Dynamical theory of classical surface plasmas

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The authors present a self-consistent approximation scheme for the calculation of the dynamical polarizability $\alpha(\mathbf{k},\omega)$ and plasmon dispersion at long wavelengths in electron films trapped on the free surface of liquid helium. The principal building blocks to the construction of the approximation scheme are the nonlinear fluctuation-dissipation theorem and linearized equations for the plasma density, fluid velocity, pressure tensor, and heat-flow tensor moments. Equilibrium three-point correlations, quadratic polarizability response functions, and the Navier-Stokes hypothesis linking the pressure tensor to its trace are all central elements in the development of the theory. At frequencies high compared with the collision frequency, the Golden-Lu (GL) formula for $\alpha(\mathbf{k},\omega)$ exactly satisfies the ω^{-4} moment sum rule. Analysis of $\alpha_{\rm GL}(\mathbf{k},\omega)$ at $\omega\simeq(2\pi ne^2k/m)^{1/2}\gg kv_{\rm th}$ leads to the description of the long-wavelength plasmon structure over the range of couping strengths spanning the entire fluid regime. The GL theory predicts that the transition from plasmonlike to longitudinal-phonon-like dispersion occurs at the critical value of the coupling parameter $\Gamma_{\rm crit}=3.22$; this transition is inextricably linked to the onset of a "liquid-state" short-range order signaled by the development of oscillations in the equilibrium pair correlation function somewhere in the range $2.2<\Gamma<2.9$. In the infinite coupling ($\Gamma\rightarrow\infty$) limit, the GL long-wavelength dispersion relation very nearly reproduces the zero-temperature Bonsall-Maradudin longitudinal-phonon dispersion for the two-dimensional hexagonal lattice.

I. INTRODUCTION

Over the past decade there has been a considerable effort directed at the understanding and prediction of the collective mode and transport properties of electron films trapped on the free surface of liquid helium. In the actual laboratory setup[1] extra electrons are deposited in a monolayer just above the free surface and are held there by the combination of the long-range image potential and short-range repulsive barrier to penetration into the liquid-helium surface. At a temperature $T\sim0.5$ K and over the range of areal densities $10^5<n<10^9$ cm^{-2} realized in the Grimes-Adams experiments,[1] such electron monolayers behave very much like a strongly coupled two-dimensional (2D) classical one-component plasma (OCP): The electrons can execute only horizontal (parallel to the free surface) motions and they interact via the Coulomb potential $\phi(r)=e^2/r$ (r is the range along the surface and $e=e_0[2/(1+\epsilon_{\rm He})]^{1/2}$ is the effective electronic charge ($e_0=4.8\times10^{-10}$ esu)). The compensating uniform positive background is provided by an electrode placed just below the surface.

Some of the more novel characteristics exhibited by the 2D classical OCP can be listed here. (i) Laboratory experiments[1(a)] confirm the existence at long wavelengths ($k\ll k_D=2\pi ne^2\beta$, $\beta^{-1}=k_BT$) of surface-plasmon excitations near the frequency $\omega=\omega_p(k)=(2\pi ne^2k/m)^{1/2}$; these low-frequency plasma modes are well developed and persist over the entire range of liquid-state coupling strengths up through crystalization.[1(b)] (ii) The mean electron-electron collision frequency $\nu\sim\omega_0=(2\pi ne^2k_D/m)^{1/2}$, even when calculated under the usual weak-coupling assumption that a test electron interacts weakly with a large population of field electrons inside the Debye circle, turns out to be entirely independent of the plasma parameter $\gamma=k_D^2/(2\pi n)$. (iii) Consequently at $\omega\simeq\omega_p(k)\ll\nu$ and for $\gamma\neq0$, collective mode behavior should be profoundly affected by dynamic collisions *however weak the coupling may be.* Indeed, Baus's long-wavelength plasmon dispersion formula[2]

$$\omega(k\ll k_D,\ \gamma\ll1)=\omega_p(k)\left[1+\frac{k}{k_D}\right] \tag{1}$$

bears this out: The non-random-phase-approximation $-k/(2k_D)$ dispersive correction is a consequence of two-dimensional adiabatic compressions in which plasma wave-induced perturbations in the longitudinal component of the pressure tensor are immediately accompanied by collision-induced perturbations in the transverse component. The collisional transfer of longitudinal momentum is controlled by viscous transport. (iv) The γ-independent viscous transport and γ-dependent correlational effects play a central role in the damping of the 2D OCP plasma mode. In the actual laboratory experiment,[1] however, it is, in fact, electron-ripplon scattering which controls the lifetime of the surface-plasma mode for $T\leq0.68$ K.

Computer experimental and theoretical efforts have already provided a great deal of information about the dynamical properties of the 2D classical OCP over a wide range of γ values. Structure function data and curves for the longitudinal and transverse collective excitations have been generated from molecular dynamics (MD) computer simulations.[3] Formulas for the dispersion and damping of the collective excitations have been derived (i) by calculating the frequency spectrum of lattice vibrations of a finite 2D electron crystal,[4] (ii) in the random-phase approximation (RPA),[5] (iii) by following the Singwi-Tosi-Land-Sjolander[6] mean-field-theory approach,[7] (iv) from the Boltzmann equation,[8] and (v) by following a microscopic hydrodynamic approach.[2]

The present paper has a twofold purpose: (1) to formulate a long-wavelength ($kv_{\rm th} \ll \omega$) dynamical theory of the 2D OCP in the language of linear and nonlinear response functions and (2) to calculate [from (1)] the dispersion of the 2D OCP plasma mode over the range of coupling strengths spanning the entire fluid regime. Our formulation of the dispersion in the fluid regime will provide the critical value $\Gamma_{\rm crit}$ of the coupling parameter $\Gamma=\beta e^2(\pi n)^{1/2}=(\gamma/2)^{1/2}$ marking the transition from plasmonlike to phononlike dispersion; MD simulations[3] predict that $\Gamma_{\rm crit}$ lies somewhat above 2.29. The accuracy of our dispersion relation will be further assessed by going to the infinite coupling ($\Gamma\rightarrow\infty$) limit and making comparison with dispersion calculations of the 2D OCP crystal.[4(b)] Our calculations are to be carried out in the velocity-average approximation (VAA) (Refs. 9 and 10)—actually an improved version of it—and in a linearized hydrodynamical framework which explicitly displays the two-dimensional compressions for $kv_{\rm th} \ll \omega \ll \nu$. The hydrodynamical equations are to be generated by taking velocity moments of the first Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) kinetic equation. We wish to emphasize that the VAA is not a central element of the theory, nor is the VAA indispensable to the derivation of the dispersion relations of this paper. The VAA is invoked because it provides a convenient and transparent way of converting mathematically intractable higher-velocity moments of the two-particle distribution function $G(12;t)$ into tractable nonequilibrium two-point functions. The procedure for incorporating the VAA structure into the 3D OCP hydrodynamical chain has been described at some length in Ref. 10. This procedure can be easily adapted to the present case with the stipulation that a Navier-Stokes description of the pressure tensor must necessarily be invoked for $\omega \ll \nu$ to take account of the γ-independent dynamic collision effects. The need for the Navier-Stokes hypothesis becomes all the more apparent when we observe (i) that even weakly coupled ($\gamma \ll 1$) two-dimensional OCP's must be in a state of *local thermodynamic equilibrium* for $\omega \ll \nu$ and (ii) that the dominant zeroth velocity moment and higher even-moment VAA projections of $G(12;t)$, because they are γ dependent, cannot possibly relate to the persistence of collisions all the way down to $\gamma \ll 1$. We shall see how the nonequilibrium two-point correlation functions enter moment by moment in the linearized hydrodynamical equations and how, in particular, the dominant γ-dependent correlation appears as the lowest-order (in k) term originating from the zeroth moment of $G(12;t)$. A central element of the theory is the *nonlinear fluctuation-dissipation theorem (*NLFDT*)* (Ref. 11) which makes it possible to trade the nonequilibrium two-point correlations for more accessible quadratic polarizability response functions. What ultimately results from the combined NLFDT-hydrodynamical equations is a compact and elegant response-function relation linking the linear and quadratic polarizabilities. Self-consistency can then be guaranteed by approximating the latter in terms of the former. A systematic decomposition scheme for making this kind of approximation has been worked out elsewhere[9(d)] for bulk OCP's and can be easily reformulated for the 2D OCP under consideration here. Finally, it should be mentioned that our treatment of the 2D OCP will not take account of electron scattering by helium-vapor atoms and by ripplons on the liquid-helium surface.[1(a)]

The plan of the paper is as follows. In Sec. II relevant structure and polarizability response functions are introduced and defined. In Sec. III velocity moments of $G(12;t)$ are introduced and linearized moment equations for the average plasma density and pressure tensor and heat-flow tensor responses are generated from the plasma kinetic equation; the fundamental zeroth and first velocity moments of $G(12;t)$ are at the same time converted from nonequilibrium two-point functions into equilibrium three-point functions. In Sec. IV the linear polarizability $\alpha(\mathbf{k},\omega)$ is calculated at high frequencies $\omega \gg \nu$ where the compressions are one dimensional. This is done by combining the longitudinal projections of the moment equations into a single equation linking the average plasma density linear response to a total (average+external) density perturbation. In Sec. V $\alpha(\mathbf{k},\omega)$ is calculated at lower frequencies $kv_{\rm th} \ll \omega \ll \nu$ characterizing the long-wavelength hydrodynamic regime where local thermodynamic equilibrium is expected to prevail. Consequently, the hydrostatic pressure p and the Navier-Stokes hypothesis connecting the pressure tensor to p play the principal role in describing the γ-independent dynamic collision processes. The calculation of $\alpha(\mathbf{k},\omega)$ in Sec. V will therefore be accomplished by incorporating the Navier-Stokes hypothesis into the full two-dimensional moment equations (of Sec. III) and then combining the latter into a single equation linking the average plasma density response to the total density perturbation. Generally speaking, our results in Secs. IV and V can be best summarized by the simple relation $\alpha(\mathbf{k},\omega;\gamma)=\alpha(\mathbf{k},\omega;\gamma \ll 1)[1+v(\mathbf{k},\omega)]$, where the dynamical coupling correction $v(\mathbf{k},\omega)$ contains all the γ-dependent correlational contributions. The identification of $\alpha(\mathbf{k},\omega;\gamma \ll 1)$ as the RPA polarizability $\alpha_0(\mathbf{k},\omega)$ in Sec. IV and as the "hydrodynamical" polarizability $\alpha_H(\mathbf{k},\omega)$ in Sec. V is entirely consistent with the dimensionality of the γ-independent compression processes described in these sections. As to the development of the γ-dependent $v(\mathbf{k},\omega)$, it will proceed in three stages. In the first stage $v(\mathbf{k},\omega)$ is expressed in terms of equilibrium three-point structure functions (Sec. IV). In particular, we shall demonstrate in Sec. IV that $v(\mathbf{k},\omega)$, when evaluated at $\omega \gg \nu$, reproduces the exact $O(1/\omega^4)$ moment sum-rule structure for the 2D OCP polarizability. The second-stage conversion of the three-point functions into quadratic polarizabilities is effected by application of the centrally important NLFDT (Sec. V). The resulting response-function relation linking the linear and quadratic polarizabilities is made self-consistent in the third stage (Sec. V) by postulating that a decomposition of the latter in terms of the former, which prevails in the $k \to 0$ limit for weak coupling, can be relied upon as a paradigm for arbitrary coupling. The formulation of our self-consistent approximation scheme in Sec. V is one of the two principal accomplishments of the present paper. The second principal accomplishment follows in Sec. VI, where the dispersion of the long-wavelength surface plasmons is calculated over the range of coupling strengths spanning the entire fluid regime. We shall see that in the infinite coupling limit, our dispersion formula very nearly reproduces the longitudinal-phonon dispersion characterizing the zero-temperature 2D hexagonal lattice in the $k \to 0$ limit. Conclusions are drawn in Sec. VII.

II. STRUCTURE FUNCTIONS AND POLARIZABILITIES

We begin by listing and defining the relevant structure and polarizability functions. The former are customarily defined by the relations

$$\langle n_{\mathbf{k}\omega} n^*_{\mathbf{p}\mu} \rangle^{(0)} = (2\pi N)\delta_{\mathbf{k}-\mathbf{p}}\delta(\omega-\mu)[S(\mathbf{k},\omega)+(2\pi N)\delta_{\mathbf{k}}\delta(\omega)] , \tag{2}$$

$$\langle n_{\mathbf{k}\omega} n^*_{\mathbf{p}\mu} n^*_{\mathbf{q}\nu} \rangle^{(0)} = (2\pi N)\delta_{\mathbf{k}-\mathbf{p}-\mathbf{q}}\delta(\omega-\mu-\nu) \times \{S(\mathbf{p},\mu;\mathbf{q},\nu)+(2\pi N)[\delta_{\mathbf{k}}\delta(\omega)S(\mathbf{p},\mu)+\delta_{\mathbf{p}}\delta(\mu)S(\mathbf{q},\nu)+\delta_{\mathbf{q}}\delta(\nu)S(\mathbf{k},\omega)]+(2\pi N)^2\delta_{\mathbf{p}}\delta_{\mathbf{q}}\delta(\mu)\delta(\nu)\} , \tag{3}$$

where the angular brackets denote ensemble averaging of the microscopic densities $n_{\mathbf{p}} = \sum_i \exp(-i\mathbf{p}\cdot\mathbf{x}_i)$; the zero superscript indicates that the averaging is to be performed over the equilibrium system; and $N = An$, where A is the large but bounded area of the system. Successive frequency integrations of (2) and (3), namely

$$S(\mathbf{k}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(\mathbf{k},\omega) , \tag{4}$$

$$S(\mathbf{p};\mathbf{q}) = \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} S(\mathbf{p},\mu;\mathbf{q},\nu) , \tag{5}$$

provide the corresponding static structure functions.

"External" polarizability response functions are defined through the 2D OCP relations

$$\langle n_{\mathbf{k}} \rangle^{(1)}(\omega) = -\hat{\alpha}(\mathbf{k},\omega)\hat{n}(k,\omega) , \tag{6}$$

$$\langle n_{\mathbf{k}} \rangle^{(2)}(\omega) = \frac{in}{2A}\sum_{\mathbf{p}} \beta^2\phi(\mathbf{p})\phi(\mathbf{k}-\mathbf{p}) \times \int_{-\infty}^{\infty} \frac{d\mu}{2\pi}\hat{a}(\mathbf{p},\mu;\mathbf{k}-\mathbf{p},\omega-\mu) \times \hat{n}(\mathbf{p},\mu)\hat{n}(\mathbf{k}-\mathbf{p},\omega-\mu) , \tag{7}$$

connecting the average density response $\langle n_k \rangle(\omega)$ to the weak external density perturbation $\hat{n}$ [$\phi(\mathbf{p}) = 2\pi e^2/p$ is the Fourier-transformed Coulomb energy]. "Total" polarizabilities, on the other hand, connect $\langle n_{\mathbf{k}} \rangle(\omega)$ to the total density perturbation $\tilde{n} = \hat{n} + \langle n \rangle$:

$$\langle n_{\mathbf{k}} \rangle^{(1)}(\omega) = -\alpha(\mathbf{k},\omega)\tilde{n}^{(1)}(\mathbf{k},\omega) , \tag{8}$$

$$\langle n_{\mathbf{k}} \rangle^{(2)}(\omega) = \frac{in}{2A}\sum_{\mathbf{p}} \beta^2\phi(\mathbf{p})\phi(\mathbf{k}-\mathbf{p}) \times \int_{-\infty}^{\infty} \frac{d\mu}{2\pi}\frac{a(\mathbf{p},\mu;\mathbf{k}-\mathbf{p},\omega-\mu)}{\epsilon(\mathbf{k},\omega)} \times \tilde{n}^{(1)}(\mathbf{p},\mu)\tilde{n}^{(1)}(\mathbf{k}-\mathbf{p},\omega-\mu) , \tag{9}$$

where $\epsilon(\mathbf{k},\omega) = 1+\alpha(\mathbf{k},\omega)$ is the dielectric response function. Since $\tilde{n}^{(1)} = \hat{n}/\epsilon$, we have from (6) to (9) that

$$\hat{\alpha}(\mathbf{k},\omega) = \frac{\alpha(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} , \tag{10}$$

$$\hat{a}(\mathbf{p},\mu;\mathbf{q},\nu) = \frac{a(\mathbf{p},\mu;\mathbf{q},\nu)}{\epsilon(\mathbf{p},\mu)\epsilon(\mathbf{q},\nu)\epsilon(\mathbf{p}+\mathbf{q},\mu+\nu)} . \tag{11}$$

III. MOMENT EQUATIONS

Let $F(\mathbf{x},\mathbf{v};t)$ and $G(\mathbf{x},\mathbf{v};\mathbf{x}',\mathbf{v}';t)$ be one- and two-particle velocity distribution functions normalized to N and $N(N-1)$, respectively. In the absence of external perturbations, the 2D OCP equilibrium distributions are given by

$$F^{(0)}(v) = \frac{\beta m n}{2\pi}\exp\left[\frac{-\beta m v^2}{2}\right] , \tag{12}$$

$$G^{(0)}(\mathbf{x},\mathbf{v};\mathbf{x}',\mathbf{v}') = F^{(0)}(v)F^{(0)}(v') \times \left[1+\frac{1}{N}\sum_{\mathbf{q}}[S(\mathbf{q})-1]\exp[i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')]\right] . \tag{13}$$

$F^{(1)}$ and $G^{(1)}$ are the corresponding first-order responses to the weak external electric field perturbation $\hat{E}$; they are linked by the linearized first BBGKY kinetic equation

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\right]F^{(1)}(\mathbf{x},\mathbf{v};t) - \frac{e}{m}\hat{\mathbf{E}}(\mathbf{x},t)\cdot\frac{\partial}{\partial \mathbf{v}}F^{(0)}(v) = \frac{1}{m}\frac{\partial}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{x}}\int d^2x' \frac{e^2}{|\mathbf{x}-\mathbf{x}'|} \times \int d^2v' G^{(1)}(\mathbf{x},\mathbf{v};\mathbf{x}',\mathbf{v}';t) . \tag{14}$$

The long-wavelength ($kv_{\text{th}} \ll \omega$) hydrodynamical equations of this paper are to be generated by taking velocity moments of (14). Thus dynamic collision effects will appear in a natural way as correlations originating from velocity moments of the two-particle distribution function. The following illustrates the generation of the fundamental correlations:

$$[G]_{\rm av}(\mathbf{x},\mathbf{x}';t)\equiv\int d^2v\int d^2v' G(\mathbf{x},\mathbf{v};\mathbf{x}',\mathbf{v}';t)$$
$$=\langle n(\mathbf{x})n(\mathbf{x}')\rangle(t)-\delta(\mathbf{x}-\mathbf{x}')\langle n(\mathbf{x})\rangle(t)\ , \tag{15}$$

$$[\mathbf{v}G]_{\rm av}(\mathbf{x},\mathbf{x}';t)\equiv\int d^2v\int d^2v'\mathbf{v}G(\mathbf{x},\mathbf{v};\mathbf{x}',\mathbf{v}';t)$$
$$=\langle \mathbf{j}(\mathbf{x})n(\mathbf{x}')\rangle(t)-\delta(\mathbf{x}-\mathbf{x}')\langle \mathbf{j}(\mathbf{x})\rangle(t)\ , \tag{16}$$

etc. The straightforward procedure of taking the 1, **v**, **vv**, etc. moments of (14) then results in the conservation equations

$$\frac{\partial}{\partial t}\langle n(\mathbf{x})\rangle(t)+nu_{\lambda,\lambda}(\mathbf{x},t)=0\ , \tag{17}$$

$$\frac{\partial}{\partial t}u_\mu(\mathbf{x},t)+\frac{e}{m}\hat{E}_\mu(\mathbf{x},t)+\frac{1}{mn}p_{\mu\nu,\nu}(\mathbf{x},t)=-\frac{1}{mn}\int d^2x'[G]_{\rm av}(\mathbf{x},\mathbf{x}';t)\frac{\partial}{\partial x_\mu}\frac{e^2}{|\mathbf{x}-\mathbf{x}'|}$$
$$=-\frac{1}{mn}\int d^2x'\langle n(\mathbf{x})n(\mathbf{x}')\rangle(t)\frac{\partial}{\partial x_\mu}\frac{e^2}{|\mathbf{x}-\mathbf{x}'|}\ , \tag{18}$$

$$\frac{\partial}{\partial t}p_{\mu\nu}(x,t)+\frac{n}{\beta}[\delta_{\mu\nu}\mu_{\lambda,\lambda}(\mathbf{x},t)+u_{\mu,\nu}(\mathbf{x},t)+\mu_{\nu,\mu}(\mathbf{x},t)]+q_{\mu\nu\lambda,\lambda}(\mathbf{x},t)$$
$$=-\int d^2x'\left[[v_\mu G]_{\rm av}(\mathbf{x},\mathbf{x}';t)\frac{\partial}{\partial x_\nu}+[v_\nu G]_{\rm av}(\mathbf{x},\mathbf{x}';t)\frac{\partial}{\partial x_\mu}\right]\frac{e^2}{|\mathbf{x}-\mathbf{x}'|}$$
$$=-\int d^2x'\left[\langle j_\mu(\mathbf{x})n(\mathbf{x}')\rangle(t)\frac{\partial}{\partial x_\nu}+\langle j_\nu(\mathbf{x})n(\mathbf{x}')\rangle(t)\frac{\partial}{\partial x_\mu}\right]\frac{e^2}{|\mathbf{x}-\mathbf{x}'|}\ , \tag{19}$$

$$\frac{\partial}{\partial t}q_{\mu\nu\eta}(\mathbf{x},t)+\frac{n}{\beta}\left[\frac{e}{m}\hat{E}_\lambda(\mathbf{x},t)+\frac{\partial}{\partial t}u_\lambda(\mathbf{x},t)\right](\delta_{\lambda\mu}\delta_{\nu\eta}+\delta_{\lambda\nu}\delta_{\mu\eta}+\delta_{\lambda\eta}\delta_{\mu\nu})+r_{\mu\nu\eta\lambda,\lambda}(\mathbf{x},t)$$
$$=-\int d^2x'\{\delta_{\lambda\mu}[v_\nu v_\eta G]_{\rm av}(\mathbf{x},\mathbf{x}';t)+\delta_{\lambda\nu}[v_\mu v_\eta G]_{\rm av}(\mathbf{x},\mathbf{x}';t)+\delta_{\lambda\eta}[v_\mu v_\nu G]_{\rm av}(\mathbf{x},\mathbf{x}';t)\}\frac{\partial}{\partial x_\lambda}\frac{e^2}{|\mathbf{x}-\mathbf{x}'|}\ , \tag{20}$$

where

$$\mathbf{u}(\mathbf{x},t)=\frac{1}{n}\langle \mathbf{j}(\mathbf{x})\rangle^{(1)}(t)$$

is the mean fluid velocity and the tensors

$$p_{\mu\nu}(\mathbf{x},t)=m\int d^2w\, w_\mu w_\nu F^{(1)}(\mathbf{x},\mathbf{w};t)\ , \tag{21}$$

$$q_{\mu\nu\eta}(\mathbf{x},t)=m\int d^2w\, w_\mu w_\nu w_\eta F^{(1)}(\mathbf{x},\mathbf{w};t)\ , \tag{22}$$

$$r_{\mu\nu\eta\lambda}(\mathbf{x},t)=m\int d^2w\, w_\mu w_\nu w_\eta w_\lambda F^{(1)}(\mathbf{x},\mathbf{w};t)\ , \tag{23}$$

etc. are defined in terms of the peculiar velocity $\mathbf{w}=\mathbf{v}-\boldsymbol{\mu}$. Note that the (1) superscript has been dropped in (17)–(20) to keep the notation from getting too cumbersome.

As was mentioned earlier, our approximation scheme is to be formulated in a hydrodynamical framework in which the dynamic collision contributions appear in a natural way as correlations originating from velocity moments of the two-particle distribution function. Indeed, the zeroth and first velocity moments are inextricably linked to nonequilibrium two-point correlation functions [see Eqs. (15) and (16)]. Unfortunately, no such two-point function links exist for the higher moments appearing in the chain of equations beyond (19). Note, however, that the original VAA ansatz

$$G(\mathbf{x},\mathbf{v};\mathbf{x}',\mathbf{v}';t)=\frac{1}{2}\frac{F(\mathbf{x},\mathbf{v};t)}{\langle n(\mathbf{x})\rangle(t)}\int d^2\bar{v}\,G(\mathbf{x},\bar{\mathbf{v}};\mathbf{x}',\mathbf{v}';t)$$
$$+\frac{1}{2}\frac{F(\mathbf{x}',\mathbf{v}',t)}{\langle n(\mathbf{x}')\rangle(t)}\int d^2\bar{v}'G(\mathbf{x},\mathbf{v};\mathbf{x}',\bar{\mathbf{v}}';t) \tag{24}$$

makes it possible to systematically factorize the (otherwise intractable) higher moments into velocity- and position-dependent products or, equivalently, into products of $1/\beta$ and the fundamental $\langle nn\rangle(t)$ or $\langle \mathbf{j}n\rangle(t)$ correlations, that is,[10]

$$[v_\mu v_\nu G]^{(1)}_{\rm av}(\mathbf{x},\mathbf{x}';t)=\delta_{\mu\nu}\frac{1}{\beta m}[G]^{(1)}_{\rm av}(\mathbf{x},\mathbf{x}';t)\ , \tag{25}$$

$$[v_\mu v_\nu \mathbf{v}G]^{(1)}_{\rm av}(\mathbf{x},\mathbf{x}';t)=\delta_{\mu\nu}\frac{1}{\beta m}[\mathbf{v}G]^{(1)}_{\rm av}(\mathbf{x},\mathbf{x}';t)\ , \tag{26}$$

$$[v_\mu v_\nu v_\eta v_\lambda G]^{(1)}_{\rm av}(\mathbf{x},\mathbf{x}';t)$$
$$=(\delta_{\mu\nu}\delta_{\eta\lambda}+\delta_{\mu\eta}\delta_{\nu\lambda}+\delta_{\mu\lambda}\delta_{\nu\eta})\frac{1}{(\beta m)^2}[G]^{(1)}_{\rm av}(\mathbf{x},\mathbf{x}';t)\ , \tag{27}$$

etc., such that

$$[vG]^{(1)}_{\rm av}(\mathbf{x},\mathbf{x}';t)=0\ . \tag{28}$$

Moreover, the VAA ansatz, when substituted into (14), generates—via the fluctuation-dissipation relations—the *exact* static BBGKY hierarchy at $\omega=0$.[9(f)] Consequently, in solving the self-consistent set of equations (54) and (59) below for the dynamical polarizability, it will be entirely justifiable to input the correlation energy density contribution to (59) with high-precision data which are assumed to be determined by computer or other experimental data or by an independent theoretical approach.

The condition (28) [which is a corollary of (24)] will be dropped in the sequel since $[\mathbf{v}G]^{(1)}_{\rm av}(\mathbf{x},\mathbf{x}';t)$ in fact plays a crucial role in the $kv_{\rm th}\ll\omega\ll\nu$ frequency domain where collisions are predominant: In a nutshell, this fundamental moment goes hand in hand with the state of local thermodynamic equilibrium which, in the 2D OCP, persists all the way down to very small γ values. The treatment of the present paper still invokes the decompositions (25)—(27), but improves on the original VAA by now considering $[\mathbf{v}G]_{\rm av}(\mathbf{x},\mathbf{x}';t)$ to be different from zero. The exactness of the theory in the static ($\omega=0$) limit is at the same time preserved since $[\mathbf{v}G]^{(1)}_{\rm av}(\mathbf{k}-\mathbf{q},\mathbf{q};t=0)=-\langle \mathbf{j}_{\mathbf{k}}\rangle^{(1)}(t=0)=0$. In Ref. 10 this improved version of the VAA is called the "velocity-moment approximation" (VMA).

Equations (17)—(20) and (25), when combined and Fourier transformed, give

$$\langle n_{\mathbf{k}}\rangle(\omega)=\frac{k^2}{m\omega^2}p_{ll}(\mathbf{k},\omega)+\frac{\omega_p^2(k)}{\omega^2}\tilde{n}(\mathbf{k},\omega)+\frac{\omega_p^2(k)}{\omega^2}\frac{1}{N}\sum_{\mathbf{q}\neq\mathbf{k}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\langle n_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)\ , \tag{29}$$

$$\begin{aligned}p_{\mu\nu}(\mathbf{k},\omega)=&\frac{n}{\beta\omega}[\delta_{\mu\nu}k_\lambda u_\lambda(\mathbf{k},\omega)+k_\nu u_\mu(\mathbf{k},\omega)+k_\mu u_\nu(\mathbf{k},\omega)]+\frac{k_\lambda}{\omega}q_{\mu\nu\lambda}(\mathbf{k},\omega)\\&+\frac{1}{\omega A}\sum_{\mathbf{q}}\phi(q)[q_\nu\langle j_{\mu_{\mathbf{k}-\mathbf{q}}}n_{\mathbf{q}}\rangle(\omega)+q_\mu\langle j_{\nu_{\mathbf{k}-\mathbf{q}}}n_{\mathbf{q}}\rangle(\omega)]\ ,\end{aligned} \tag{30}$$

$$\begin{aligned}q_{\mu\nu\eta}(\mathbf{k},\omega)=&\frac{\omega}{\beta k}(\delta_{\lambda\mu}\delta_{\nu\eta}+\delta_{\lambda\nu}\delta_{\mu\eta}+\delta_{\lambda\eta}\delta_{\mu\nu})\\&\times\left[\frac{\omega_p^2(k)}{\omega^2}\frac{k_\lambda}{k}\tilde{n}(\mathbf{k},\omega)-\frac{nk}{\omega}u_\lambda(\mathbf{k},\omega)+\frac{\omega_p^2(k)}{\omega^2}\frac{1}{N}\sum_{\substack{\mathbf{q}\\ q\neq k}}(q_\lambda/q)\langle n_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)\right]+\frac{k_\lambda}{\omega}r_{\mu\nu\eta\lambda}(\mathbf{k},\omega)\ ,\end{aligned} \tag{31}$$

where $p_{ll}=(k_\mu k_\nu/k^2)p_{\mu\nu}$ is the longitudinal component of the pressure tensor. The passage from $\hat{E}$ language to the more convenient *total* density $\tilde{n}$ language was effected first by exploiting the 2D OCP relation $\hat{\mathbf{E}}(\mathbf{k},\omega)=2\pi ie\hat{n}(\mathbf{k},\omega)(\mathbf{k}/k)$ and then by separating out the $\mathbf{q}=\mathbf{k}$ contribution to the $\sum_{\mathbf{q}}\phi(\mathbf{q})\mathbf{q}\langle n_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)$ terms in (29) and (31) and combining it with the $\hat{n}$ terms. As to the $\mathbf{q}\neq\mathbf{k}$ nonequilibrium two-point function contributions, they can be converted into equilibrium three-point functions: straightforward statistical mechanical calculations provide the fluctuation-dissipation relations

$$\langle n_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)\,|_{q\neq k}=-\frac{k_D}{k}\hat{n}(\mathbf{k},\omega)\left[\frac{i\omega}{2\pi}\int_{-\infty}^{\infty}d\mu\int_{-\infty}^{\infty}d\nu\,\delta_+(\omega-\mu-\nu)S(\mathbf{q},\mu;\mathbf{k}-\mathbf{q},\nu)+S(\mathbf{q};\mathbf{k}-\mathbf{q})\right]\ , \tag{32}$$

$$\langle \mathbf{j}_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)=-\frac{k_D}{k}\hat{n}(\mathbf{k},\omega)\frac{i\omega}{2\pi}\int_{-\infty}^{\infty}d\mu\int_{-\infty}^{\infty}d\nu\,\delta_+(\omega-\mu-\nu)\mathbf{T}(\mathbf{q},\mu;\mathbf{k}-\mathbf{q},\nu)\ , \tag{33}$$

where

$$\mathbf{T}(\mathbf{q},\mu;\mathbf{k}-\mathbf{q},\nu)=\frac{1}{N}\int_{-\infty}^{\infty}\frac{d\chi}{2\pi}\langle n_{\mathbf{q},\mu}j_{\mathbf{k}-\mathbf{q},\nu}n^*_{\mathbf{k},\chi}\rangle^{(0)}\ . \tag{34}$$

Elsewhere,[12] it has been demonstrated that $S(\mathbf{q},\mu;\mathbf{p},\nu)$ is real. $\mathbf{T}(\mathbf{q},\mu;\mathbf{p},\nu)$ is also real, this follows from the fact that $\mathbf{T}$ changes sign both (i) under space inversion, i.e., $\mathbf{T}(\mathbf{q},\mu;\mathbf{p},\nu)=-\mathbf{T}(-\mathbf{q},\mu;-\mathbf{p},\nu)$, and (ii) under microscopic time reversal, i.e., $\mathbf{T}(-\mathbf{q},\mu;-\mathbf{p},\nu)=-\mathbf{T}(-\mathbf{q},-\mu;-\mathbf{p},-\nu)=-\mathbf{T}^*(\mathbf{q},\mu;\mathbf{p},\nu)$. Hence, $\mathbf{T}(\mathbf{q},\mu;\mathbf{p},\nu)=\mathbf{T}^*(\mathbf{q},\mu;\mathbf{p},\nu)$.

IV. HIGH-FREQUENCY BEHAVIOR

At high frequencies where collisions are unimportant (i.e., $\omega\gg\nu$), perturbations in the average plasma density are necessarily one dimensional and adiabatic. One therefore combines the *longitudinal (l) projections* of (30) and (31) with (29) to readily obtain

$$\begin{aligned}\langle n_{\mathbf{k}}\rangle(\omega)=&\frac{\omega_p^2(k)}{\omega^2}\left[1+3\frac{\omega_p^2(k)}{\omega^2}\frac{k}{k_D}\right]\\&\times\left[\tilde{n}(\mathbf{k},\omega)+\frac{1}{N}\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\langle n_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)\right]\\&+\frac{\omega_p^2(k)}{\omega^2}\frac{2}{\omega}\mathbf{k}\cdot\frac{1}{N}\sum_{\mathbf{q}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\langle \mathbf{j}_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)\\&+\frac{k^4}{m\omega^4}r_{llll}(\mathbf{k},\omega)\ .\end{aligned} \tag{35}$$

Subsequent calculations of the higher-order moments beginning with $r_{llll}(\mathbf{k},\omega)$ provide higher-order (in k^2/ω^2) contributions to the coefficients $[\omega_p^2(k)/\omega^2]\{1+3[\omega_p^2(k)/\omega^2][k/k_D]\}$ and $[\omega_p^2(k)/\omega^2]$ of the first and second right-hand side (rhs) members, respectively, of

(35). These coefficients ultimately sum to the RPA polarizability

$$\alpha_0'(\mathbf{k},\omega\gg\nu)\equiv\mathrm{Re}\alpha_0(\mathbf{k},\omega\gg\nu)$$

$$=-\frac{\omega_p^2(k)}{\omega^2}\left[1+3\frac{\omega_p^2(k)}{\omega^2}\frac{k}{k_D}+15\frac{\omega_p^4(k)}{\omega^4}\frac{k^2}{k_D^2}+\cdots\right], \tag{36}$$

whence (43) becomes

$$\langle n_{\mathbf{k}}\rangle(\omega)=-\alpha_0'(\mathbf{k},\omega\gg\nu)\times\left[\tilde{n}(\mathbf{k},\omega)+\frac{1}{N}\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\langle n_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)+\frac{2}{\omega}\mathbf{k}\cdot\frac{1}{N}\sum_{\mathbf{q}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\langle \mathbf{j}_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)\right]. \tag{37}$$

Substitution of (32) and (33) into (37) and comparison with the $\langle n\rangle-\tilde{n}$ constitutive relation (8) then results in the high-frequency polarizability

$$\alpha(\mathbf{k},\omega\gg\nu)=\alpha_0'(\mathbf{k},\omega\gg\nu)\times[1+v(\mathbf{k},\omega)+w(\mathbf{k},\omega)], \tag{38}$$

where the *dynamical coupling function* $v(\mathbf{k},\omega)$ is given by

$$\frac{v(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}=-\frac{k_D}{k}\frac{1}{N}\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\left[\frac{i\omega}{2\pi}\int_{-\infty}^{\infty}d\mu\int_{-\infty}^{\infty}d\nu\,\delta_+(\omega-\mu-\nu)S(\mathbf{q},\mu;\mathbf{k}-\mathbf{q},\nu)+S(\mathbf{q};\mathbf{k}-\mathbf{q})\right], \tag{39}$$

and

$$\frac{w(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}=-\frac{ik_D}{\pi k}\frac{1}{N}\sum_{\mathbf{q}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\left[\int_{-\infty}^{\infty}d\mu\int_{-\infty}^{\infty}d\nu\,\delta_+(\omega-\mu-\nu)\mathbf{k}\cdot\mathbf{T}(\mathbf{q},\mu;\mathbf{k}-\mathbf{q},\nu)\right]. \tag{40}$$

The expression

$$\mathrm{Re}\frac{v(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}=\frac{k_D}{k}\frac{1}{N}\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\left[\omega P\int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\frac{S(\mathbf{q},\mu;\mathbf{k}-\mathbf{q},\nu)}{\omega-\mu-\nu}-S(\mathbf{q};\mathbf{k}-\mathbf{q})\right], \tag{41}$$

when further evaluated at $\omega\to\infty$ according to the procedure of Ref. 9(d), gives

$$\mathrm{Re}\frac{v(\mathbf{k},\omega\to\infty)}{\epsilon(\mathbf{k},\omega\to\infty)}=\frac{k_D}{k}\frac{1}{N\omega^2}\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\int_{-\infty}^{\infty}\frac{d\mu}{2\pi}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}(\mu+\nu)^2S(\mathbf{q},\mu;\mathbf{k}-\mathbf{q},\nu)$$

$$=-\frac{k_D}{k}\frac{1}{N\omega^2}\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\left[\left[\frac{\partial}{\partial t'}+\frac{\partial}{\partial t''}\right]^2S(\mathbf{q},t';\mathbf{k}-\mathbf{q},t'')\right]_{t'=t''=0}$$

$$=\frac{\omega_p^2(k)}{\omega^2}\frac{1}{N}\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{(\mathbf{k}\cdot\mathbf{q})^2}{k^3q}[S(\mathbf{k}-\mathbf{q})-S(\mathbf{q})]. \tag{42}$$

As to (40), its evaluation at $\omega\to\infty$ according to the procedure of Ref. 10 gives

$$\mathrm{Re}\frac{w(\mathbf{k}\omega\to\infty)}{\epsilon(\mathbf{k}\omega\to\infty)}=\frac{\omega_p^2(k)}{\omega^2}\frac{2}{N}\sum_{\mathbf{q}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}[S(\mathbf{q})+N\delta_{\mathbf{q}}]=0 \tag{43}$$

through $O(\omega_p^2(k)/\omega^2)$. Equations (38), (42), and (43) then combine into the high-frequency expression

$$\alpha'(\mathbf{k},\omega\gg\nu)=-\frac{\omega_p^2(k)}{\omega^2}-\frac{\Omega^{(4)}(k)}{\omega^4}-\cdots, \tag{44}$$

$$\Omega^{(4)}(k)=\omega_p^4(k)\left[3\frac{k}{k_D}+\frac{1}{N}\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{(\mathbf{k}\cdot\mathbf{q})^2}{k^3q}[S(\mathbf{k}-\mathbf{q})-S(\mathbf{q})]\right], \tag{45}$$

which is known to be exact through $O(\omega_p^4(k)/\omega^4)$.[13]

Finally, note that at such high frequencies, the longitudinal projection of (30) simplifies to

$$p(\mathbf{k},\omega)=p_{ll}(\mathbf{k},\omega)\simeq\frac{3}{\beta}\langle n_{\mathbf{k}}\rangle(\omega)+\frac{k}{\omega}q_{lll}(\mathbf{k},\omega); \tag{46}$$

this follows from the fact that w and, consequently, $\langle \mathbf{j}_{\mathbf{k}-\mathbf{q}}n_q\rangle(\omega)$ contribute nothing through $O(\omega_p^4(k)/\omega^4)$ to $\mathrm{Re}\alpha(\mathbf{k},\omega\gg\nu)$. The *hydrostatic* pressure p has been introduced since, for a one-dimensional compression, p and the longitudinal pressure p_{ll} are, by definition, one and the same.

V. POLARIZABILITY FORMULATION IN THE HYDRODYNAMIC REGIME

We come now to the task of formulating the fundamental response-function relation in the hydrodynamic regime $kv_{\mathrm{th}}\ll\omega\ll\nu$. Our development proceeds in two stages:

First, we establish a relation which links linear and quadratic polarizabilities; this relation is then made self-consistent by approximating the latter in terms of the former. At the lower frequencies $\omega \ll \nu$ where collisions are predominant, the 2D OCP is in a state of local thermodynamic equilibrium and perturbations in the longitudinal projection of the pressure tensor are immediately accompanied by collision-induced perturbations in its transverse (t) projection. Consequently, the hydrostatic pressure $p(\mathbf{k},\omega)=\frac{1}{2}p_{\mu\mu}(\mathbf{k},\omega)$ is expected to play a central role in a hydrodynamic description of the 2D OCP; from (30), (31), and Appendix A,

$$p(\mathbf{k},\omega)=\frac{2}{\beta}\langle n_{\mathbf{k}}\rangle(\omega)+\frac{k}{2\omega}q_{\mu\mu l}(\mathbf{k},\omega) \tag{47a}$$

$$=\frac{2}{\beta}\frac{\omega_p^2(k)}{\omega^2}[1+v(\mathbf{k},\omega)]\tilde{n}(\mathbf{k},\omega)+\frac{k^2}{2\omega^2}[r_{llll}(\mathbf{k},\omega)+r_{ttll}(\mathbf{k},\omega)]\;. \tag{47b}$$

Equations (46) and (47a) describe entirely different processes even though they are structurally quite similar: The former highlights the one dimensionality of high-frequency compressions where γ-independent dynamic collisions can have no significance; the latter, on the other hand, highlights the two dimensionality of compressions at the lower frequencies where γ-independent collisional transport effects become all-important.

The momentum-energy conservation equations (29) and (47b) are the principal velocity-moment equations in our derivation. The third rhs member of (29),

$$\frac{\omega_p^2(k)}{\omega^2}\frac{1}{N}\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\langle n_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)=\frac{\omega_p^2(k)}{\omega^2}v(\mathbf{k},\omega)\tilde{n}(\mathbf{k},\omega)\;, \tag{48}$$

contributes the fundamental γ-dependent correlational correction to the linear polarizability in the approximation scheme. The subsequent next-higher-order (in k^2/ω^2) correction (which is the lowest-order VAA projection (25) of the $[V_l^2G]_{\rm av}^{(1)}$ moment) appears in (47b) as the contribution proportional to $v(\mathbf{k},\omega)$; successively higher-order VAA projections will operate the higher velocity-moment contributions to the r's. Now, for the sake of closure and in order to properly account for the γ-independent collisional transport processes which persist all the way down to very small γ values $(0\neq\gamma\ll 1)$, we suppose in the sequel that the 2D OCP pressure tensor has the Navier-Stokes structure

$$p_{ll}(\mathbf{k},\omega)=p(\mathbf{k},\omega)-ik\eta u_l(\mathbf{k},\omega)=p(\mathbf{k},\omega)-\frac{i\eta\omega}{n}\langle n_{\mathbf{k}}\rangle(\omega)\;, \tag{49}$$

$$p_{tt}(\mathbf{k},\omega)=p(\mathbf{k},\omega)+ik\eta u_l(\mathbf{k},\omega)\;, \tag{50}$$

$$p_{lt}(\mathbf{k},\omega)=p_{tl}(\mathbf{k},\omega)=0 \tag{51}$$

consistent with our remarks in Appendix A; $\eta\propto(m/\pi\beta)^{1/2}(2/\beta e^2)$ is the coefficient of viscosity. The Navier-Stokes hypothesis (49) now makes it possible to join (29) and (47b), resulting in

$$\langle n_{\mathbf{k}}\rangle(\omega)=\frac{\omega_p^2(k)}{\omega^2}\left[1+2\frac{\omega_p^2(k)}{\omega^2}\frac{k}{k_D}-\frac{i\eta k^2}{mn\omega}\right]\times[1+v(\mathbf{k},\omega)]\tilde{n}(\mathbf{k},\omega)+\frac{k^4}{2m\omega^4}[r_{llll}(\mathbf{k},\omega)+r_{ttll}(\mathbf{k},\omega)]=-\alpha_H(\mathbf{k},\omega)[1+v(\mathbf{k},\omega)]\tilde{n}(\mathbf{k},\omega)\;, \tag{52}$$

where

$$\alpha_H(\mathbf{k},\omega)=-\frac{\omega_p^2(k)}{\omega^2}-2\frac{\omega_p^4(k)}{\omega^4}\frac{k}{k_D}+\frac{i\eta k^2\omega_p^2(k)}{mn\omega^3}-\cdots\;. \tag{53}$$

From (52) and (8), one then obtains the compact formula

$$\alpha(\mathbf{k},\omega)=\alpha_H(\mathbf{k},\omega)[1+v(\mathbf{k},\omega)] \tag{54}$$

linking the linear polarizability to dynamical three-point structure functions [cf. (39)].

We next convert the Eq. (39) three-point structure functions into quadratic polarizabilities (defined in Sec. II) by application of the NLFDT [Ref. 11(b)]

$$S(\mathbf{p},\mu;\mathbf{q},\nu)=-2\,\mathrm{Im}\left[\frac{\hat{a}(\mathbf{p},\mu;\mathbf{q},\nu)}{\mu\nu}-\frac{\hat{a}(-\mathbf{p}-\mathbf{q},-\mu-\nu;\mathbf{p},\mu)}{\mu(\mu+\nu)}-\frac{\hat{a}(\mathbf{q},\nu;-\mathbf{p}-\mathbf{q},-\mu-\nu)}{(\mu+\nu)\nu}\right] \tag{55}$$

which is a central element of this theory. As the straightforward but involved mathematical steps for the conversion are already detailed in Ref. 9(d), we need only state the final result here:

$$v(\mathbf{k},\omega)=\frac{ik_D}{k}\frac{1}{N}\sum_{\mathbf{p}}\frac{\mathbf{k}\cdot\mathbf{p}}{kp}\int_{-\infty}^{\infty}d\mu\,\delta_-(\mu)\left[\frac{a(p,\mu;\mathbf{k}-\mathbf{p},\omega-\mu)}{\epsilon(\mathbf{p},\mu)\epsilon(\mathbf{k}-\mathbf{p},\omega-\mu)}+\frac{a(\mathbf{p},\omega-\mu;\mathbf{k}-\mathbf{p},\mu)}{\epsilon(\mathbf{p},\omega-\mu)\epsilon(\mathbf{k}-\mathbf{p},\mu)}\right]. \tag{56}$$

The appearance of the screening $\epsilon\epsilon$ clusters indicates that the dynamical coupling function has a superposition structure reminiscent of the well-known cluster expansions of statistical mechanics. The completes the first stage in our derivation: Eq. (54): with $v(\mathbf{k},\omega)$ given by (56) is the fundamental response-function relation linking linear and quadratic polarizabilities in the regime $kv_{\rm th}\ll\omega\ll\nu$.

The second stage in the formulation of the approximation scheme consists in making (54) and (56) self-consistent by approximating the quadratic polarizability in terms of linear ones. To accomplish this, we suppose that the quadratic polarizability $a(\mathbf{p},\mu;\mathbf{q},\nu)$ has the RPA structure

$$a_0(\mathbf{p},\mu;\mathbf{q},\nu)=\frac{i}{\beta mn}\int d^2v\frac{F^{(0)}(v)}{(\omega-\mathbf{k}\cdot\mathbf{v})^2}\left[\mathbf{k}\cdot\mathbf{p}\frac{\mathbf{q}\cdot\mathbf{v}}{\nu-\mathbf{q}\cdot\mathbf{v}}+\mathbf{k}\cdot\mathbf{q}\frac{\mathbf{p}\cdot\mathbf{v}}{\mu-\mathbf{p}\cdot\mathbf{v}}\right](\mathbf{q}=\mathbf{k}-\mathbf{p},\ \nu=\omega-\mu)\ , \tag{57}$$

consistent with the fact that $v(\mathbf{k},\omega)$ is γ dependent. Following the procedure of Ref. 9(d), we next develop (57) at long wavelengths ($|\mathbf{k}\cdot\mathbf{v}|\ll\omega$) and introduce the resulting expression into (56). After some algebra [see Ref. 9(d) for the details], one obtains

$$v_0(k\to0,\,\omega)\simeq\frac{\omega_p^2(k)}{\omega^2}\frac{1}{N}\sum_{\mathbf{p}}\frac{(\mathbf{k}\cdot\mathbf{p})^2}{k^3p}[S_0(\mathbf{k}-\mathbf{p})-S_0(\mathbf{p})]$$

$$-\frac{\omega_p^2(k)}{\omega^2}\frac{k}{k_D}\frac{1}{N}\sum_{\mathbf{p}}(1-4\chi^2+5\chi^4)\int_{-\infty}^{\infty}d\mu\,\delta_-(\mu)\hat{\alpha}_0(\mathbf{p},\mu)\hat{\alpha}_0(\mathbf{p},\omega-\mu)\ , \tag{58}$$

where $\chi\equiv(\mathbf{k}\cdot\mathbf{p})/kp$. We now postulate the expression (58) to be valid for *arbitrary coupling* and accordingly drop the "0" subscripts. This done, the resulting expression is then put into the more compact form

$$v(k\to0,\,\omega)\simeq\frac{\omega_p^2(k)}{\omega^2}\frac{k}{k_D}\left[\frac{5}{8}\beta E_c(\gamma)-\frac{7}{8}\frac{1}{N}\sum_{\mathbf{p}}\int_{-\infty}^{\infty}d\mu\,\delta_-(\mu)\hat{\alpha}(\mathbf{p},\mu)\,\hat{\alpha}(\mathbf{p},\omega-\mu)\right], \tag{59}$$

where

$$E_c(\gamma)=\frac{1}{2A}\sum_{\mathbf{p}}[S(\mathbf{p})-1]\phi(p) \tag{60}$$

is the correlation energy per particle. Notice how the correlational part

$$\lim_{k\to0}\left[\frac{1}{N}\sum_{p}\frac{(\mathbf{k}\cdot\mathbf{p})^2}{k^3p}[S(\mathbf{k}-\mathbf{p})-S(\mathbf{p})]\right]=\frac{k}{k_D}\frac{5}{8}\beta E_c(\gamma)$$

of the third frequency-moment sum-rule coefficient $\Omega^{(4)}(k)$ [cf. Eq. (45)] appears explicitly in (59): *This contribution to $v(k\to0,\omega)$ is assuredly exact.* Consequently, it is only the second rhs contribution (proportional to the $\hat{\alpha}\hat{\alpha}$ cluster) which is RPA-like. Equations (54) and (59) comprise the proposed self-consistent approximation scheme in which the correlation energy density is considered to be a given known input. This completes the task of the present section.

VI. PLASMON STRUCTURE

For the calculation of the dispersion and damping rate of the long-wavelength surface plasmons, we let

$$\omega(k)=\omega_p(k)+\Delta\omega(k)\ , \tag{61}$$

where $\omega(k)$ satisfies the dispersion relation

$$\alpha(\mathbf{k},\omega(k))=-1\ , \tag{62}$$

and $|\Delta\omega(k)|\ll\omega_p(k)$ is the long-wavelength correction which is to be calculated. The development of $\alpha(\mathbf{k},\omega(k))$ in what will amount to a small-k expansion about the point $\omega=\omega_p(k)$ gives [through $O(k^{3/2})$]

$$\alpha(k\to0,\,\omega(k))=\alpha(k\to0,\,\omega_p(k))+\Delta\omega(k)\left[\frac{\partial}{\partial\omega}\alpha(k\to0,\,\omega)\right]_{\omega=\omega_p(k)}+\cdots$$

$$\simeq-1-2\frac{k}{k_D}-v(k\to0,\,\omega_p(k))+\frac{i\eta k^2}{mn\omega_p(k)}+2\frac{\Delta\omega(k)}{\omega_p(k)}\ . \tag{63}$$

Taking account of (62), one then obtains

$$\mathrm{Re}\Delta\omega(k)\simeq\left[\frac{k}{k_D}+\tfrac{1}{2}\mathrm{Re}v(k\to0,\,\omega_p(k))\right]\omega_p(k)\ , \tag{64}$$

$$\mathrm{Im}\Delta\omega(k)\simeq-\frac{\eta k^2}{2mn}+\tfrac{1}{2}\mathrm{Im}v(k\to0,\,\omega_p(k))\omega_p(k)\ . \tag{65}$$

Now, from the Appendix B calculations,

$$v(k\to0,\,\omega_p(k))\approx\frac{5}{8}\frac{k}{k_D}\beta E_c(\gamma)-\frac{7}{16}\frac{1}{N}\sum_{\mathbf{p}}\frac{kk_D}{p^2}S^2(p)$$

$$-\frac{7i}{16}\frac{1}{N}\sum_{\mathbf{p}}\frac{kk_D}{p^2}\omega_p(k)\int_0^{\infty}\frac{d\mu}{2\pi}S^2(\mathbf{p},\mu) \tag{66}$$

so that (64)–(66), when substituted into (61), gives

$$\mathrm{Re}\,\omega(k\to 0)\simeq\left[1+\lambda(\gamma)\frac{k}{k_D}\right]\omega_p(k)\ , \tag{67}$$

$$\mathrm{Im}\,\omega(k\to 0)\simeq-\delta(\gamma)\left[\frac{k}{k_D}\right]^{3/2}\omega_p(k)\ , \tag{68}$$

where

$$\lambda(\gamma)=1+\frac{5}{16}\beta E_c(\gamma)-\frac{7}{32}\frac{1}{N}\sum_{\mathbf{p}}\frac{k_D^2}{p^2}S^2(p)\ , \tag{69}$$

$$\delta(\gamma)=\frac{\pi e^2\beta^{3/2}\eta}{\sqrt{m}}+\frac{7}{32}\frac{1}{N}\sum_{\mathbf{p}}\frac{k_D^2}{p^2}\omega_0\int_0^\infty\frac{d\mu}{2\pi}S^2(\mathbf{p},\mu)\ . \tag{70}$$

Equations (67)–(70) describe the dispersion and damping of long-wavelength surface-plasmon excitations over a wide range of γ values. The γ-independent collisional contribution to the dispersion is readily identified by observing that

$$\lambda_{\mathrm{coll}}(\gamma\ll 1)=\lambda(\gamma\ll 1)-\lambda_{\mathrm{RPA}}\simeq 1-\tfrac{3}{2}=-\tfrac{1}{2}\ ; \tag{71}$$

the first rhs viscosity term in (70) is the corresponding γ-independent collisional contribution to the damping rate. Considering the low-frequency character of the excitations, it is hardly surprising that the dispersive corrections turn out to be wholly thermodynamic, a feature which is not shared by the high-frequency electron plasmons in bulk systems. Since the improved VAA (Ref. 14) is exact at $\omega=0$, it is entirely justifiable to input (69) with correlation energy density data which are assumed to be determined by computer or other experimental data or by an independent theoretical approach. For example, Totsuji's cluster expansion calculations[8,15(b)] and Monte Carlo simulations[15(a)] provide

$$\beta E_c(\gamma)=(\gamma/2)[\ln(2\gamma)+0.1544]\quad\text{for }\gamma\ll 1\ , \tag{72}$$

$$\begin{aligned}\beta E_c(\gamma)&=-0.79\gamma^{1/2}+0.65\gamma^{1/8}-0.38\quad\text{for }1<\gamma<5\times10^3\\&=-1.12\Gamma+0.71\Gamma^{1/4}-0.38\quad\text{for }0.707<\Gamma<50\ \ (\Gamma=\beta e^2\sqrt{\pi n}=\sqrt{\gamma/2})\ ,\end{aligned} \tag{73}$$

whereas Lado's hypernetted-chain (HNC) calculations[16] provide

$$\beta E_c(\Gamma)=-1.0952\Gamma+0.9851\quad\text{for }\Gamma>30\ . \tag{74}$$

As to static structure function inputs, the Debye-Hückel formula (B8) is appropriate for $\gamma\ll 1$, while Lado's HNC data[16] or the somewhat less accurate compressibility formula

$$S(p)\simeq\frac{p}{\beta(\partial P/\partial n)_\beta p+k_D} \tag{75}$$

should suffice for $\gamma>2$. Equations (72), (B8), (73), and (75), when substituted into (69), result in the weak- and strong-coupling-regime expressions

$$\lambda(\gamma\ll 1)\simeq 1+0.351\gamma-0.375\gamma\ln\gamma^{-1}\ , \tag{76}$$

$$\lambda(\Gamma\gg 1)=1+\frac{5}{16}\frac{k_D}{\sqrt{\pi n}}\left[-0.559+\frac{0.354}{\Gamma^{3/4}}-\frac{0.3685}{\Gamma}\right]\ . \tag{77}$$

Note that the contribution from the third rhs cluster term in (69), while it is prominent for γ values up to unity, becomes dominated at stronger coupling by the $(\frac{5}{16}\beta E_c)$ third frequency-moment sum-rule contribution. Equations (67), (76), and (77) show how at low γ values the dispersion is controlled by k/k_D and γ, whereas at high γ values it is $k/\sqrt{\pi n}$ and Γ which control the dispersion, viz.

$$\mathrm{Re}\,\omega(k\to 0)\,|_{\substack{\Gamma\gg 1\\ \text{fluid}}}\simeq\left\{1+\frac{k}{\sqrt{\pi n}}\left[\frac{1}{2\Gamma}+\frac{5}{16}\left[-0.559+\frac{0.354}{\Gamma^{3/4}}-\frac{0.3685}{\Gamma}\right]\right]\right\}\omega_p(k)\ . \tag{78}$$

It is especially interesting to examine our fluid dispersion formula (67) in the infinite coupling ($\Gamma\to\infty$) limit. Strictly speaking, our theory can reach this limit only by inputting (69) with the fluid HNC expression (74). The resulting dispersion then saturates according to

$$\mathrm{Re}\,\omega(k\to 0)\,|_{\substack{\Gamma\to\infty\\ \text{HNC}}}\simeq\left[1-0.1711\frac{k}{\sqrt{\pi n}}\right]\omega_p(k)\ . \tag{79}$$

If, on the other hand, one inputs (69) with the Gann-Chakravarty-Chester solid phase Monte Carlo (MC) correlation energy expression[17]

$$\beta E_c(\Gamma)=-1.106\Gamma+\frac{5}{\Gamma}+\frac{560}{\Gamma^2} \tag{80}$$

and then goes to the $\Gamma\to\infty$ limit, one obtains

$$\mathrm{Re}\,\omega(k\to 0)\,|_{\substack{\Gamma\to\infty\\ \text{MC}}}=\left[1-0.1728\frac{k}{\sqrt{\pi n}}\right]\omega_p(k)\ . \tag{81}$$

Equations (79) and (81) are in excellent quantitative agreement with both (i) the Bonsall-Maradudin longitudinal-phonon dispersion formula[4(b)]

$$\mathrm{Re}\,\omega(k\to 0)\,|_{\substack{T=0\mathrm{K}\\ \text{crystal}}}\simeq\left[1-0.173\frac{k}{\sqrt{\pi n}}\right]\omega_p(k) \tag{82}$$

for the 2*D hexagonal lattice*[18] in the long-wavelength limit and (ii) the 2D OCP liquid-state dispersion data pertaining to the longest wavelengths ($k=0.425\sqrt{\pi n}$ and $0.847\sqrt{\pi n}$) which could be realized in the Totsuji-Kakeya simulations.[3]

As the plasma parameter increases from just above zero, λ decreases from its maximum value, $\lambda(\gamma\ll 1)\simeq 1$, to zero and then becomes negative. Equation (69), when inputted with (73), predicts that this transition from plasmonlike ($\lambda>0$) to longitudinal-phonon-like ($\lambda<0$) dispersion will occur at $\Gamma_{\mathrm{crit}}=3.22$. Our result compares favorably both with the Totsuji-Kakeya simulations[3] which predict the occurrence of the $\lambda=0$ boundary just beyond $\Gamma=2.29$ and with Baus's $\Gamma_{\rm crit}=3.55$ which we have calculated by inputting his dispersion formula[2]

$$\mathrm{Re}\,\omega(k\to 0)\,|_{\rm Baus}=\left[1+\frac{c_p}{c_v}\frac{\beta}{2}\left[\frac{\partial P}{\partial n}\right]_\beta \frac{k}{k_D}\right]\omega_p(k) \tag{83}$$

with the same correlation energy density formula (73).[19] This transition from "positive" to "negative" dispersion which, incidentally, is observed in bulk OCP's as well, is inextricably linked to the onset of a liquid-state short-range order signaled by the development of oscillations in the equilibrium pair correlation function,[15(a),16]

$$g(r)=\frac{1}{N}\sum_{\mathbf{q}}[S(\mathbf{q})-1]\exp(i\mathbf{q}\cdot\mathbf{r})\ , \tag{84}$$

somewhere in the range $2.2<\Gamma<2.9$.

VII. CONCLUSIONS

The twofold purpose of the present paper has been (i) to formulate a self-consistent approximation scheme for the calculation of the 2D OCP dynamical polarizability at long wavelengths and arbitrary coupling and (ii) to calculate the dispersion of the low-frequency plasma mode.

The principal building blocks to the construction of the approximation scheme are the nonlinear fluctuation-dissipation theorem[11(b)] and linearized equations for the plasma density, fluid velocity, pressure tensor, and heat-flow tensor moments. Equilibrium three-point correlations, quadratic polarizability response functions, and the Navier-Stokes hypothesis linking the pressure tensor to its trace are all central elements in the development of the theory. The polarizability calculations of this paper can be best stated in terms of the simple relation

$$\alpha(\mathbf{k},\omega;\gamma)=\alpha(\mathbf{k},\omega;\gamma\ll 1)[1+v(\mathbf{k},\omega)]\ , \tag{85}$$

where the dynamical coupling correction $v(\mathbf{k},\omega)$ contains all the γ-dependent correlational contributions. Elsewhere,[9(d)] in a treatment of bulk OCP's, it was shown that the 3D version of (85) has the static ($\omega=0$) counterpart

$$\alpha(\mathbf{k},0;\gamma)=\alpha_0(\mathbf{k},0)[1+v(\mathbf{k},0)] \tag{86}$$

which turns out to be identical to the *exact* second BBGKY static equation linking the equilibrium pair and ternary correlation functions.[9(d),9(f)] As one might expect, Eq. (86) with

$$\alpha_0(k,0)=\frac{k_D}{k} \tag{87}$$

and [cf. (56)]

$$v(k,0)=\frac{ik_D}{k}\frac{1}{N}\sum_{\mathbf{p}}\frac{\mathbf{k}\cdot\mathbf{p}}{kp}\frac{a(\mathbf{p},0;\mathbf{k}-\mathbf{p},0)}{\epsilon(\mathbf{p},0)\epsilon(\mathbf{k}-\mathbf{p},0)} \tag{88}$$

is also the *exact* static counterpart of (85) in two dimensions. This further attests to the universality of (85) over the entire frequency domain. At frequencies large compared with the collision frequency, compressions are necessarily one dimensional and $\alpha(\mathbf{k},\omega\gg\nu;\gamma\ll 1)$ is identified as the Vlasov polarizability (36); $v(\mathbf{k}\omega)$, when evaluated at $\omega\gg\nu$, simplifies to the known exact correlational contribution (42) to the third frequency-moment sum rule; consequently, our expressions (44) and (45) for $\mathrm{Re}\,\alpha(\mathbf{k},\omega\gg\nu;\gamma)$ are *exact* through $O[\omega_p^4(k)/\omega^4]$. In the lower-frequency band $kv_{\rm th}\ll\omega\ll\nu$ characterizing the long-wavelength hydrodynamic regime, the compressions are two dimensional and $\alpha(\mathbf{k},\omega;\gamma\ll 1)$ is identified as the hydrodynamic polarizability (53); the conversion of $v(\mathbf{k},\omega)$ from its three-point structure function representation (39) to quadratic polarizability representation (56) is effected by application of the NLFDT (55). Thus, Eqs. (54) and (56) combine into a compact and elegant response-function relation linking the linear and quadratic polarizabilities. Self-consistency is then guaranteed by postulating that a decomposition of the latter in terms of the former, which prevails in the $k\to 0$ limit for weak coupling, can be relied upon as a paradigm for arbitrary coupling. The resulting nonlinear integral equation for the dynamical polarizability [Eqs. (54) and (59)] we now propose to be valid at long wavelengths and arbitrary γ values.

We have analyzed Eqs. (54) and (59) near $\omega=\omega_p(k)$ to determine the structure of the long-wavelength surface plasmons. Equations (67)—(70) describe the dispersion and damping over the range of coupling strengths spanning the entire fluid regime. Considering the low-frequency character of the excitations, it is hardly surprising that the γ-dependent dispersive corrections turn out to be wholly thermodynamic, a feature which is not shared by high-frequency electron plasma modes in bulk systems. Our formula (69) for the dispersive coefficient λ, when inputted with Totsuji's correlation energy density expression (73),[15(a)] predicts that the transition from plasmonlike ($\lambda>0$) to longitudinal-phonon-like ($\lambda<0$) dispersion will occur at $\Gamma_{\rm crit}=3.22$. This compares reasonably well both with the Totsuji-Kakeya MD simulations[3] which predict the occurrence of the $\lambda=0$ boundary just beyond $\Gamma=2.29$ and with Baus's $\Gamma_{\rm crit}=3.55$ which we have calculated by inputting his dispersion formula[2] with the same Totsuji correlation energy density formula (73). The transition from positive to negative dispersion which, incidentally, is observed in bulk OCP's as well, is inextricably linked to the onset of a "liquid-state" short-range order signaled by the development of oscillations in the equilibrium pair correlation function[15(a),16] somewhere in the range $2.2<\Gamma<2.9$. Finally, in the infinite coupling ($\Gamma\to\infty$) limit, our dispersion formulas (79) and (81) very nearly reproduce the zero-temperature Bonsall-Maradudin longitudinal-phonon formula (82) for the two-dimensional hexagonal lattice.[4(b)] This suggests that it is the correlational contribution (42) to the third frequency-moment sum rule which controls the long-wavelength dispersion of 2D OCP longitudinal-phonon excitations.

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APPENDIX A: 2D OCP PRESSURE TENSOR

From Eqs. (30) and (31), the longitudinal (l: in the direction of $\mathbf{k}$) and transverse (t: perpendicular to $\mathbf{k}$) elements of the pressure are calculated to be

$$p_{ll}(k,\omega)=\frac{3}{\beta}\frac{\omega_p^2(k)}{\omega^2}[1+v(\mathbf{k},\omega)]\tilde{n}(\mathbf{k},\omega)+\frac{2}{\omega A}\sum_{q}\phi(q)q_l\langle j_{l_{\mathbf{k}-\mathbf{q}}}n_{\mathbf{q}}\rangle(\omega)+\frac{k^2}{\omega^2}r_{llll}(\mathbf{k},\omega)\ , \tag{A1}$$

$$p_{tt}(k,\omega)=\frac{1}{\beta}\frac{\omega_p^2(k)}{\omega^2}[1+v(\mathbf{k},\omega)]\tilde{n}(\mathbf{k},\omega)+\frac{2}{\omega A}\sum_{q}\phi(q)q_t\langle j_{t_{\mathbf{k}-\mathbf{q}}}n_{\mathbf{q}}\rangle(\omega)+\frac{k^2}{\omega^2}r_{ttll}(\mathbf{k},\omega)\ , \tag{A2}$$

$$p_{lt}(k,\omega)=\frac{1}{\omega A}\sum_{q}\phi(q)[q_t\langle j_{l_{\mathbf{k}-\mathbf{q}}}n_{\mathbf{q}}\rangle(\omega)+q_l\langle j_{t_{\mathbf{k}-\mathbf{q}}}n_{\mathbf{q}}\rangle(\omega)]+\frac{\omega_p^2(k)}{\beta\omega^2}\frac{1}{N}\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{q_t}{q}\langle n_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)\ . \tag{A3}$$

To see that $p_{lt}(\mathbf{k},\omega)=0$, we observe that the equilibrium ternary correlation

$$\langle j_{t_{\mathbf{k}-\mathbf{q}}}(t=0)n_{\mathbf{q}}(t=0)n_{-\mathbf{k}}(t)\rangle^{(0)}$$

changes sign under the *transverse* spatial inversion $q_t\to-q_t$; consequently,

$$\sum_{\mathbf{q}}\phi(q)q_l\langle j_{t_{\mathbf{k}-\mathbf{q}_l-\mathbf{q}_t}}n_{\mathbf{q}_l+\mathbf{q}_t}\rangle(\omega)=\sum_{\mathbf{q}}\phi(q)q_l\langle j_{t_{\mathbf{k}-\mathbf{q}_l+\mathbf{q}_t}}n_{\mathbf{q}_l-\mathbf{q}_t}\rangle(\omega)=-\sum_{\mathbf{q}}\phi(q)q_l\langle j_{t_{\mathbf{k}-\mathbf{q}_l-\mathbf{q}_t}}n_{\mathbf{q}_l+\mathbf{q}_t}\rangle(\omega)\ ,$$

whence

$$\sum_{\mathbf{q}}\phi(q)q_l\langle j_{t_{\mathbf{k}-\mathbf{q}}}n_{\mathbf{q}}\rangle(\omega)=0\ . \tag{A4}$$

On the other hand, the fact that the correlations

$$\langle j_{l_{\mathbf{k}-\mathbf{q}}}(t=0)n_{\mathbf{q}}(t=0)n_{-\mathbf{k}}(t)\rangle^{(0)}\ ,$$

$$\langle n_{\mathbf{k}-\mathbf{q}}(t=0)n_{\mathbf{q}}(t=0)n_{-\mathbf{k}}(t)\rangle^{(0)}$$

do not change sign under $q_t\to-q_t$ inversion guarantees that

$$\sum_{\mathbf{q}}\phi(q)q_t\langle j_{l_{\mathbf{k}-\mathbf{q}}}n_{\mathbf{q}}\rangle(\omega)=0\ , \tag{A5}$$

$$\sum_{\substack{\mathbf{q}\\ \mathbf{q}\neq\mathbf{k}}}\frac{q_t}{q}\langle n_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)=0\ , \tag{A6}$$

as well. Momentum conservation [cf. (18)] then requires

$$u_t(k,\omega)=\frac{k}{mn\omega}p_{lt}(\mathbf{k},\omega)=0\ , \tag{A7}$$

consistent with the fact that symmetry considerations preclude any possibility of a nonzero average transverse fluid velocity response to a longitudinal perturbation.

Turning next to Eqs. (A1) and (A2), the tendency for p_{ll} and p_{tt} to equilibrate is a direct consequence of the fact that

$$\sum_{\mathbf{q}}\phi(q)\mathbf{q}\cdot\langle \mathbf{j}_{\mathbf{k}-\mathbf{q}}n_{\mathbf{q}}\rangle(\omega)=0\ , \tag{A8}$$

i.e., collisions do not lead to a net gain or loss of total (longitudinal + transverse) energy; they affect only exchange in energy between longitudinal and transverse degrees of freedom of the particle motions. In light of (A8), Eqs. (A1) and (A2) can be restated in the forms

$$p_{ll}(\mathbf{k},\omega)=\frac{3}{\beta}\frac{\omega_p^2(k)}{\omega^2}[1+v(\mathbf{k},\omega)]\tilde{n}(\mathbf{k},\omega)+\frac{1}{\beta}\frac{k_D}{k}w(\mathbf{k},\omega)\tilde{n}(\mathbf{k}\omega)+\frac{k^2}{\omega^2}r_{llll}(\mathbf{k},\omega)\ , \tag{A9}$$

$$p_{tt}(\mathbf{k},\omega)=\frac{1}{\beta}\frac{\omega_p^2(k)}{\omega^2}[1+v(\mathbf{k},\omega)]\tilde{n}(\mathbf{k},\omega)-\frac{1}{\beta}\frac{k_D}{k}w(\mathbf{k},\omega)\tilde{n}(\mathbf{k}\omega)+\frac{k^2}{\omega^2}r_{ttll}(\mathbf{k},\omega)\ . \tag{A10}$$

The collision exchange term $(1/\beta)(k_D/k)w(\mathbf{k},\omega)\tilde{n}(\mathbf{k},\omega)$ evidently tends to reduce the numerical coefficient of the first rhs member of (A9) from 3 to 2 while at the same time raising from 1 to 2 the corresponding numerical coefficient in (A10). The *hydrostatic* pressure is accordingly given by

$$p(\mathbf{k},\omega)=\tfrac{1}{2}[p_{ll}(\mathbf{k},\omega)+p_{tt}(\mathbf{k},\omega)]=\frac{2}{\beta}\frac{\omega_p^2(k)}{\omega^2}[1+v(\mathbf{k},\omega)]\tilde{n}(\mathbf{k},\omega)+\frac{k^2}{2\omega^2}[r_{llll}(\mathbf{k},\omega)+r_{ttll}(\mathbf{k},\omega)]\ . \tag{A11}$$

APPENDIX B: DYNAMICAL COUPLING FUNCTION

In this appendix we evaluate the dynamical coupling function

$$v_{\text{dyn}}(k\to0,\omega)=-\frac{7}{8}\frac{\omega_p^2(k)}{\omega^2}\frac{k}{k_D}\frac{1}{N}\sum_{p}\int_{-\infty}^{\infty}d\mu\,\delta_-(\mu)\hat{\alpha}(\mathbf{p},\mu)\hat{\alpha}(\mathbf{p},\omega-\mu)\ , \tag{B1}$$

at $\omega=\omega_p(k)$. To facilitate the calculation, the function

$$\hat{H}(\mathbf{p},\omega)=\int_{-\infty}^{\infty}d\mu\,\delta_{-}(\mu)\hat{\alpha}(\mathbf{p},\mu)\hat{\alpha}(\mathbf{p},\omega-\mu) \tag{B2}$$

is first of all put into the somewhat more tractable form

$$\hat{H}(\mathbf{p},\omega)=\frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{d\mu}{\mu}\hat{\alpha}''(\mathbf{p},\mu)\hat{\alpha}(\mathbf{p},\omega-\mu)\ . \tag{B3}$$

Both Eq. (B3) and the useful Hilbert transform

$$\hat{\alpha}(p,0)\hat{\alpha}^{*}(\mathbf{p},\omega)=-\frac{i}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{d\mu}{\omega-\mu}\hat{\alpha}^{*}(\mathbf{p},\mu)\hat{\alpha}(\mathbf{p},\omega-\mu) \tag{B4}$$

are derived from (B2) by exploiting the fact that $\hat{\alpha}(\mathbf{p},\mu)=\hat{\alpha}'(\mathbf{p},\mu)+i\hat{\alpha}''(\mathbf{p}\mu)\equiv\hat{\alpha}^{*}(\mathbf{p},\mu)+2i\hat{\alpha}''(\mathbf{p},\mu)$ is a plus function and therefore $\hat{\alpha}^{*}(\mathbf{p},\mu)$ as well as $\hat{\alpha}(\mathbf{p},\omega-\mu)$ are minus functions of μ.

We next observe that for $\omega_p(k)$ small,

$$\hat{H}(\mathbf{p},\omega_p(k))=\mathrm{Re}\hat{H}(\mathbf{p},0)+i\omega_p(k)\left[\frac{\partial}{\partial\omega}\mathrm{Im}\hat{H}(\mathbf{p},\omega)\right]_{\omega=0}+\tfrac{1}{2}\omega_p^2(k)\left[\frac{\partial^2}{\partial\omega^2}\mathrm{Re}\hat{H}(\mathbf{p},\omega)\right]_{\omega=0}+\cdots$$
$$\simeq\tfrac{1}{2}\hat{\alpha}^{2}(\mathbf{p},0)+\frac{i\omega_p(k)}{2\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{d\mu}{\mu}\frac{\partial}{\partial\mu}\hat{\alpha}''^{2}(\mathbf{p},\mu)+\frac{\omega_p^2(k)}{2\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{d\mu}{\mu}\hat{\alpha}''(\mathbf{p}\mu)\frac{\partial^2}{\partial\mu^2}\hat{\alpha}'(\mathbf{p},\mu)+\cdots\ , \tag{B5}$$

whence from (B1) and the linear fluctuation-dissipation theorem, one obtains to *lowest order in* k

$$\mathrm{Re}v_{\mathrm{dyn}}(k\to0,\ \omega_p(k))\simeq-\frac{7}{16}\frac{1}{N}\sum_{\mathbf{p}}\frac{kk_D}{p^2}S^2(p)<0\ , \tag{B6}$$

$$\mathrm{Im}v_{\mathrm{dyn}}(k\to0,\ \omega_p(k))\simeq-\frac{7}{32}\frac{1}{N}\sum_{\mathbf{p}}\frac{kk_D}{p^2}\omega_p(k)\int_0^{\infty}\frac{d\mu}{\pi}S^2(\mathbf{p},\mu)<0\ . \tag{B7}$$

Equation (B6) provides an $O(k)$ correlational correction to the dispersion of the low-frequency plasma mode while (B7) provides an $O(k^{3/2})$ correction to the damping. The dispersive correction turns out to be especially prominent in the weak-coupling regime for γ values up to unity.

For $\gamma\ll1$, (B6) and (B7) can be analytically evaluated when inputted with the RPA structure functions

$$S_0(x)=\frac{x}{1+x}\ , \tag{B8}$$

$$S_0(x,\mu)=\frac{\sqrt{2\pi}}{x\omega_0\,|\,\epsilon_0(\mathbf{p},\mu)\,|^2}\exp\left[-\frac{\mu^2}{2\omega_0^2x^2}\right]\ , \tag{B9}$$

where

$$\epsilon_0(p,\mu)=1+\frac{\phi(p)}{m}\int d^2v\frac{\mathbf{p}\cdot\partial F^{(0)}(v)/\partial\mathbf{v}}{\mu-\mathbf{p}\cdot\mathbf{v}} \tag{B10}$$

and $x=p/k_D$. Equation (B6) readily integrates to

$$\mathrm{Re}v_{\mathrm{dyn}}(k\to0,\ \omega_p(k))\,|_{\gamma\ll1}\simeq-\frac{7}{16}\frac{k}{k_D}\gamma(\ln\gamma^{-1}-1) \tag{B11}$$

with the understanding that the summation in (B6) is to be cut off at $p_{\max}=1/(\beta e^2)=k_D/\gamma$ in order to avoid the unphysical logarithmic divergence arising, most likely, from the inherent RPA character of (B6) (see Sec. V). Note that for γ values above unity, the cutoff $p_{\max}=(\pi n)^{1/2}\gg1/(\beta e^2)$ is more appropriate.

The evaluation of (B7) at $\gamma\ll1$ or, equivalently, of

$$\mathrm{Im}v_{\mathrm{dyn}}(k\to0,\ \omega_p(k))\,|_{\gamma\ll1}\simeq-\frac{7}{16}\left[\frac{k}{k_D}\right]^{3/2}\gamma\int_0^{\infty}\frac{dx}{x^3}\int_0^{\infty}\frac{d\mu}{\omega_0}\frac{1}{|\,\epsilon_0(\mathbf{x},\mu)\,|^4}\times\exp\left[-\frac{\mu^2}{\omega_0^2x^2}\right] \tag{B12}$$

is quite straightforward when screening effects are entirely ignored, for then

$$\mathrm{Im}v_{\mathrm{dyn}}(k\to0,\ \omega_p(k))\,|_{\substack{\gamma\ll1\\|\epsilon|=1}}\simeq-\frac{7}{16}\gamma\left[\frac{k}{k_D}\right]^{3/2}\int_1^{\infty}\frac{dx}{x^3}\int_0^{\infty}\frac{d\mu}{\omega_0}\exp\left[-\frac{\mu^2}{\omega_0^2x^2}\right]=-\frac{7}{32}\frac{k}{k_D}\left[\frac{\pi k}{k_D}\right]^{1/2}\gamma\ . \tag{B13}$$

Note that the $p_{\min}=k_D$ lower limit has been imposed on the x integration to compensate for the removal of the screening. Should the screening be fully retained, then the subsequently more involved calculations lead to

$$\mathrm{Im}v_{\mathrm{dyn}}(k\to0,\ \omega_p(k))\,|_{\gamma\ll1}\simeq-\frac{7}{16}\frac{k}{k_D}\left[\frac{\pi k}{k_D}\right]^{1/2}\gamma\left[\frac{25e^{-1}}{6\sqrt{\pi}}+\tfrac{1}{6}\mathrm{erf}(1)+\tfrac{1}{3}\{\mathrm{erf}[(k_D/k)^{1/4}]-\mathrm{erf}(1)\}\right]\ . \tag{B14}$$

Equations (B13) and (B14) can be stated in the more compact form

$$\mathrm{Im}v_{\mathrm{dyn}}(k\to0,\ \omega_p(k))\,|_{\gamma\ll1}\simeq\frac{7\bar{A}}{16}\frac{k}{k_D}\left[\frac{\pi k}{k_D}\right]^{1/2}\gamma\ , \tag{B15}$$

where

$$\overline{A}_{\substack{\text{zero}\\\text{screening}}} = 0.5 \ , \qquad \text{(B16)}$$

$$\overline{A}_{\text{screening}} = 1.00 - 1.05 \ , \qquad \text{(B17)}$$

depending on k_D/k in the range $1 < (k_D/k) < \infty$. To summarize: (B6) and (B7) provide $O(k)$ and $O(k^{3/2})$ *correlational* corrections to the dispersion and damping of the long-wavelength plasmon modes at arbitrary coupling; (B11) and (B15) provide the corresponding corrections in the weak-coupling limit.

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$$\omega(k)\,|_{\substack{\Gamma\to\infty\\\text{Baus}}} = \omega_p(k)\left[1 - 0.2059\frac{k}{\sqrt{\pi n}}\right]$$

derived from (83) and (74) in the infinite coupling limit agrees less satisfactorily with the Bonsall-Maradudin result (82) for the two-dimensional hexagonal lattice.