry out a detailed investigation of hypothesis (1) and establish a set of results relevant to the subject matter of this paper; this includes the determination of the scale factors $C_1$ and $C_2$ on the basis of the bulk results for an ideal relativistic Bose gas with pair production, as obtained earlier by Singh and Pandita.[7] Predictions for the finite system are thereby laid out. Next, these predictions are verified against actual, analytical results derived in Sec. IV; details of the process of verification are given in Secs. V and VI. Wherever possible, a comparison is made with the previous analytical results on this problem, which are generally some special cases of the ones reported here. In Sec. VII we make a brief reference to the field-dependent case ($x_2 \neq 0$) and describe how the scale factors $l_1$ and $l_2$ typically employed in that case can be obtained from the scale factors $C_1$ and $C_2$ of the present treatment.

## II. FORMULATION OF THE PROBLEM

In accordance with (1), the singular part of the specific heat per unit volume of the system will be given by

$$c_\rho^{(s)}(t,h;L) \approx C_1^2 L^{\alpha/\nu} Y_{(1)}(C_1 t L^{1/\nu}, C_2 h L^{\Delta/\nu}) \tag{7}$$

and that of the order parameter by

$$\Psi^{(s)}(t,h;L) \approx C_2 L^{-\beta/\nu} Y_{(2)}(C_1 t L^{1/\nu}, C_2 h L^{\Delta/\nu}) , \tag{8}$$

where $Y_{(1)}$ and $Y_{(2)}$ are appropriate derivatives of the original function $Y(x_1,x_2)$, while use has been made of the relationships

$$d\nu = 2-\alpha, \quad \Delta = \beta+\gamma, \quad \alpha+2\beta+\gamma=2 . \tag{9}$$

It will be noted that Eqs. (7) and (8) are consistent with the standard bulk behavior, viz.

$$c_\rho^{(s)}(t,0;\infty) \propto |t|^{-\alpha}, \quad \Psi^{(s)}(t,0;\infty) \propto |t|^{\beta} . \tag{10}$$

In the case of the Bose system it seems natural to study the condensate density

$$\rho_0(t,0;L) \propto [\Psi^{(s)}(t,0;L)]^2 \approx C_2^2 L^{-2\beta/\nu} [Y_{(2)}(C_1 t L^{1/\nu}, 0)]^2 . \tag{11}$$

In the following the specific heat $c_\rho^{(s)}$ will also pertain to the field-free situation $h=0$; in view of this, the variable $x_2$ may not be displayed explicitly in the subsequent expressions.

For the determination of $C_1$ and $C_2$ for the relativistic Bose gas we find it convenient to draw on the bulk behavior of the condensate density $\rho_0$ and the specific heat $c_\rho^{(s)}$, namely

$$\rho_0(t;\infty) = \begin{cases} B^2 |t|^{2\beta} \quad (t<0) & \text{(12a)} \\ 0 \quad (t>0) & \text{(12b)} \end{cases}$$

and

$$c_\rho^{(s)}(t;\infty) = \begin{cases} 0 \quad (t<0) & \text{(13a)} \\ -E_+ t^{-\alpha} \quad (t>0) , & \text{(13b)} \end{cases}$$

where[7]

$$B^2 = p_d \beta_c m^d \left| \frac{dW}{d\beta} \right|_c \tag{14}$$

and

$$E_+ = q_d \left[ \beta_c^2 m^{d-1} \left| \frac{dW}{d\beta} \right|_c \right]^{d/(d-2)} , \tag{15}$$

with

$$p_d = [2^{d-1} \pi^{d/2} \Gamma(d/2)]^{-1} , \tag{16}$$

$$q_d = \left[ \frac{d-2}{2} \right]^{(4-d)/(d-2)} \{ (2\sqrt{\pi})^d [\Gamma(d/2)]^{d/(d-2)} \times [\Gamma(2-d/2)]^{2/(d-2)} \}^{-1} , \tag{17}$$

and

$$W_d(\beta,\mu) = 2^{(d+1)/2} \pi^{-1/2} \Gamma(d/2) \times \sum_{j=1}^{\infty} \sinh(j\beta\mu) \frac{K_{(d+1)/2}(j\beta m)}{(j\beta m)^{(d-1)/2}} , \tag{18}$$

while

$$W \equiv W_d(\beta,m) .$$

Here, $d$ stands for the dimensionality of the system ($2<d<4$), $\beta$ for the inverse temperature, $\mu$ for the chemical potential, $m$ for the particle mass, and $K_\nu(z)$ for the modified Bessel function, while the critical point $\beta_c$ is determined by the condition

$$(W_d)_c \equiv W_d(\beta_c, m) = 2^{d-1} \pi^{d/2} \Gamma(d/2) (\rho/m^d) , \tag{19}$$

$\rho$ ($\equiv Q/V$) being the "charge" density in the system; note that the units employed are such that $\hbar = c = k_B = 1$.

To reproduce (12a) from (11), the scaling function $[Y_{(2)}(x_1)]^2$ must behave as

$$P_- |x_1|^{2\beta} \quad (x_1 \to -\infty) , \tag{20}$$

with

$$P_- = \frac{B^2}{C_1^{2\beta} C_2^2} . \tag{21}$$

Similarly, to reproduce (13b) from (7), the scaling function $Y_{(1)}(x_1)$ must behave as

$$-C_+ x_1^{-\alpha} \quad (x_1 \to +\infty) , \tag{22}$$

with

$$C_+ = \frac{E_+}{C_1^{2-\alpha}} . \tag{23}$$

It follows that

$$C_1 = \left[ \frac{E_+}{C_+} \right]^{1/(2-\alpha)}, \quad C_2 = \frac{B}{P_-^{1/2}} \left[ \frac{C_+}{E_+} \right]^{\beta/(2-\alpha)} . \tag{24}$$

Substituting from Eqs. (14) and (15), and noting that the critical indices $\alpha$ and $\beta$ for the system under study are given by[7,8]

$$\alpha = \frac{4-d}{2-d}, \quad \beta = \tfrac{1}{2} \quad (2 < d < 4) , \tag{25}$$

we obtain

$$C_1 = \left[ \frac{q_d}{C_+} \right]^{(d-2)/d} \beta_c^2 m^{d-1} \left| \frac{dW}{d\beta} \right|_c \tag{26}$$

and

$$C_2 = \left[ \frac{p_d}{P_-} \right]^{1/2} \left[ \frac{C_+}{q_d} \right]^{(d-2)/2d} \left[ \frac{m}{\beta_c} \right]^{1/2} . \tag{27}$$

For simplicity, we may choose the normalization of the universal function $Y$ such that the coefficients $P_-$ and $C_+$ appearing in the asymptotic expressions (20) and (22) are exactly equal to the numbers $p_d$ and $q_d$, respectively; note that, for $d=3$, $p_d = 1/(2\pi^2)$ and $q_d = 1/(2\pi^4)$. With this choice, $C_1$ and $C_2$ assume the simplified form

$$C_1 = \beta_c^2 m^{d-1} \left| \frac{dW}{d\beta} \right|_c , \quad C_2 = (m/\beta_c)^{1/2} , \tag{28}$$

which is clearly system dependent.

Once $C_1$ and $C_2$ are known, no more nonuniversal amplitudes are needed to describe the critical behavior of the system—regardless of whether it is finite or infinite in extent; all amplitudes appearing in the expressions for the various physical properties of the system will be related to $C_1$ and $C_2$ through universal factors alone. Before we proceed to carry out a detailed evaluation of the various quantities of interest, we shall examine the most salient consequences of the Privman-Fisher hypothesis.

## III. CONSEQUENCES OF THE PRIVMAN-FISHER HYPOTHESIS

We start with the free-energy density $f^{(s)}(t;L)$, as given by Eq. (1) or (3) with $h=0$, and write

$$f^{(s)}(t;L) \approx F_+ t^{2-\alpha} \quad (t>0, \ L \to \infty) . \tag{29}$$

Comparing (29) with (13b), we find that

$$F_+ = E_+ / (2-\alpha)(1-\alpha) . \tag{30}$$

At the same time, we conclude that the scaling function $Y(x_1)$ in (1) must behave as

$$Y_+ x_1^{2-\alpha} \quad (x_1 \to +\infty) , \tag{31}$$

with the universal coefficient

$$Y_+ = F_+ / C_1^{2-\alpha} ; \tag{32}$$

cf. the corresponding equation (23) for the specific-heat function $c_\rho^{(s)}$.

For $t<0$ and $L \to \infty$, there are two possibilities of interest in the present study.

$$\text{(i)} \quad Y(x_1) \to Y_- \, |x_1|^{\nu(d-\epsilon)} \quad (x_1 \to -\infty) , \tag{33a}$$

so that

$$f^{(s)}(t;L) \approx Y_- C_1^{\nu(d-\epsilon)} \, |t|^{\nu(d-\epsilon)} L^{-\epsilon} , \tag{34a}$$

where the index $\epsilon$ is as yet undetermined but is expected to be geometry dependent.

$$\text{(ii)} \quad Y(x_1) \to Y_-^* (\ln|x_1| + \text{const}) \quad (x_1 \to -\infty) , \tag{33b}$$

so that

$$f^{(s)}(t;L) \approx Y_-^* \left[ \ln C_1 + \ln|t| + \frac{1}{\nu} \ln L + \text{const} \right] L^{-d} . \tag{34b}$$

In each case the coefficient $Y_-$ or $Y_-^*$ is universal. The repercussion of this on the specific heat of the system is that for $\epsilon \neq d$,

$$c_\rho^{(s)}(t;L) \propto |t|^{-(\alpha+\nu\epsilon)} L^{-\epsilon} \quad (t<0, \ L \to \infty) . \tag{35a}$$

The special case $\epsilon \to d$ corresponds to possibility (ii) above, for which $c_\rho^{(s)}(t;L)$ is still given by (35a), i.e.,

$$c_\rho^{(s)}(t;L) \propto |t|^{-2} L^{-d} . \tag{35b}$$

For $\epsilon = d$, the leading term in $f^{(s)}(t;L)$ would be independent of $t$, with the result that no $L^{-d}$ term would appear in the specific heat of the system.

We shall now examine the condensate density $\rho_0(t;L)$. Since, for $t>0$ and $L \to \infty$, the total number of particles in the ground state is expected to be $O(1)$, we conclude that, for a hypercube of volume $L^d$, the function $[Y_{(2)}(x_1)]^2$ in (11) must behave as

$$P_+ x_1^{-\gamma} \quad (x_1 \to +\infty) , \tag{36}$$

with $P_+$ universal, so that

$$\rho_0(t;L) \approx P_+ C_1^{-\gamma} C_2^2 t^{-\gamma} L^{-d} \quad (t>0, \ L \to \infty) , \tag{37}$$

as desired; here again, use has been made of relationships (9) among the various bulk indices.

For $t<0$ and $L \to \infty$, we may write on general grounds[9,10]

$$\Psi^{(s)}(t;L) \approx B|t|^{\beta} + B^{\times} L^{-1} |t|^{\beta^{\times}} , \tag{38}$$

with the result that

$$\rho_0(t;L) \approx B^2 |t|^{2\beta} + 2BB^{\times} L^{-1} |t|^{\beta+\beta^{\times}} ; \tag{39}$$

cf. Eq. (12a) for $\rho_0(t;\infty)$. (Here, the superscript $\times$ refers to the surface properties, as usual.) In view of (39), the function $[Y_{(2)}(x_1)]^2$ must behave as

$$P_- |x_1|^{2\beta}+P_-^{\times} |x_1|^{\phi} \quad (x_1 \to -\infty) , \tag{40}$$

with both $P_-$ and $P_-^{\times}$ universal; again, cf. the corresponding bulk Eq. (20). In order that Eqs. (11) and (40) lead to (39), we require, apart from the relation (21),

$$P_-^{\times}=\frac{2BB^{\times}}{C_1^{2\beta-\nu}C_2^2}, \quad \phi=2\beta-\nu, \quad \beta^{\times}=\beta-\nu . \tag{41}$$

In view of the known results,[7,8]

$$\beta=\tfrac{1}{2}, \quad \nu=1/(d-2) , \tag{42}$$

it follows that

$$\phi=(d-3)/(d-2), \quad \beta^{\times}=(d-4)/2(d-2) \qquad (2<d<4) . \tag{43}$$

Finally, in the "core" region, where $|x_1|=O(1)$ and hence $|t|=O(L^{-1/\nu})$, the functions $f^{(s)}(t,L)$, $c_\rho^{(s)}(t;L)$, and $\rho_0(t;L)$, for a fixed value of $x_1$, are proportional to $L^{-d}$, $L^{\alpha/\nu}$, and $L^{-2\beta/\nu}$, respectively; see Eqs. (1), (7), and (11). It follows that the quantities

$$U=f^{(s)}(0;L)L^d , \tag{44}$$

$$U_{(1)}=c_\rho^{(s)}(0;L)L^{-\alpha/\nu}C_1^{-2} , \tag{45}$$

and

$$U_{(2)}=\rho_0(0;L)L^{2\beta/\nu}C_2^{-2} , \tag{46}$$

evaluated at the erstwhile critical point ($t=0$), which clearly lies in the core region, must be universal. This completes the set of predictions, on the basis of the Privman-Fisher hypothesis, which we propose to test at length in the sections ahead.

## IV. THERMODYNAMICS OF AN IDEAL RELATIVISTIC BOSE GAS WITH PAIR PRODUCTION

We consider an ideal Bose gas composed of $N_1$ particles and $N_2$ antiparticles, each of mass $m$, confined to a three-dimensional cuboidal enclosure of sides $L_1$, $L_2$, and $L_3$. Since particles and antiparticles are supposed to be created in pairs, the system is governed by the conservation of the number $Q$ $(=N_1-N_2)$, rather than of the numbers $N_1$ and $N_2$ separately; the conserved quantity $Q$ may be looked upon as a kind of generalized "charge." In equilibrium the chemical potentials of the two species will be equal and opposite: $\mu_1=-\mu_2=\mu$, say, with the result that[3]

$$N_1=\sum_{\epsilon}(e^{\beta(\epsilon-\mu)}-1)^{-1} , \qquad N_2=\sum_{\epsilon}(e^{\beta(\epsilon+\mu)}-1)^{-1} , \tag{47}$$

where $\beta=1/T$ and $\epsilon=(\mathbf{k}^2+m^2)^{1/2}$; remember that we are using units such that $\hbar=c=k_B=1$. Both $\epsilon$ and $\mu$ here include the rest energy $m$ of the particle (or the antiparticle) and, for the mean occupation numbers in the various states to be positive definite, we must have $|\mu|\le m$. Assuming that, to begin with, $\mu>0$, it readily follows that $N_1>N_2$ and hence $Q>0$. In view of the conservation of $Q$, $\mu$ then stays positive under all circumstances. Without loss of generality, we shall assume this to be the case.

Under periodic boundary conditions, the eigenvalues $k_i$ $(i=1,2,3)$ of the wave vector $\mathbf{k}$ are given by

$$k_i=(2\pi/L_i)n_i \quad (n_i=0,\pm1,\pm2,\dots) ; \tag{48}$$

the pressure $\mathscr{P}$ in the grand canonical ensemble may then be written as

$$\mathscr{P}=-\frac{1}{\beta V}\sum_{\epsilon_n}[\ln(1-e^{-\beta(\epsilon_n-\mu)})+\ln(1-e^{-\beta(\epsilon_n+\mu)})] = \frac{2}{\beta V}\sum_{j=1}^{\infty}\frac{\cosh(j\beta\mu)}{j}\sum_{n_1,n_2,n_3=-\infty}^{\infty}\exp\left\{-j\beta m\left[1+\frac{4\pi^2}{m^2}\sum_{i=1}^{3}\left[\frac{n_i}{L_i}\right]^2\right]^{1/2}\right\} . \tag{49}$$

Following the techniques of I and II, we obtain (correct to *all* powers of the parameters $\lambda/L_i$, where $\lambda$ denotes the mean thermal wavelength $\sqrt{2\pi\beta/m}$ or the Compton wavelength $1/m$ of the particles)

$$\mathscr{P}=\frac{m^4}{2\pi^2}X(\beta,\mu)+\frac{1}{2\pi\beta}[(m^2-\mu^2)^{1/2}H_2(\mu)+H_3(\mu)] , \tag{50}$$

where

$$X(\beta,\mu)=2\sum_{j=1}^{\infty}\cosh(j\beta\mu)\frac{K_2(j\beta m)}{(j\beta m)^2} \tag{51}$$

and

$$H_n(\mu)=\sum_{\mathbf{q}}{}'\frac{e^{-(m^2-\mu^2)^{1/2}\gamma(\mathbf{q})}}{\gamma^n(\mathbf{q})} , \tag{52}$$

with

$$\gamma(\mathbf{q})=(q_1^2L_1^2+q_2^2L_2^2+q_3^2L_3^2)^{1/2} . \tag{53}$$

The primed summation in (52) implies that the term with $\mathbf{q}=0$ is excluded; accordingly, $\gamma(\mathbf{q})>0$. Using the standard thermodynamic relation[11] $\rho=(\partial\mathscr{P}/\partial\mu)_T$ we obtain for the charge density

$$\rho\equiv\frac{Q}{V}=\frac{m^3}{2\pi^2}W(\beta,\mu)+\frac{\mu}{2\pi\beta}H_1(\mu) , \tag{54}$$

where

$$W(\beta,\mu)=m\left[\frac{\partial X}{\partial \mu}\right]_\beta =2\sum_{j=1}^{\infty}\sinh(j\beta\mu)\frac{K_2(j\beta m)}{j\beta m}\,, \tag{55}$$

while use has been made of the recurrence relation

$$\frac{d}{dx}H_n(x)=-H_{n-1}(x)\quad [x\equiv(m^2-\mu^2)^{1/2}]\,; \tag{56}$$

Eq. (54) agrees with the corresponding result, viz. Eq. (29), of I. The *thermal* free-energy density of the system is then given by

$$\frac{\bar F}{V}\equiv\frac{F-mQ}{V}=(\mu-m)\rho-\mathscr{P} =-\frac{m^4}{2\pi^2}\left[\frac{(m-\mu)}{m}W(\beta,\mu)+X(\beta,\mu)\right] -\frac{1}{2\pi\beta}[\mu(m-\mu)H_1(\mu) +(m^2-\mu^2)^{1/2}H_2(\mu)+H_3(\mu)]\,. \tag{57}$$

In the region of phase transition ($\mu\simeq m$), we invoke the expansions[7]

$$W(\beta,\mu)=W(\beta,m)-\frac{\pi}{\beta m^2}(m^2-\mu^2)^{1/2}+O(m^2-\mu^2) \tag{58}$$

and

$$X(\beta,\mu)=X(\beta,m)+\frac{\mu-m}{m}W(\beta,m) +\frac{\pi}{3\beta m^4}(m^2-\mu^2)^{3/2}+O((m^2-\mu^2)^2)\,, \tag{59}$$

whence Eqs. (54) and (57) take the form

$$\rho=\frac{m^3}{2\pi^2}W(\beta,m)-\frac{m}{2\pi\beta}[(m^2-\mu^2)^{1/2} -H_1(\mu)]+O(m^2-\mu^2) \tag{60}$$

and

$$\frac{\bar F}{V}=-\frac{m^4}{2\pi^2}X(\beta,m) +\frac{1}{2\pi\beta}\{\tfrac{1}{6}(m^2-\mu^2)^{3/2} -[\tfrac{1}{2}(m^2-\mu^2)H_1(\mu)+(m^2-\mu^2)^{1/2}H_2(\mu) +H_3(\mu)]\}+O((m^2-\mu^2)^2)\,. \tag{61}$$

We readily identify the singular part of the free-energy density, viz.

$$f^{(s)}(t;L)\equiv\frac{\beta\bar F^{(s)}}{V}=\frac{1}{2\pi}\{\tfrac{1}{6}(m^2-\mu^2)^{3/2} -[\tfrac{1}{2}(m^2-\mu^2)H_1(\mu) +(m^2-\mu^2)^{1/2}H_2(\mu) +H_3(\mu)]\}\,, \tag{62}$$

where $\mu(t)$, for a given $\rho$, is determined by Eq. (60).

Introducing the thermogeometric parameters $y_i$, as defined in I, namely

$$y_i=\tfrac{1}{2}(m^2-\mu^2)^{1/2}L_i\quad (i=1,2,3)\,, \tag{63}$$

and remembering that the bulk critical point $\beta_c$ is determined by the condition

$$\rho=\frac{m^3}{2\pi^2}W(\beta_c,m)\,, \tag{64}$$

Eq. (60) becomes, to leading order in $(m^2-\mu^2)$,

$$W(\beta,m)-W(\beta_c,m)=\frac{\pi}{\beta m^2}\frac{y_<}{L_<}[2-S_1(y_i)]\,, \tag{65}$$

where $S_n(y_i)$ denote the familiar sums[12]

$$S_n(y_i)=\sum_{\mathbf q}{}'\frac{e^{-2R(\mathbf q)}}{R^n(\mathbf q)}\quad [R(\mathbf q)=(q_1^2y_1^2+q_2^2y_2^2+q_3^2y_3^2)^{1/2}]\,, =2^n(m^2-\mu^2)^{-n/2}H_n(\mu) \tag{66}$$

while $L_<$ denotes, for convenience, the shortest side of the container. At the same time, Eq. (62) takes the form

$$f^{(s)}(t;L)=\frac{1}{2\pi}\frac{y_<^3}{L_<^3}\{\tfrac{4}{3}-[2S_1(y_i)+2S_2(y_i)+S_3(y_i)]\}\,. \tag{67}$$

If we now specialize to the geometry of (i) a cube ($L_1=L_2=L_3=L$), (ii) a square channel ($L_1\to\infty$, $L_2=L_3=L$), or (iii) a film ($L_1,L_2\to\infty$, $L_3=L$), then we have to deal with only one $y$, viz.

$$y=\tfrac{1}{2}(m^2-\mu^2)^{1/2}L\,. \tag{68}$$

Furthermore, if we restrict ourselves to temperatures close to the bulk critical point, $\beta\simeq\beta_c$, Eq. (65) reduces to

$$y[2-S_1(y)]=\frac{\beta_c^2m^2}{\pi}\left|\frac{dW}{d\beta}\right|_c Lt\quad (|t|\ll 1)\,. \tag{69}$$

Recalling expression (28) for $C_1$, with $d=3$, the right-hand side of (69) becomes $C_1Lt/\pi$ and, since the index $\nu$ for the system under study is equal to 1, it further reduces to $x_1/\pi$, where $x_1$ $(=C_1tL^{1/\nu})$ is the scaled variable appropriate to the present analysis. Expression (67) is thus manifestly in conformity with the Privman-Fisher hypothesis (1), for

$$f^{(s)}(t;L)=L^{-3}Y(x_1)\,, \tag{70}$$

where $Y(x_1)$ is given by the *parametric equations*

$$Y(y)=\frac{1}{2\pi}y^3\{\tfrac{4}{3}-[2S_1(y)+2S_2(y)+S_3(y)]\}\ , \tag{71}$$

$$x_1(y)=\pi y[2-S_1(y)]\ , \tag{72}$$

among which the parameter $y$ is supposed to be eliminated. Note that no nonuniversal metric factor $C_0$ appears in front of the function $Y(x_1)$ in Eq. (70). We are now in a position to verify the set of predictions made in Sec. III.

## V. TESTS OF THE PRIVMAN-FISHER HYPOTHESIS

First of all, we shall consider the behavior of the scaling function $Y(x_1)$ in different regimes of $t$ and $L$ and for different geometries of the enclosure.

*(a)* $t>0$, $L\to\infty$. In this regime, $x_1\to+\infty$, with the result that $y$ diverges while the functions $S_n(y)$ vanish exponentially. Equations (71) and (72) then give

$$Y(x_1)\simeq\frac{x_1^3}{12\pi^4}\ , \tag{73}$$

which agrees with prediction (31), with $\alpha=-1$ and $Y_+=1/(12\pi^4)$. While the value of $\alpha$ obtained here agrees with the bulk result (25), with $d=3$, the value of $Y_+$ agrees with the prediction [see Eqs. (30), (32), (15), and (28)]

$$Y_+=\frac{E_+}{6C_1^3}=\frac{q_3}{6}=\frac{1}{12\pi^4}\ . \tag{74}$$

*(b)* $t<0$, $L\to\infty$. In this regime, $x_1\to-\infty$, with the result that $y$ tends to zero while the functions $S_n(y)$ diverge. It is not difficult to show that, for $y\to0$,

$$S_n(y)\to\begin{cases}\dfrac{\pi^{d^*/2}\Gamma(d^*-n)}{2^{d^*-1-n}\Gamma(d^*/2)}\dfrac{1}{y^{d^*}} & (n<d^*) \quad (75\text{a})\\[2ex] \dfrac{2\pi^{d^*/2}}{\Gamma(d^*/2)}\dfrac{\ln(1/y)+\text{const}}{y^{d^*}} & (n=d^*) \quad (75\text{b})\\[2ex] \left[\sum_{\mathbf{q}}{}'\dfrac{1}{|\mathbf{q}|^n}\right]\dfrac{1}{y^n} & (n>d^*)\ , \quad (75\text{c})\end{cases}$$

where $d^*$ $(\le d)$ is the number of dimensions in which the system is finite; $d^*=3$ for a cube, 2 for a square channel, and 1 for a film. The asymptotic forms of the function $S_n(y)$, for relevant values of $n$ and $d^*$ are given in Table I; for the evaluation of $S_3(y)$, with $d^*=2$, we had to make use of the Hardy sum[13]

$$\sum_{q_1,q_2=-\infty}^{\infty}{}'(q_1^2+q_2^2)^{-s}=4\zeta(s)\beta(s)\quad(s>1)\ ,$$

where

$$\zeta(s)=\sum_{l=0}^{\infty}(l+1)^{-s}\ ,$$

$$\beta(s)=\sum_{l=0}^{\infty}(-1)^l(2l+1)^{-s}\ .$$

Applying Eqs. (71) and (72) to the three geometries of interest, we obtain the following results.

*(i) cube* ($d^*=3$):

$$Y(x_1)\simeq(-\ln|x_1|+\text{const})\ , \tag{76}$$

which conforms to Eq. (33b), with $Y_-^*=-1$. According to (34b), this will lead to a finite-size term, $-1/(t^2L^3)$, in the specific heat of the system, which is indeed true; see Eq. (31) of II.

*(ii) square channel* ($d^*=2$):

$$Y(x_1)\simeq-\frac{2}{\pi}\zeta(\tfrac{3}{2})\beta(\tfrac{3}{2})\ , \tag{77}$$

which conforms to Eq. (33a), with $\epsilon=3$. The leading term in $f^{(s)}(t;L)$ being independent of $t$, no $L^{-3}$ term is expected to appear in the specific heat of the system in this case. This is again true; see Eq. (35) of II, which shows that the leading finite-size term in $c_p^{(s)}(t;L)$ in this case is of order $|t|^{-3}L^{-4}$.

*(iii) film* ($d^*=1$):

$$Y(x_1)\simeq-\frac{1}{\pi}\zeta(3)\ , \tag{78}$$

which is qualitatively similar to the case of the square channel. Here, too, we do not expect an $L^{-3}$ term in the specific heat. A reference to Eq. (39) of II shows that the finite-size effect in this case is exponentially small.

TABLE I. Asymptotic forms of the function $S_n(y)$ for $y\to0$.

| $n$ \ $d^*$ | 1 | 2 | 3 |
|---|---|---|---|
| 0 | $\frac{1}{y}$ | $\frac{\pi}{2y^2}$ | $\frac{\pi}{y^3}$ |
| 1 | $\frac{2}{y}\left[\ln\left[\frac{1}{y}\right]+\text{const}\right]$ | $\frac{\pi}{y^2}$ | $\frac{\pi}{y^3}$ |
| 2 | $\frac{\pi^2}{3y^2}$ | $\frac{2\pi}{y^2}\left[\ln\left[\frac{1}{y}\right]+\text{const}\right]$ | $\frac{2\pi}{y^3}$ |
| 3 | $\frac{2\zeta(3)}{y^3}$ | $\frac{4\zeta(\frac{3}{2})\beta(\frac{3}{2})}{y^3}$ | $\frac{4\pi}{y^3}\left[\ln\left[\frac{1}{y}\right]+\text{const}\right]$ |

*(c) The core region* $|tL|=O(1)$. In this region both $|x_1|$ and $y$ are of order 1, so Eqs. (71) and (72) have to be solved numerically. At the bulk critical point ($x_1=0$), we have

$$S_1(y_0)=2\ , \tag{79}$$

which gives $y_0=0.97$, 0.76, and 0.48 for $d^*=3$, 2, and 1, respectively.[12] This leads to the manifestly universal number [see Eq. (44)]

$$U=-\frac{y_0^3}{2\pi}[\tfrac{8}{3}+2S_2(y_0)+S_3(y_0)]\ , \tag{80}$$

which depends on $d^*$ only.

We shall now consider the specific-heat function

$$c_p^{(s)}(t;L)=-\left[\frac{\partial^2 f^{(s)}(t;L)}{\partial t^2}\right]_L = -\frac{C_1^2}{L}\frac{Y''x_1'-Y'x_1''}{x_1'^3}\ ,$$

where primes denote differentiation with respect to $y$. Making use of the relation

$$\frac{dS_n(y)}{dy}=-\frac{2S_{n-1}(y)+nS_n(y)}{y}\ , \tag{81}$$

we obtain

$$c_p^{(s)}(t;L)=-\frac{C_1^2}{\pi^3 L}\frac{y}{1+S_0(y)}\ , \tag{82}$$

which agrees with Eq. (25) of II. The scaling function $Y_{(1)}(x_1)$ of Eq. (7) thus turns out to be

$$Y_{(1)}(x_1)=-\frac{y}{\pi^3[1+S_0(y)]}\ , \tag{83}$$

coupled, of course, with Eq. (72) for $x_1(y)$. Again, we shall examine different regions one by one.

*(a)* $t>0$, $L\to\infty$. In this region $y\simeq x_1/(2\pi)$, whence

$$c_p^{(s)}(t;L)\simeq-\frac{C_1^3}{2\pi^4}t\ , \tag{84}$$

which reproduces the bulk result (13b), with $E_+$ given by (15). It will be noted that finite-size effects in this region are exponentially small.

*(b)* $t<0$, $L\to\infty$. Using Eqs. (72) and (75), we find that in this region

$$c_p^{(s)}(t;L)\simeq-\left[\frac{\pi^{(4-d^*)}}{(d^*-1)^{(d^*+1)}}\left[\frac{\Gamma(d^*)}{\Gamma(d^*/2)}\right]^2\left[\frac{C_1}{2}\right]^{d^*-3}\frac{1}{|t|^{d^*+1}L^{2d^*}}\right]^{1/(d^*-1)} \tag{85a}$$

for $d^*>1$, and

$$c_p^{(s)}(t;L)\simeq-\text{const}\,\frac{C_1^2}{\pi^3 L}\exp\left[-\frac{C_1}{\pi}|t|L\right] \tag{85b}$$

for $d^*=1$. Equations (85) give finite-size effects in the specific heat of the system for temperatures below, but close to, the bulk critical temperature $T_c$. It is significant to observe that *only* in the case of a film does one get exponential effects; for other geometries, one obtains a power law instead. For all relevant values of $d^*$, the present results agree with the ones reported in II.

*(c)* In the core region further progress can only be made numerically. Of course, at $t=0$, we encounter the universal number

$$U_{(1)}=-\frac{y_0}{\pi^3[1+S_0(y_0)]}\ ; \tag{86}$$

see Eq. (45).

At this stage it seems worthwhile to point out that the scaling function $Y_{(1)}(x_1)$ in the case of film can be expressed in a closed form which holds over a considerable range of temperatures—in fact, from $T\gtrsim T_c$ down to $T=0$ K. Noting that, for $d^*=1$,

$$S_1(y)=-\frac{2}{y}\ln(1-e^{-2y}),\quad S_0(y)=\frac{2}{e^{2y}-1}\ , \tag{87}$$

Eqs. (72) and (83) yield

$$Y_{(1)}(x_1)=-\frac{1}{\pi^3}y\tanh y\ ,\qquad y=\sinh^{-1}(\tfrac{1}{2}e^{x_1/2\pi})\ . \tag{88}$$

It follows that

$$c_p^{(s)}(t;L)=-\frac{C_1^2}{\pi^3 L}y\tanh y\ . \tag{89}$$

In the *nonrelativistic* (NR) limit the scale factor $C_1$, as given by Eq. (28), takes the form

$$(C_1)_{\text{NR}}=\begin{cases}(4\pi)^{(d-2)/2}\Gamma\left[\frac{d}{2}+1\right]\left[\zeta\left[\frac{d}{2}\right]\right]^{2/d}\rho^{(d-2)/d}\quad(d>2) & (90)\\[6pt] \frac{3\pi}{2}[\zeta(\tfrac{3}{2})]^{2/3}\rho^{1/3}\quad(d=3)\ , & (91)\end{cases}$$

whence one obtains

$$c_\rho^{(s)}(t;L) = -\frac{9[\zeta(\frac{3}{2})]^{4/3}\rho^{2/3}}{4\pi L} y \tanh y \ , \tag{92}$$

where

$$y = \sinh^{-1}(\tfrac{1}{2}\exp\{\tfrac{3}{4}[\zeta(\tfrac{3}{2})]^{2/3}\rho^{1/3}tL\}) \ . \tag{93}$$

Equations (92) and (93) are in complete agreement with the corresponding results of Barber and Fisher reported earlier.[14]

## VI. SCALING FUNCTION FOR THE CONDENSATE

For the analysis of the condensate we may restrict ourselves to the case of the cube. Since the condensate density $\rho_0$, for $|t| \ll 1$, is given by

$$\rho_0 = \frac{m}{2\beta_c L y^2} \tag{94}$$

[see Eq. (40) of I], the scaling function $[Y_2(x_1)]^2$ in (11) has the simple form

$$[Y_{(2)}(x_1)]^2 = \frac{1}{2y^2} \ , \tag{95}$$

where $y(x_1)$, as before, is determined by Eq. (72). In region (a) we obtain

$$[Y_{(2)}(x_1)]^2 \simeq \frac{2\pi^2}{x_1^2} \ ,$$
$$\rho_0(t;L) \simeq 2\pi^2 \left[\frac{C_2}{C_1}\right]^2 \frac{1}{t^2 L^3} \ , \tag{96}$$

which agree with predictions (36) and (37), with $\gamma=2$ and $P_+ = 2\pi^2$. In region (b) we obtain

$$[Y_{(2)}(x_1)]^2 \simeq \frac{|x_1|}{2\pi^2} \ , \tag{97}$$

which agrees with (20), with $\beta = \frac{1}{2}$ and $P_- = 1/(2\pi^2)$.

To go beyond the leading term in (97), we invoke the asymptotic expansion[15,16]

$$S_1(y) = \frac{\pi}{y^3} + \frac{C_3}{\pi y} + 2 + O(y) \quad (d^* = 3) \ ,$$

where

$$C_3 = \pi \lim_{y\to 0} \left[ {\sum_{\mathbf{q}}}' \frac{e^{-2yq}}{q} - \int_{\text{all } \mathbf{q}} \frac{e^{-2yq}}{q} d^3q \right]$$
$$= -8.913\,633 \ldots \ ,$$

with the result that

$$[Y_{(2)}(x_1)]^2 \simeq \frac{1}{2\pi^2}(|x_1| + |C_3|) \ . \tag{98}$$

This agrees with prediction (40), with $\phi = 0$ and $P_- = |C_3|/(2\pi^2)$; see also Eq. (43) for $\phi$. One is thereby led to a "surface" condensate of the form

$$\rho_0^\times = \frac{|C_3| m}{2\pi^2 \beta_c L} \ , \tag{99}$$

which agrees with Eqs. (38) and (39), with $\beta^\times = -\frac{1}{2}$; see also Eq. (43) for $\beta^\times$. Finally, in the core region, we encounter the universal number

$$U_{(2)} = \frac{1}{2y_0^2} \ ; \tag{100}$$

see Eq. (46).

## VII. CONCLUDING REMARKS

We have shown analytically that the various predictions of the Privman-Fisher hypothesis on the hyperuniversality of finite systems are fully borne out in the case of an ideal, relativistic Bose gas confined to restricted geometries. With pair production included, the scaling functions governing the behavior of the system in the vicinity of the bulk critical point $T = T_c$ are found to be universal—irrespective of the severity of the relativistic effects. The influence of the latter enters only through the nonuniversal scale factors $C_1$ and $C_2$, which depend on the particle mass $m$ and density $\rho$ as well; see Eqs. (28). Once $C_1$ and $C_2$ are determined, no more nonuniversal amplitudes are needed to describe the critical behavior of the system, regardless of whether it is finite or infinite in extent.

The explicit expression for $C_1$ in the nonrelativistic limit has been quoted in (90); for completeness, we place on record other limiting forms as well:

$$(C_2)_{\text{NR}} = (2\pi)^{1/2}[\zeta(d/2)]^{-1/d}\rho^{1/d} \ , \tag{101}$$

$$(C_1)_{\text{ER}} = (d-1)\Gamma(d) \left[ \frac{2\zeta(d-1)\pi^{(d-2)(d+1)/2}}{\{\Gamma[(d+1)/2]\}^{d-2}} \left[\frac{\rho}{m}\right]^{d-2} \right]^{1/(d-1)} \ , \tag{102}$$

and

$$(C_2)_{\text{ER}} = \left[ \frac{\pi^{(d+1)/2} m^{d-2}}{2\Gamma[(d+1)/2]\zeta(d-1)} \right]^{1/2(d-1)} \ , \tag{103}$$

all for $d > 2$. Here, NR denotes the nonrelativistic limit $(\rho/m^3 \ll 1)$ while ER denotes the extreme relativistic limit $(\rho/m^3 \gg 1)$.

Finally, we would like to remark that while the calculations reported in this paper have been carried out for the field-free case $(h=0)$, something can be said about the manner in which $h$ will enter into the scaling functions governing the critical behavior of the system. For this, we rewrite (8) in the form

$$\frac{\Psi^{(s)}(t,h;L)}{C_2 C_1^{\beta} |t|^{\beta}} \approx F\left[C_1 t L^{1/\nu}, \frac{C_2 h}{C_1^{\Delta} |t|^{\Delta}}\right] \tag{104}$$

and let $L \to \infty$. We thus obtain for the bulk system

$$\frac{\Psi^{(s)}(t,h;\infty)}{C_2 C_1^{\beta} |t|^{\beta}} \approx F^{\pm}\left[\frac{C_2 h}{C_1^{\Delta} |t|^{\Delta}}\right], \tag{105}$$

where $F^{\pm}$ denote the limiting forms of the function $F$ as $x_1 \to \pm\infty$. Taking the inverse of (105), multiplying both sides by $(\Psi^{(s)}/C_2 C_1^{\beta} |t|^{\beta})^{-\delta}$, where $\delta\beta = \Delta$, we arrive at the bulk scaling form of Singh and Pandita,[7] viz.

$$h = l_1 \Psi^{(s)\delta} f(l_2 |t| / \Psi^{(s)1/\beta}), \tag{106}$$

where

$$l_1 \propto C_2^{-(1+\delta)}, \quad l_2 \propto C_1 C_2^{1/\beta}. \tag{107}$$

Recalling that the index $\delta$ for this system is given by

$$\delta = (d+2)/(d-2) \quad (2 < d < 4), \tag{108}$$

and using the general expression for $C_2$ [as given by Eq. (28)], we obtain

$$l_1 \propto (\beta_c/m)^{d/(d-2)}, \tag{109}$$

the constant of proportionality being universal. Next, making use of Eq. (21) and remembering that $\beta = \frac{1}{2}$, we obtain

$$l_2 \propto B^2, \tag{110}$$

again the constant of proportionality being universal. Equations (109) and (110) are in complete agreement with the analytical results of Singh and Pandita;[7] see their Eqs. (48) and (49). A full investigation of the Privman-Fisher hypothesis for a finite system with $h \neq 0$ is clearly a matter of some interest. We hope to return to this investigation at a later date.

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