

Relativistic modulational instability of an electromagnetic wave in a magnetized plasma

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The relativistic modulational instability of an intense circularly polarized electromagnetic wave propagating along an external magnetic field in a plasma is investigated. The method of investigation is based on the derivation of a nonlinear Schrödinger equation. The effect of the external magnetic field is to diminish the modulational instability of the circularly polarized wave. Actually, the modulational instability can be completely eliminated if the external magnetic field is sufficiently strong.

I. INTRODUCTION

Nonlinear interaction of very intense laser pulses with plasma is of great interest in laboratory plasmas. At very high laser intensities of current interest, the directed velocity of the electrons become relativistic. The consequent electron-mass variation can lead to modulational instability of electromagnetic waves propagating through the plasma (Max, Arons, and Langdon¹). The latter enables the incident laser pulse to propagate into denser regions of the plasma, and therefore, plays an important role in transferring the laser energy to the plasma. The usual ponderomotive-force driven modulational instabilities (Drake *et al.*² and Kaw, Schmidt, and Wilcox³) involving ion motions occur with characteristic frequencies $|\omega^2| \leq \omega_{pi}^2$. However, the relativistic modulational instability arising with high laser intensities involves only electron motions. This is because this instability can grow so quickly in time that ion inertia prevents the ions from following along.

Experiments of Stamper *et al.*⁴ and some others showed that intense spontaneously generated magnetic fields are present in laser-produced plasmas. Therefore, some studies have been made of the laser interaction in a magnetized plasma (Aliev and Kuznetsov,⁵ Berezhiani, Paverman, and Tskhakaya,⁶ and Rao, Shukla, and Yu⁷). We investigate here the modulational instability of an intense circularly polarized electromagnetic wave propagating along a constant external magnetic field in the plasma. The method of investigation is different from the one used by Max *et al.*,¹ and is based on the derivation of a nonlinear Schrödinger equation.

II. GOVERNING EQUATIONS

Consider the propagation of an intense circularly polarized electromagnetic wave of frequency ω_0 and wave number k_0 in a plasma subjected to a state external magnetic field \mathbf{B}_0 . In the modulational interaction problem the plasma response is two-pronged: the high-frequency motion of the electrons in the wave field, and the low-frequency motion caused by the ponderomotive force exerted by the self-interaction of the wave. The relativistic modulational processes that occur in cases involving high

laser intensities occur at a rate too rapid for the ions to participate in them. Therefore, we consider the ions to remain immobile and to form a neutralizing background, and consider only the electron response. Further, we shall neglect the thermal motion of the electrons in comparison with the directed motion. We then have the following equations governing the motion of the electron fluid:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{V}) = 0, \quad (1)$$

$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{p} = -e \left[\mathbf{E} + \frac{1}{c} \mathbf{V} \times (\mathbf{B}_0 + \mathbf{B}) \right], \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (3)$$

$$\nabla \times \mathbf{B} = -\frac{4\pi ne}{c} \mathbf{V} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \quad (4)$$

$$\nabla \cdot \mathbf{E} = -4\pi e(n - n_0), \quad (5)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (6)$$

where n is the number density, \mathbf{V} is the velocity of the fluid, c is the speed of light,

$$\mathbf{p} = \frac{m_0 \mathbf{V}}{(1 - V^2/c^2)^{1/2}}, \quad (7)$$

m_0 is the rest mass of an electron, and n_0 is the unperturbed number density. Let us introduce the scalar and vector potentials ϕ and \mathbf{A} to describe the wave fields \mathbf{E} and \mathbf{B} :

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}. \quad (8)$$

ϕ turns out to be the low-frequency ambipolar potential.

Let us consider a left-hand circularly polarized wave given by

$$\mathbf{A} = A(z, t)(\hat{\mathbf{i}}_z - i\hat{\mathbf{i}}_y)\cos\theta, \quad \theta = k_0 z - \omega_0 t \quad (9)$$

to propagate along $\mathbf{B}_0 = B_0 \hat{\mathbf{i}}_z$, with the linear wave propagation relation

$$\omega_0^2 = c^2 k_0^2 + \frac{\omega_p^2 \omega}{\omega + \Omega_0}, \quad (10)$$

where

$$\omega_p^2 = \frac{4\pi n_0 e^2}{m_0}, \quad \Omega_0 = \frac{eB_0}{m_0 c}.$$

Using (7) and (8), Eq. (2) can be written as

$$\begin{aligned} \frac{\partial \mathbf{p}}{\partial t} + \frac{1}{m_0(1+p^2/m_0^2 c^2)^{1/2}} (\mathbf{p} \cdot \nabla) \mathbf{p} \\ = \frac{e}{c} \frac{\partial \mathbf{A}}{\partial t} + e \nabla \phi \\ - \frac{e/m_0 c}{(1+p^2/m_0^2 c^2)^{1/2}} \{ \mathbf{p} \times [(\nabla \times \mathbf{A}) + \mathbf{B}_0] \} \end{aligned} \quad (11)$$

from which it follows that (Aliev *et al.*,⁵ and Rao *et al.*⁷)

$$A \cos \theta = \frac{cp}{e} + \alpha \frac{cp/e}{(1+p^2/m_0^2 c^2)^{1/2}}, \quad (12)$$

$$\nabla \phi = c^2 \nabla \left[(1+p^2/m_0^2 c^2)^{1/2} - \frac{\alpha/2}{1+p^2/m_0^2 c^2} \right] = 0, \quad (13)$$

where

$$\alpha \equiv \frac{\Omega_0}{\omega_0}$$

and Eq. (13) has been obtained by ignoring the electron momentum in the wave propagation direction in the low-frequency plasma response.

III. NONLINEAR SCHRÖDINGER EQUATION

We obtain from Eqs. (3), (4), and (6) the wave equation

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} - c^2 \nabla^2 \mathbf{A} = -4\pi n e c \mathbf{V}. \quad (14)$$

Using (9) and (10) and noting that $A(Z, t)$ is a slowly-varying function of Z and t , we obtain from Eq. (14),

$$\begin{aligned} \left[i \left[\frac{\partial A}{\partial t} + \frac{k_0 c^2}{\omega_0} \frac{\partial A}{\partial z} \right] + \frac{c^2}{2\omega_0} \frac{\partial^2 A}{\partial z^2} \right] \cos \theta \\ = \frac{\omega_p^2}{2\omega_0} \left[\frac{(c\mathbf{p}/e)(n/n_0)}{(1+p^2/m_0^2 c^2)^{1/2}} - \frac{A \cos \theta}{1+\alpha} \right]. \end{aligned} \quad (15)$$

Here,

$$n = n_0 + \delta n \quad (16)$$

and the density perturbation δn can be calculated from the dynamics of the plasma motion parallel to the wave propagation direction,

$$\frac{\partial \delta n}{\partial t} + n_0 \frac{\partial \delta V}{\partial z} = 0, \quad (17)$$

$$\frac{\partial \delta V}{\partial t} = -\frac{e}{m} \left[-i \frac{V_0 - \delta B_+}{c} + \delta E_z \right], \quad (18)$$

$$\frac{\partial \delta E_z}{\partial z} = -4\pi e \delta n, \quad (19)$$

where $q_{\pm} = q_x \pm iq_y$, and

$$V_{0-} = -\frac{eE_0 e^{i\theta}}{im_0 \omega_0 (1+\alpha)}, \quad \delta B_+ = \frac{ick_0}{\omega_0} E_+ e^{i\theta}. \quad (20)$$

Equations (17)–(20) give

$$\frac{\delta n}{n_0} = -\frac{e^2 E_0^2}{m_0^2 \omega_0^2 c^2 (1+\alpha)} \frac{\omega_0^2 - \omega_p^2 / (1+\alpha)}{4\omega_0^2 - \omega_p^2} \cos(2\theta) \quad (21)$$

or

$$\frac{\delta n}{n_0} = \frac{e^2 A^2}{m_0^2 c^4 (1+\alpha)} \frac{\omega_0^2 - \omega_p^2 / (1+\alpha)}{4\omega_0^2 - \omega_p^2} \cos(2\theta).$$

Next, assuming that $p^2/m_0^2 c^2 \ll 1$, we obtain from (12),

$$p \approx \frac{eA/c}{1+\alpha} \cos \theta + \frac{\alpha}{2} \frac{e^3 A^2 / m_0^2 c^5}{(1+\alpha)^3} \cos^3 \theta. \quad (22)$$

Using (16), (21), and (22), Eq. (15) becomes

$$i \left[\frac{\partial A}{\partial t} + \frac{k_0 c^2}{\omega_0} \frac{\partial A}{\partial z} \right] + \frac{c^2}{2\omega_0} \frac{\partial^2 A}{\partial z^2} + \frac{\omega_p^2 q}{4\omega_0} |A|^2 A = 0, \quad (23)$$

where

$$q = \frac{e^2 / m_0^2 c^4}{(1+\alpha)^2} \left[\frac{3/4}{1+\alpha} - \frac{\omega_0^2 - \omega_p^2 / (1+\alpha)}{4\omega_0^2 - \omega_p^2} \right]. \quad (24)$$

For the case of an unmagnetized plasma (with $\alpha=0$, Eq. (23) can be seen to arise through the familiar nonlinear frequency shift due to the relativistic effects (Sluijter and Montgomery⁸).

Upon transferring to a frame moving with speed $k_0 c^2 / \omega_0$, Eq. (23) reduces to the standard form of the nonlinear Schrödinger equation (Hasegawa⁹):

$$i \frac{\partial A}{\partial t} + \frac{c^2}{2\omega_0} \frac{\partial^2 A}{\partial \xi^2} + \frac{\omega_p^2 q}{4\omega_0} |A|^2 A = 0, \quad (25)$$

where

$$\xi = z - \frac{k_0 c^2}{\omega_0} t.$$

IV. MODULATIONAL INSTABILITY

In order to investigate stability with respect to low-frequency modulation, let us rewrite Eq. (25) as follows:

$$i \frac{\partial A}{\partial t} + \frac{c^2}{2\omega_0} \frac{\partial^2 A}{\partial \xi^2} + \frac{\omega_p^2 q}{4\omega_0} (|A|^2 - |A_0|^2) A = 0 \quad (26)$$

and set (Hasegawa⁹)

$$A = \sqrt{P}(\xi, t) e^{i\sigma(\xi, t)}, \quad (27)$$

$$P = P_0 + P_1(\xi, t), \quad \sigma = \sigma_1(\xi, t)$$

and linearize in P_1, σ_1 , and consider

$$P_1(\xi, t) \text{ and } \sigma_1(\xi, t) \sim e^{i(K\xi - \Omega t)}. \quad (28)$$

We then obtain

$$\Omega^2 = \frac{c^2 K^2}{4\omega_0} \left[\frac{c^2 K^2}{\omega_0} - \frac{\omega_p^2 q}{\omega_0} |A_0|^2 \right] \quad (29)$$

so that the modulational instability arises if

$$|E_0|^2 > \frac{c^4 K^2 \omega_0^2}{\omega_p^2 q}. \quad (30)$$

The maximum growth rate is given by

$$\gamma_{\max} = \frac{\omega_p^2 c^2 q}{4\omega_0^3} |E_0|^2. \quad (31)$$

Equations (30) and (31) show that the larger the external magnetic field (so that q will be smaller) the larger will be the threshold value for the wave intensity to cause modulational instability, and the smaller will be the maximum growth rate. Actually, if the external magnetic field is sufficiently strong (so that $q < 0$), the modulation instability will not arise at all!

Physically this can be understood as follows. The

modulational instability arises because the group velocity of the wave increases in the direction of the increasing wave field intensity so that there is an energy transport from the region of small field intensity to the region of large field intensity. It is seen that the effect of an external magnetic field is to diminish this trend and even reverse it if sufficiently strong.

The results corresponding to the right circularly polarized wave can be obtained simply by letting α be negative. Because of possible gyroresonance, however, the validity of the fluid model becomes then questionable (Dimonte, Lamb, and Morales¹⁰).

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