Generation of squeezed states via degenerate four-wave mixing

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A microscopic model of degenerate four-wave mixing including the quantization of the medium is given. Thus the full effects of loss and spontaneous emission on the squeezing attainable are analyzed. We examine separately the squeezing in the output fields for counterpropagating four-wave mixing, copropagating four-wave mixing, and four-wave mixing in a single-ended optical ring cavity. Good squeezing is possible only in certain limits of atomic parameters.

I. INTRODUCTION

There has been much recent interest in the generation of squeezed or two-photon coherent states.[1,2] Such states have less noise than a coherent state in one of the field quadratures and can exhibit a number of distinctly quantum features, such as sub-Poissonian statistics. They have potential applications in optical communication systems[3] and gravitational radiation detectors.[4] Potential schemes for producing squeezed light include the degenerate parametric amplifier,[5–10] degenerate four-wave mixing,[11–13] resonance fluorescence,[14–16] and two-photon optical bistability.[17,18] More recently Collett and Walls[19] have suggested squeezing to be possible in an output mode for dispersive optical bistability and second-harmonic and subharmonic generation.

Degenerate four-wave mixing was first proposed as a way to produce squeezed states by Yuen and Shapiro.[11] Their analysis assumed an ideal $\chi^{(3)}$ nonlinear polarizability and made assumptions such as nondepletion of the pump modes and zero loss. The work of Bondurant *et al.*[12] and Milburn, Levenson, and Walls[20] has considered the effect of quantization of the pumps and suggests this to be, in appropriate limits, no real barrier to squeezing. The effect of loss has been studied by Reid and Walls[13] and Bondurant *et al.*[12] and has been shown to significantly reduce squeezing. It has recently been suggested by Kumar and Shapiro[21] that one can attain good squeezing even in a lossy medium by using a forward or copropagating beam geometry. Their work, however, assumes an ideal polarization and does not include additional atomic fluctuation terms present at higher pump intensities. The recent treatments of the quantum statistics of four-wave mixing by Perina *et al.*[22] and Jansky and Yuskin[23] also neglect these terms.

In this paper we present a unified treatment of degenerate four-wave-mixing systems in which the effects of loss and spontaneous emission on the squeezing are accounted for. The medium is modeled as an ensemble of two-level atoms characterized by an electric dipole moment, transverse and longitudinal relaxation times, and an atomic resonance frequency. Thus we develop a microscopic model enabling calculation of squeezing in terms of atomic variables. This is in contrast to the macroscopic model of the nonlinear polarization used in the work of Bondurant *et al.*[12] and Kumar and Shapiro[21] and which we show to be valid only in certain limits. In Sec. II we follow the approach of Reid and Walls[13] to derive an expression for the phase-matched polarization including quantum fluctuation terms. The pump modes are treated classically and are assumed nondepleting.

In Sec. III we restrict attention to the original four-wave-mixing scheme proposed by Yuen and Shapiro[11] in which the two pump waves and two probe waves are counterpropagating. The squeezing in the appropriately combined output waves is calculated. Only in a certain limit (henceforth referred to as the ideal-noise limit) corresponding to large detuning and low pump intensity is the quantum noise that derivable from the idealized macroscopic model. If we assume ideal noise but include the loss, our results coincide with those of Bondurant *et al.*[12] and indicate good squeezing only if the absolute value of loss is suitably small. However, the additional atomic fluctuation terms destroy squeezing prematurely at higher intensities and further requirements are placed on atomic parameters. We provide numerical values for parameters to attain good squeezing.

In Sec. IV the forward or copropagating type of four-wave mixing suggested by Kumar and Shapiro[21] is analyzed. Once again, in the limit of ideal noise, our results agree with Kumar and Shapiro and one can improve squeezing in a lossy medium by increasing the pump intensity. However, because the effect of additional nonideal atomic fluctuations is to destroy squeezing at higher pump intensities, the validity of the Kumar and Shapiro claim is sensitive to the nature of these terms. We present a solution including atomic fluctuations and thus analyze the true upper limit of squeezing possible for a given detuning.

Alternative schemes to produce squeezed states have involved interaction of the medium with the steady-state field in an optical cavity. Studies by Milburn and Walls[5] and Lugiato and Strini[6] of the degenerate parametric amplifier in a cavity showed a maximum squeezing of only

50% at threshold in the internal field. However, recent analyses by Yurke,[7] Collett and Gardiner,[8] and Gardiner and Savage[9] of the field outside a single-ended cavity reveal ideal squeezing to be possible in the resonant mode at threshold. See also Reynaud and Heidmann[24] for a discussion of four-wave mixing in a cavity. In Sec. V we examine the four-wave configuration in a single-ended ring cavity and compute the squeezing in an appropriately combined mode outside the cavity. As in the degenerate parametric amplifier, if one assumes a macroscopic polarization model with zero loss, ideal squeezing is possible at threshold in the appropriately filtered resonant mode outside the cavity. The effect of spontaneous emission on the squeezing in this mode is also analyzed.

II. ATOMIC MODEL OF THE MEDIUM

Consider four modes of the same frequency and polarization interacting with a nonlinear medium. The quantized modes of frequency ω and propagation vectors $\mathbf{k}_j$ are described by boson operators a_j $(j=1,\ldots,4)$. The medium is modeled as N two-level atoms uniformly distributed, with density n, throughout a total volume V. The atoms are assumed stationary, Doppler broadening being ignored in this treatment. Following the approach of Reid and Walls,[13] we consider a microscopic volume element δV containing N_0 atoms at position $\mathbf{r}$ in the medium. The Hamiltonian for this element in the electric dipole and rotating-wave approximations is, in the Schrödinger picture,

$$H_r=\sum_{j=1}^{4}\hbar\omega a_j^\dagger a_j+\sum_{i=1}^{N_0}\hbar\frac{\omega_0}{2}\sigma_{zi}+i\hbar g\sum_{i=1}^{N_0}(\sigma_i a_{\mathbf{r}}^\dagger-\sigma_i^\dagger a_{\mathbf{r}})+\sum_{i=1}^{N_0}(\sigma_i\Gamma^\dagger+\sigma_i^\dagger\Gamma)\,,$$
$$a_{\mathbf{r}}=\sum_{j=1}^{4}a_j e^{i\mathbf{k}_j\cdot\mathbf{r}}\,. \tag{1}$$

$\sigma^\dagger$, σ, and σ_Z are Pauli spin operators and ω_0 is the atomic resonance frequency. g describes the electric dipole coupling between the atoms and field. Γ and $\Gamma^\dagger$ are atomic reservoir operators describing radiative decay or spontaneous emission. The total Hamiltonian for the system is found by integrating over $\mathbf{r}$

$$H=\int\frac{d^3r}{\delta v}H_{\mathbf{r}}\,. \tag{2}$$

The averaging over a volume δV at position $\mathbf{r}$ allows us to consider a high-density medium for which the wavelength of the field is much larger than the mean distance between the atoms.

We wish to derive first an expression for the field-induced polarization at position $\mathbf{r}$ of the atomic medium. Since we are interested in the quantum statistics of the final field, a quantum-mechanical derivation of the atomic polarization is required. We follow the approach developed by Haken[25] for laser theory and later adapted for a quantum theory of optical bistability by Drummond and Walls.[26]

To summarize briefly, a master equation for the density operator ρ is derived using standard techniques.[27] A normally ordered characteristic function χ is defined as

$$\chi=\mathrm{Tr}(O\rho)\,, \tag{3}$$

where

$$O=e^{i\epsilon^\dagger S^\dagger}e^{i\eta S_z}e^{i\epsilon S}e^{i\beta^\dagger a_{\mathbf{r}}^\dagger}e^{i\beta a_{\mathbf{r}}}\,,$$
$$S=\sum_{i=1}^{N_0}\sigma_i,\quad S_z=\sum_{i=1}^{N_0}\sigma_{zi}\,.$$

A distribution function f is the Fourier transform of the characteristic function

$$f=\int\chi e^{iv'^\dagger\epsilon^\dagger}e^{iD\eta}e^{iv'\epsilon}e^{i\bar{\alpha}'^\dagger\beta^\dagger}e^{i\bar{\alpha}'\beta}d^2\epsilon\, d^2\epsilon^\dagger\, d^2\eta\, d^2\beta\, d^2\beta^\dagger\,. \tag{4}$$

Because the standard representation used in laser theory does not in general provide a Fokker-Planck equation with a positive-definite diffusion matrix, we use a generalized representation in which the pairs $(v',v'^\dagger)$ and $(\bar{\alpha}',\bar{\alpha}'^\dagger)$ are independent complex variables (not complex-conjugate) and D can be complex.[26] Thus one establishes a correspondence between c numbers and operators as follows:

$$v'\leftrightarrow S\,,\quad v'^\dagger\leftrightarrow S^\dagger\,,\quad D\leftrightarrow S_z\,,\quad \bar{\alpha}'^\dagger\leftrightarrow a_r^\dagger\,,\quad \bar{\alpha}'\leftrightarrow a_r\,. \tag{5}$$

Operator rules enable us to derive an equation of motion for f. The initial equation involves infinite-order derivatives, With N_0 reasonably large one can use a scaling argument to ignore all but first- and second-order derivatives. After substituting $\bar{\alpha}=\bar{\alpha}'e^{i\omega t}$ and $v=v'e^{-i\omega t}$, the final equation is

$$\frac{df}{dt}=\left[-\frac{\partial}{\partial\bar{\alpha}}gv-\frac{\partial}{\partial v}(-\gamma v+gD\bar{\alpha})-\frac{\partial}{\partial D}[-\gamma_{\parallel}(D+N_0)-2g(v^\dagger\bar{\alpha}+\bar{\alpha}^\dagger v)]+\frac{1}{2}\frac{\partial^2}{\partial v^2}(2gv\bar{\alpha})+\frac{1}{2}\frac{\partial^2}{\partial D^2}[2\gamma_{\parallel}(D+N_0)-4g(v^\dagger\bar{\alpha}+v\bar{\alpha}^\dagger)]+\mathrm{c.c.}\right]f\,, \tag{6}$$

where $\delta=(\omega_0-\omega)/\gamma_\perp$ is the normalized detuning from line center and $\gamma=\gamma_\perp(1+i\delta)$. $\gamma_\perp$ and $\gamma_{||}$ are the atomic transverse and longitudinal relaxation parameters, respectively. c.c. refers to the previous terms repeated but interchanging $\bar{\alpha}^\dagger\leftrightarrow\bar{\alpha}$, $v^\dagger\leftrightarrow v$.

The equation derived is for a nonactive medium. To describe a lasing medium, one allows for a nonzero transition rate from the lower to the upper atomic levels due to incoherent pumping. The result is an additional fluctuation term which destroys squeezing.[18] For this reason we do not include the possibility of a pumped medium in this work.

The next step is to write equivalent c-number Langevin equations and to obtain an expression for the polarization v. If one may assume that the atomic variables relax at a much greater rate than the field variables ($\gamma_\perp,\gamma_{||}\gg\gamma_F$ where γ_F is the field damping rate), it is possible to adiabatically eliminate the atomic variables by setting $\dot{v}=\dot{D}=0$ and solving for the steady-state polarization v. Such a procedure is valid for the case we are considering provided we can safely ignore all field loss except that due to absorption via the interacting two-level atoms. The elimination procedure will also be valid where the medium is enclosed in a high Q optical cavity. In this case one cannot ignore an additional cavity damping rate which describes loss through the cavity mirrors, but one can still adiabatically eliminate provided the above condition ($\gamma_F\ll\gamma_\perp,\gamma_{||}$) holds.

Considerable simplification is obtained if one is able to write the field in terms of large and small components as follows:

$$\begin{aligned}\bar{\alpha}&=\epsilon+\alpha\ ,\\ \epsilon&=\epsilon_1e^{i\mathbf{k}_1\cdot\mathbf{r}}+\epsilon_2e^{i\mathbf{k}_2\cdot\mathbf{r}}\ ,\\ \alpha&=\alpha_3e^{i\mathbf{k}_3\cdot\mathbf{r}}+\alpha_4e^{i\mathbf{k}_4\cdot\mathbf{r}}\ .\end{aligned}\tag{7}$$

The pump fields α_1 and α_2 of Eq. (1) are of large intensity and are assumed to be nondepleting, their intensities taking on constant values $|\epsilon_1|^2$ and $|\epsilon_2|^2$, respectively. The fields α_3 and α_4 are of smaller intensity. We are thus able to expand our steady-state polarization to first order in α. The final expression for the atomic polarization at $\mathbf{r}$ is derivable from results obtained in Drummond and Walls[26] and is

$$v_{\mathbf{r}}=-g\frac{N_0}{\gamma\pi}(\epsilon+\alpha)\left[1-\frac{(\epsilon^\dagger\alpha+\epsilon\alpha^\dagger)}{I_s\Pi}\right]+\Gamma_{\mathbf{r}}\ ,\tag{8}$$

where $n_0=\gamma_\perp\gamma_{||}/4g^2$ is the line-center saturation intensity, $I_s=n_0(1+\delta^2)$, and $\Pi=1+|\epsilon|^2/I_s$.

The nonzero noise correlations are (taking the dominant terms depending on ϵ only)

$$\begin{aligned}&\langle\Gamma_{\mathbf{r}}(t)\Gamma_{\mathbf{r}}(t')\rangle=D_1\delta(t'-t)\ ,\\ &D_1=\frac{-N_0\epsilon^2}{\gamma_\perp I_s(1+\delta^2)^2\Pi^3}\left[(1-i\delta)^3+\frac{|\epsilon|^4}{2n_0^2}\right]\ ,\\ &\langle\Gamma_{\mathbf{r}}^\dagger(t)\Gamma_{\mathbf{r}}(t')\rangle=D_2\delta(t'-t)\ ,\\ &D_2=\frac{N_0|\epsilon|^2}{\gamma_\perp I_s(1+\delta^2)^2\Pi^3}\left[\frac{2}{n_0}|\epsilon|^2+\frac{|\epsilon|^4}{2n_0^2}\right]\ ,\end{aligned}\tag{9}$$

where we have assumed for simplicity pure radiative damping ($\gamma_{||}=2\gamma_\perp$). The effect of collisional damping on the squeezing is presently being investigated.

The expression (8) is to first order in α but includes all orders of $|\epsilon|^2$ and thus accounts for saturation and related effects of the medium. I_s is the detuning-dependent pump intensity at which saturation effects in the drift become important. Also contributing important effects at larger pump intensities are the additional noise terms, of order higher than ϵ^2, in D_1 and D_2.

The next step is to describe the behavior of the individual field amplitudes α_j. The field interacts with the entire medium (N atoms) and thus we use the Hamiltonian (2) to derive the Heisenberg equations of motion for a_j:

$$\dot{a}_j=\frac{1}{i\hbar}\int\frac{d^3r}{\delta v}[a_j,H_{\mathbf{r}}]=\int\frac{d^3r}{\delta v}gSe^{-i\mathbf{k}_j\cdot\mathbf{r}}\ .\tag{10}$$

Thus the equation for the amplitude is of the form

$$\dot{\alpha}_j(t)=\int\frac{d^3r}{\delta v}e^{-i\mathbf{k}_j\cdot\mathbf{r}}gv_{\mathbf{r}}(t)\ .\tag{11}$$

It is readily seen that terms in the integrand which are perfectly phase matched (independent of $\mathbf{r}$) will contribute more significantly to the integral than terms which have a phase mismatch $\Delta\mathbf{k}$ (i.e., a $e^{i\Delta\mathbf{k}\cdot\mathbf{r}}$ dependence), at least for the situation where the interaction lengths L are much greater than $2\pi/\Delta\mathbf{k}$. Thus one arrives at the phase-matching condition

$$\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4\ .\tag{12}$$

To evaluate the pump saturation terms relating to Π one writes (assuming $\epsilon_1=\epsilon_2$ is real for convenience and $I=|\epsilon_1|^2$)

$$|\epsilon|^2=2I[1+\cos(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}]\tag{13}$$

and notes the integrals may be approximated, provided the interaction length is much greater than $2\pi/(\mathbf{k}_2-\mathbf{k}_1)$, by computing the cycle average of Π^{-1}.

The integrated noise force is of the form

$$F_j(t)=\int d^3r\frac{e^{-i\mathbf{k}_j\cdot\mathbf{r}}}{\delta v}\Gamma_{\mathbf{r}}(t)\ .\tag{14}$$

Assuming the noise terms at different $\mathbf{r}$ are uncorrelated we obtain

$$\begin{aligned}&\frac{\langle\Gamma_{\mathbf{r}}(t)\Gamma_{\mathbf{r}'}(t')\rangle}{\delta v\,\delta v}=\mathscr{D}_1\delta(t'-t)\delta^3(\mathbf{r}-\mathbf{r}')\ ,\\ &\frac{\langle\Gamma_{\mathbf{r}}^\dagger(t)\Gamma_{\mathbf{r}'}(t')\rangle}{\delta v\,\delta v}=\mathscr{D}_2\delta(t'-t)\delta^3(\mathbf{r}-\mathbf{r}')\ ,\\ &\mathscr{D}_i=D_i/\delta v\ .\end{aligned}\tag{15}$$

To determine noise correlations consider

$$\langle F_j(t)F_{j'}(t')\rangle = \int d^3r\, d^3r' e^{-i\mathbf{k}_j\cdot\mathbf{r}} e^{-i\mathbf{k}_{j'}\cdot\mathbf{r}'} \frac{\langle \Gamma_{\mathbf{r}}(t)\Gamma_{\mathbf{r}'}(t')\rangle}{\delta v\, \delta v}$$
$$= \int d^3r\, d^3r' e^{-i\mathbf{k}_j\cdot\mathbf{r}} e^{-i\mathbf{k}_{j'}\cdot\mathbf{r}'} \mathscr{D}_1 \delta^3(\mathbf{r}-\mathbf{r}')\delta(t'-t)$$
$$= \int d^3r\, e^{-i(\mathbf{k}_j+\mathbf{k}_{j'})\cdot\mathbf{r}} \mathscr{D}_1 \delta(t'-t) \; . \tag{16}$$

Thus only terms for which $\mathbf{k}_j = -\mathbf{k}_{j'}$ will contribute significantly after integration. Similarly, one may calculate $\langle F_j(t)F^{\dagger}_{j'}(t')\rangle$.

The final phase-matched coupled equations for α_3 and α_4 are

$$\begin{aligned} \dot{\alpha}_3 &= \gamma'\alpha_3 + \chi\alpha_4^{\dagger} + \Gamma_3(t) \; , \\ \dot{\alpha}_4 &= \gamma'\alpha_4 + \chi\alpha_3^{\dagger} + \Gamma_4(t) \; , \end{aligned} \tag{17}$$

where

$$\gamma' = \frac{-2C'\left[1+\dfrac{2I}{I_s}\right]}{(1+i\delta)\left[1+\dfrac{4I}{I_s}\right]^{3/2}} = -\gamma_R + i\gamma_I \; ,$$

$$\chi = \frac{2C'\dfrac{2I}{I_s}}{(1+i\delta)\left[1+\dfrac{4I}{I_s}\right]^{3/2}} = \chi_R + i\chi_I \; ,$$

$$2C' = \frac{g^2 N}{\gamma_\perp}$$

(corresponding equations hold for $\dot{\alpha}_3^{\dagger}$ and $\dot{\alpha}_4^{\dagger}$). The nonzero noise correlations are

$$\begin{aligned} \langle \Gamma_3(t)\Gamma_4(t')\rangle &= R^*\delta(t-t') \; , \\ \langle \Gamma_3^{\dagger}(t)\Gamma_4^{\dagger}(t')\rangle &= R\,\delta(t-t') \; , \\ \langle \Gamma_3(t)\Gamma_3^{\dagger}(t')\rangle &= \langle \Gamma_4(t)\Gamma_4^{\dagger}(t')\rangle \\ &= \Lambda\delta(t-t') \; , \end{aligned} \tag{18}$$

where

$$R = R_R + iR_I \; ,$$

$$R_R = \frac{-2C'}{(1+\delta^2)^2\left[1+\dfrac{4I}{I_s}\right]^{5/2}} \left\{ (1-3\delta^2)\frac{2I}{I_s}\left[1+\frac{I}{I_s}\right] + (1+\delta^2)^2\left[\frac{1}{2}\left[1+\frac{4I}{I_s}\right]^{5/2} - \frac{1}{2} - \frac{5I}{I_s} - \frac{15I^2}{I_s^2}\right]\right\} ,$$

$$R_I = \frac{-2C'(3\delta-\delta^3)}{(1+\delta^2)^2\left[1+\dfrac{4I}{I_s}\right]^{5/2}} \frac{2I}{I_s}\left[1+\frac{I}{I_s}\right] ,$$

$$\Lambda = \frac{2C'}{(1+\delta^2)^2\left[1+\dfrac{4I}{I_s}\right]^{5/2}} \left\{ 12(1+\delta^2)\frac{I^2}{I_s^2} + (1+\delta^2)^2\left[\frac{1}{2}\left[1+\frac{4I}{I_s}\right]^{5/2} - \frac{1}{2} - \frac{5I}{I_s} - \frac{15I^2}{I_s^2}\right]\right\} .$$

III. DEGENERATE BACKWARD FOUR-WAVE MIXING

The first type of four-wave-mixing system to be proposed as a squeezed-state generator was the backward degenerate four-wave mixer. This particular system has been studied classically in some depth for its phase conjugation properties.[28] It was first suggested as a possible ideal squeezer by Yuen and Shapiro.[11] To describe the backward four-wave mixer, we follow the geometry of Yariv and Pepper[28] (Fig. 1). One has two counterpropagating pump waves of frequency ω and propagation vectors $\mathbf{k}_1$ and $-\mathbf{k}_1$, described by amplitudes ϵ_1 and ϵ_2, respectively. The weak fields α_3 and α_4 are also counterpropagating, with propagation vectors $\mathbf{k}_3$ and $-\mathbf{k}_3$ (different from $\mathbf{k}_1$), respectively, and also of frequency ω. The fields interact with a nonlinear medium and, using the two-level atomic model presented in Sec. II, the expression (8) derived for the polarization is directly applied.

We wish to describe the slowly varying spatial behavior of the field amplitude α_j, as the waves propagate through the medium, keeping the full polarization expression (8). Classically, this has been done by Abrams and Lind[29] via Maxwell's equations by writing

$$\nabla^2 E - \mu_0\epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2} , \qquad (19)$$

where P is the polarization. One substitutes the strong-weak–field approximation (7) into (19) and assumes the strong pump fields ϵ_1 and ϵ_2 to be nondepleting. The z direction is taken as that of $\mathbf{k}_4$, and we introduce the slowly varying envelope approximation $|d^2\alpha_i/dz^2| \ll |k_i(d\alpha_i/dz)|$. Further simplification is possible by noting that upon integration over the medium only the phase-matched terms for each amplitude as derived in Sec. II will give significant contributions. The final coupled equations for the weak-field amplitudes take the form[29]

$$\begin{aligned} \frac{d}{dz}\alpha_3(z) &= \left[\alpha - \frac{i\gamma_I}{c}\right]\alpha_3(z) - \bar{\chi}\alpha_4^*(z) , \\ \frac{d}{dz}\alpha_4^*(z) &= \left[-\alpha - \frac{i\gamma_I}{c}\right]\alpha_4^*(z) + \bar{\chi}^*\alpha_3(z) , \end{aligned} \qquad (20)$$

where

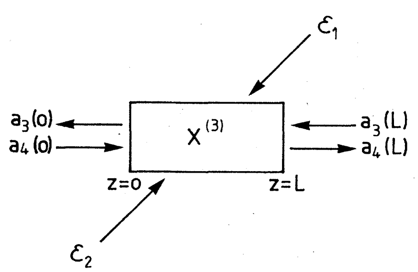

FIG. 1. Backward degenerate four-wave-mixing scheme.

$$\alpha = \frac{\gamma_R}{c} = \frac{\alpha_0\left[1+\dfrac{2I}{I_s}\right]}{(1+\delta^2)\left[1+\dfrac{4I}{I_s}\right]^{3/2}} ,$$

$$\bar{\chi} = \frac{\chi}{c} = \frac{\alpha_0\left[\dfrac{2I}{I_s}\right](1-i\delta)}{(1+\delta^2)\left[1+\dfrac{4I}{I_s}\right]^{3/2}} ,$$

and $\alpha_0 = 2C'/c$ is the line-center small-signal-field attenuation coefficient. The equations include loss, saturation, and dispersive effects due to the medium. In the idealized limit of zero loss and on making the substitutions $\alpha_3' = \alpha_3\exp(-i\gamma_I z/c)$ and $\alpha_4' = \alpha_4\exp(-i\gamma_I z/c)$ one could write

$$\begin{aligned} \frac{d\alpha_3'(z)}{dz} &= i\kappa^*\alpha_4'^*(z) , \\ \frac{d\alpha_4'^*(z)}{dz} &= i\kappa\alpha_3'(z) , \end{aligned} \qquad (21)$$

where

$$\kappa = -i\chi^*/c ,$$

which are the coupled equations first analyzed by Yariv and Pepper.[28]

Because of the large number of modes involved, the quantized treatment of the four-wave propagation problem of Abrams and Lind becomes much more complex. However, in order to indicate how atomic parameters will affect the field statistics, we approximate the quantum-mechanical problem by deriving temporal differential equations for each c-number amplitude α_j (as described in Sec. II) and replacing $t=-z/c$ and $t=z/c$ for α_3 and α_4, respectively. This amounts to solving the classical equations (20) of Abrams and Lind, but with additional fluctuation terms present to describe the quantization of the medium. The original quantized treatment by Yuen and Shapiro[11] followed a $z \leftrightarrow ct$ analysis as applied to Eq. (21), the amplitudes α_3 and α_4 becoming boson operators a_3 and a_4, respectively:

$$\begin{aligned} \frac{da_3}{dz}(z) &= i\kappa^* a_4^\dagger(z) , \\ \frac{da_4^\dagger}{dz}(z) &= i\kappa a_3(z) . \end{aligned} \qquad (22)$$

These equations may be solved for the output waves $a_3(0)$ and $a_4(L)$ in terms of the inputs $a_3(L)$ and $a_4(0)$. If one considers the squeezing in the following combined mode (as may be produced by the mirror arrangement described in Ref. 11),

$$e = [a_3(0) - ia_4(L)]/\sqrt{2} = X_1 + iX_2 , \qquad (23)$$

one finds, with the assumption of coherent inputs,

$$\Delta X_1^2 = [\sec(|\kappa| L) - \tan(|\kappa| L)]^2/4 . \qquad (24)$$

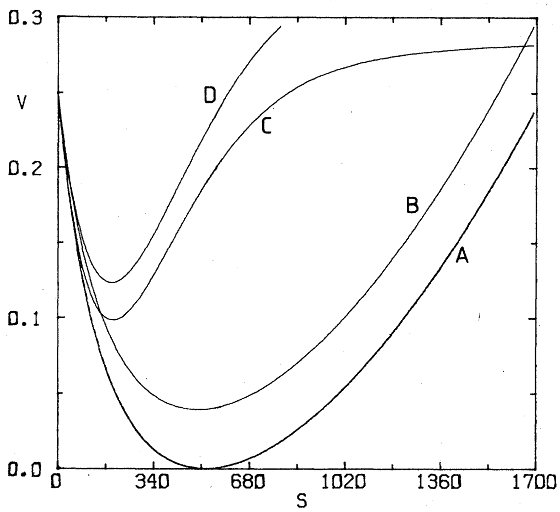

FIG. 2. Backward degenerate four-wave mixing: the low-loss limit. Variance $V=\Delta X_1^2$ vs normalized pump intensity $S=|\epsilon|^2/n_0$: $\delta=100$, $\alpha_0 L=2000$. *A*, ideal limit of zero loss and ideal noise; *B*, loss included but ideal noise assumed; *C*, zero loss but nonideal noise included; *D*, both loss and nonideal noise included.

Clearly, ideal squeezing ($\Delta X_1^2 \rightarrow 0$) is predicted as $|\kappa| L \rightarrow \pi/2$ (Ref. 11) (Figs. 2 and 3).

It is our aim to quantize the treatment of Abrams and Lind, keeping the full atomic saturation and loss terms and thus accounting for the effect of nonideal atomic noise and loss on the squeezing (24) predicted. With the

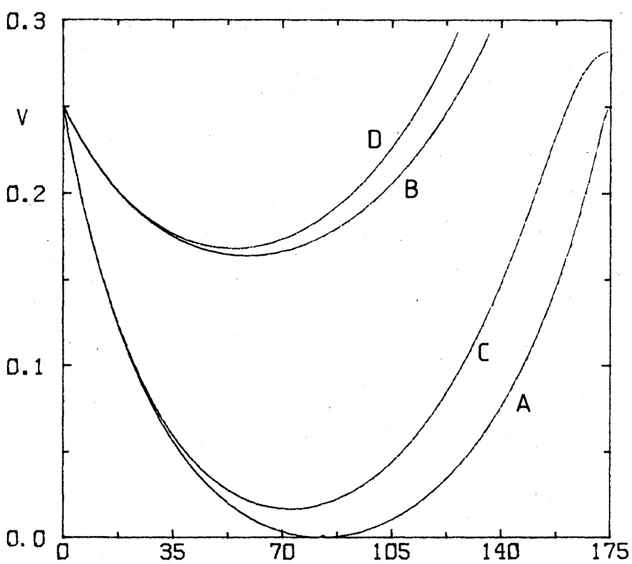

FIG. 3. Backward degenerate four-wave mixing: the high-loss limit. $V=\Delta X_1^2$ vs S: $\delta=100$, $\alpha_0 L=10^4$. *A*, ideal limit of zero loss and ideal noise; *B*, loss included but ideal noise assumed; *C*, zero loss but nonideal noise included; *D*, both loss and nonideal noise included.

assumptions outlined above, the equations we wish to solve are

$$\frac{d}{dz}\alpha_3(z)=\left[\alpha-\frac{i\gamma_I}{c}\right]\alpha_3(z)-\bar{\chi}\alpha_4^\dagger(z)+G_3(z)\ ,$$
$$\frac{d}{dz}\alpha_4^\dagger(z)=\left[-\alpha-\frac{i\gamma_I}{c}\right]\alpha_4^\dagger(z)+\bar{\chi}^*\alpha_3(z)+G_4^\dagger(z)\ , \tag{25}$$

where

$$\langle G_3(z)G_4(z')\rangle=-\bar{R}^*\delta(z-z')\ ,$$
$$\langle G_3^\dagger(z)G_4^\dagger(z')\rangle=-\bar{R}\delta(z-z')\ ,$$
$$\langle G_3(z)G_3^\dagger(z')\rangle=\langle G_4(z)G_4^\dagger(z')\rangle$$
$$=\bar{\Lambda}\delta(z-z')\ ,$$

and $\bar{R}=\bar{R}_R+i\bar{R}_I=R/c$ and $\bar{\Lambda}=\Lambda/c$. On taking the expectation value of each term in (25), one reproduces the classical equations (20) derived by Abrams and Lind [$\langle G_3(z)\rangle=\langle G_4(z)\rangle=0$]. The noise terms $G_i(z)$, however, are essential to describe the quantum effects (squeezing) present in the system. The term in γ_I relates to the intensity-dependent refractive index of the medium, as may be seen by considering the substitutions

$$\alpha_3'=\alpha_3 e^{\frac{i\gamma_I}{c}z}\ ,$$
$$\alpha_4'=\alpha_4 e^{\frac{-i\gamma_I}{c}z}\ . \tag{26}$$

Provided one is in the limit of nondepleting classical pump fields, it does not change the maximum squeezing attainable. It does, however (in the backward four-wave mixing case), alter the relative phase difference required between the outgoing fields to attain this maximum.[20] Of more interest is the effect of the loss parameter α which has been shown to reduce the magnitude of squeezing attainable.[12,13]

The coupled linear stochastic differential equations (25), subject to boundary conditions at $z=0$ and L, may be solved using standard methods.[30] The final solutions for the outgoing amplitudes $\alpha_3(0)$ and $\alpha_4(L)$ in terms of the ingoing amplitudes $\alpha_3(L)$ and $\alpha_4(0)$ are

$$\alpha_3(0)=Te^{i\gamma_I L/c}\alpha_3(L)+r\bar{\chi}\alpha_4^\dagger(0)+F_3\ ,$$
$$\alpha_4(L)=Te^{i\gamma_I L/c}\alpha_4(0)+r\bar{\chi}\alpha_3^\dagger(L)+F_4\ , \tag{27}$$

where

$$F_3=\int_0^L[G_3(z)a(z)+G_4^\dagger(z)b(z)]dz\ ,$$
$$F_4=\int_0^L[G_3^\dagger(z)c(z)+G_4(z)d(z)]dz\ ,$$

and

$$a(z)=-Tf_2(L-z)\ ,$$
$$b(z)=Tf_1(L-z)\ ,$$
$$c(z)=re^{i\gamma_I L/c}\bar{\chi}f(L-z)+f_1(L-z)\ ,$$
$$d(z)=re^{i\gamma_I L/c}\bar{\chi}f_1^*(L-z)+f_2(z-L)\ ,$$

where

$$T=\frac{\lambda}{\lambda\cos(\lambda L)+\alpha\sin(\lambda L)}\ ,$$
$$r=\frac{\sin(\lambda L)}{\lambda\cos(\lambda L)+\alpha\sin(\lambda L)}\ ,$$
$$f_1(x)=\frac{\bar{\chi}}{\lambda}\sin(\lambda x)\ ,$$
$$f_2(x)=\cos(\lambda x)+\frac{\alpha}{\lambda}\sin(\lambda x)\ ,$$
$$\lambda=(\,|\bar{\chi}|-\alpha^2)^{1/2}\ .$$

Our c-number equations and solutions are identical in form to the operator equations examined by Bondurant *et al.*[12]

The covariance matrix

$$\underset{\sim}{C}=(\langle\alpha_i\alpha_j\rangle-\langle\alpha_i\rangle\langle\alpha_j\rangle)\ , \tag{28}$$

where $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\equiv(\alpha_3(0),\alpha_3^\dagger(0),\alpha_4(L),\alpha_4^\dagger(L))$ is derivable directly from the correlations of $G_3(z)$ and $G_4(z)$. Assuming the initial states to be coherent states, with negligible intensity, and assuming the initial state to be independent of the noise sources $G_1(z)$, we find

$$C_{12}=\langle F_3F_3^\dagger\rangle=\langle F_4F_4^\dagger\rangle\ ,$$
$$C_{13}=\langle F_3F_4\rangle\ . \tag{29}$$

Thus, to calculate C_{12}, for example, consider

$$C_{12}=\langle F_4F_4^\dagger\rangle=\int_0^L dz\int_0^L dz'\langle[c(z)G_3^\dagger(z)+d(z)G_4(z)]\times[c^*(z')G_3(z')+d^*(z')G_4^\dagger(z')]\rangle\ . \tag{30}$$

Recalling the correlations (25), we find

$$C_{12}=\bar{\Lambda}\int_0^L dz[\,|c(z)|^2+|d(z)|^2]-\bar{R}\int_0^L dz\,c(z)d^*(z)-\bar{R}^*\int_0^L dz\,c^*(z)d(z)\ . \tag{31}$$

One calculates C_{13} similarly. The final solution for the covariance matrix is

$$\underset{\sim}{C}=\begin{bmatrix}0 & C_{12} & C_{13} & 0\\ C_{12} & 0 & 0 & C_{13}^*\\ C_{13} & 0 & 0 & C_{12}\\ 0 & C_{13}^* & C_{12} & 0\end{bmatrix}, \tag{32}$$

where

$$C_{12}=\left[\bar{\Lambda}\frac{T^2}{\lambda^2}(\,|\bar{\chi}|^2+\alpha^2)+2T^2\frac{\alpha}{\lambda^2}(\bar{\chi}_R\bar{R}_R-\bar{\chi}_I\bar{R}_I)\right]I_1+\bar{\Lambda}T^2I_2+\left[\bar{\Lambda}\frac{T^2}{\lambda}\alpha+\frac{T^2}{\lambda}(\bar{\chi}_R\bar{R}_R-\bar{\chi}_I\bar{R}_I)\right]I_3\ ,$$
$$C_{13}=e^{i\gamma_I L/c}C'_{13}\ ,$$
$$C'_{13}=\left[\frac{\bar{\Lambda}}{\lambda^2}T\alpha\bar{\chi}(\alpha r-1)+\frac{\bar{\Lambda}}{\lambda^2}T\bar{\chi}(r\,|\bar{\chi}|^2-\alpha)+\frac{T}{\lambda^2}\alpha(\alpha-r\,|\bar{\chi}|^2)(-\bar{R}_R+i\bar{R}_I)+\frac{I}{\lambda^2}\bar{\chi}^2(1-\alpha r)(-\bar{R}_R-i\bar{R}_I)\right]I_1$$
$$+[\bar{\Lambda}\mathrm{Tr}\bar{\chi}+T(\bar{R}_R-i\bar{R}_I)]I_2+\left[\frac{\bar{\Lambda}T}{\lambda}\bar{\chi}\alpha r+\frac{T}{2\lambda}\,|\bar{\chi}|^2r(\bar{R}_R-i\bar{R}_I)+\frac{T}{2\lambda}\bar{\chi}^2r(\bar{R}_R+i\bar{R}_I)\right]I_3\ ,$$

and

$$I_1=\int_0^L\sin^2[\lambda(L-z)]dz=\frac{L}{2}-\frac{\sin(2\lambda L)}{4\lambda}\ ,$$
$$I_2=\int_0^L\cos^2[\lambda(L-z)]dz=\frac{L}{2}+\frac{\sin(2\lambda L)}{4\lambda}\ ,$$
$$I_3=\int_0^L\sin[2\lambda(L-z)]dz=\frac{\sin^2(\lambda L)}{\lambda}\ .$$

The squeezing in the combined mode

$$e=\frac{a_3(0)+e^{i\theta}a_4(L)}{\sqrt{2}}=X_1+iX_2 \tag{33}$$

is given by

$$\Delta X_{\frac{1}{2}}^2-\tfrac{1}{4}=\tfrac{1}{2}[C_{12}\pm(\cos\psi\,\mathrm{Re}C'_{13}-\sin\psi\,\mathrm{Im}C'_{13})]\ , \tag{34}$$

where $\psi=\theta+\gamma_I L/C$. The value C_{12} is positive and real, and optimal squeezing is attained for $\cos\psi=-\mathrm{Re}C'_{13}/|C'_{13}|$ and $\sin\psi=\mathrm{Im}C'_{13}/|C'_{13}|$, in which case

$$\Delta X_{\frac{1}{2}}^2-\tfrac{1}{4}=\tfrac{1}{2}(C_{12}\mp|C_{13}|)\ . \tag{35}$$

At this stage it is convenient to present the idealized limits of zero loss

$$\alpha=\gamma_R=0 \tag{36a}$$

and ideal noise

$$\bar{\Lambda}=\bar{\chi}_R=\bar{R}_R=0\ ,$$
$$\bar{\chi}_I=-|\bar{\chi}|=-\bar{R}_I\ . \tag{36b}$$

The latter ideal-noise limit (36b) holds only for large detuning

$$\delta\gg1 \tag{37a}$$

and low saturation

$$S/\delta^2 \ll 1 \text{ and } 10S^2/\delta^3 \ll 1 , \tag{37b}$$

where $S=I/n_0$. The second low-saturation condition $10S^2/\delta^3 \ll 1$ ensures that the second bracketed term in R_R and Λ

$$(1+\delta^2)^2 \left[\frac{1}{2} \left[1+\frac{4I}{I_s} \right]^{5/2} - \frac{1}{2} - \frac{5I}{I_s} - \frac{15I^2}{I_s^2} \right] \tag{38}$$

is small compared to R_I. It is an additional condition resulting from the nature of the quantum fluctuation terms (9) and could not have been predicted from the drift part of the polarization, which requires $S/\delta^2 \ll 1$ only. The ideal-noise limit (36b) is equivalent to deriving the polarization part of the equations from the following Hamiltonian often used to describe four-wave mixing systems:

$$H_I = \hbar c\kappa(a_3 a_4 + a_3^\dagger a_4^\dagger) . \tag{39}$$

With the simplifications (36), the equations (25) reduce to

$$\frac{d}{dz}\alpha_3(z) = \frac{-i\gamma_I}{c}\alpha_3(z) + i\kappa\alpha_4^\dagger(z) + G_3(z) ,$$
$$\frac{d}{dz}\alpha_4^\dagger(z) = \frac{-i\gamma_I}{c}\alpha_4^\dagger(z) + i\kappa\alpha_3(z) + G_4^\dagger(z) , \tag{40}$$

and the nonzero correlations are

$$\langle G_3(z)G_4(z')\rangle = i\kappa\delta(z-z') ,$$
$$\langle G_3^\dagger(z)G_4^\dagger(z')\rangle = -i\kappa\delta(z-z') ,$$

where we have taken $\kappa = |\bar{X}|$. These c-number equations are equivalent to the operator equations (22) (apart from the term in γ_I), and the solution may be shown to reduce to (24) (thus giving perfect squeezing) in the ideal limit (36) being considered.

To simplify the analysis of the full solution (35), we examine in the first instance the effect of loss in γ_R alone. That is, we relax condition (36a) but still assume the ideal-noise result (36b), regardless of whether the requirement (37) is satisfied or not. This approach is equivalent to deriving the polarization part of the equations from the phenomenological Hamiltonian (in the interaction model)

$$H = H_I + H_R ,$$
$$H_R = a_3^\dagger\Gamma_3 + a_3\Gamma_3^\dagger + a_4^\dagger\Gamma + a_4\Gamma^\dagger , \tag{41}$$

where Γ are reservoir operators and H_R models the loss mechanism. The Heisenberg equations of motion may be derived for a_3 and a_4, and the spatial differential equations are obtained by the change of variable $z\leftrightarrow -ct$ ($z\leftrightarrow ct$) for a_3 (a_4), respectively. This is the procedure used by Bondurant *et al.*[12] to describe the effect of loss in backward four-wave mixing. As shown in their work, even moderate values of loss will markedly decrease the squeezing attainable. With saturation negligible, values of attenuation coefficient $\alpha_0 L$ such that $\alpha_0 L/\delta^2 > 1$ imply values of loss $\alpha L > 1$, which reduces the best squeezing attainable by at least 60% (Fig. 2). It is possible to decrease the loss αL by, for a fixed δ, decreasing α_0. For $\alpha L/\delta^2 \sim \frac{1}{5}$, one obtains only a 20% reduction in squeezing (Fig. 3).

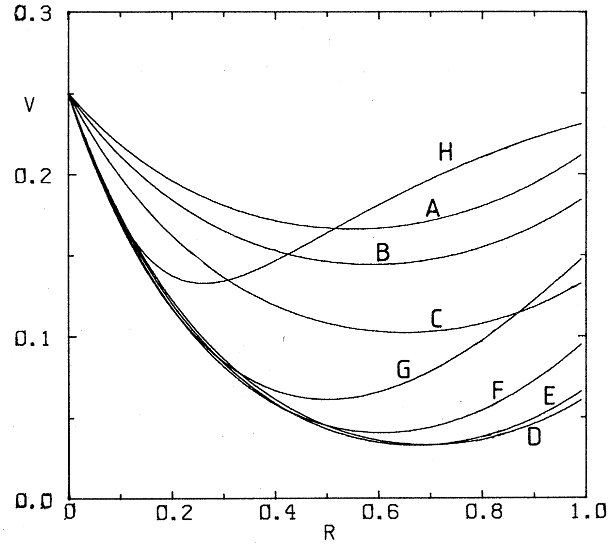

FIG. 4. Backward degenerate four-wave mixing: full loss and noise solutions. $V=\Delta X_1^2$ vs $R=(\alpha_0 L/\delta^2)(S/\delta)$: $\delta=10^4$. A, $\alpha_0 L=10^8$; B, $\alpha_0 L=8\times10^7$; C, $\alpha_0 L=5\times10^7$; D, $\alpha_0 L=10^7$; E, $\alpha_0 L=8\times10^6$; F, $\alpha_0 L=5\times10^6$; G, $\alpha_0 L=3\times10^6$; H, $\alpha_0 L=10^6$.

Unfortunately, while the loss becomes insignificant for relatively low values of $\alpha_0 L$, the normalized pump intensities S required for the maximum squeezing are increased, and it is no longer justifiable to assume the ideal-noise conditions (36) and (37). Upon increasing S, it is the condition $10S^2/\delta^3 \ll 1$ which breaks down first, enabling Λ to be nonzero, and hence determining the maximum squeezing achievable. This transition in behavior going from high to low $\alpha_0 L$ values is shown in Figs. 2 and 3.

The solution (35) including both loss and complete atomic fluctuations has been analyzed, as a function of S, for various values of detuning and attenuation coefficient $\alpha_0 L$. As described above, the maximum squeezing attained is always a compromise between the desqueezing effects of loss (at low S) and nonideal noise (at higher S). The best possible squeezing attainable, at favorable $\alpha_0 L$, and the value of intensity S for which this value is attained is critically dependent on the detuning δ. We define a detuning-dependent pump intensity parameter as $R=(\alpha_0 L/\delta^2)(S/\delta)$ ($\sim|\bar{X}|/2$). For $\delta=10$ the minimum variance ΔX_1^2 is ~ 0.18 (for $\alpha_0/\delta^2 \sim 0.8$) at a value of $R\sim 0.4$. For $\delta=100$ a minimum variance of 0.12 ($\alpha_0 L/\delta^2 \sim 0.3$) is possible at $R\sim 0.5$. Increasing the normalized detuning to $\delta=10^4$, ΔX_1^2 drops to 0.4 ($\alpha_0 L/\delta^2 \sim 0.1$) at $R\simeq 0.7$ (Fig. 4). Thus our model developed to show the effect of a nonideal atomic medium in a backward degenerate four-wave mixer shows good squeezing to be possible only in very large δ, $\alpha_0 L$, and S limits.

IV. DEGENERATE FORWARD FOUR-WAVE MIXING

It was shown in Sec. III that the effect of loss is to possibly minimize the squeezing attainable in backward four-wave mixing. Recently Kumar and Shapiro[21] have suggested that the effect of loss will be less substantial in forward degenerate four-wave mixing. Their model, however, like that of Bondurant *et al.*[12] for the backward case, does not include the full effects of atomic quantum noise. It is our objective to calculate the squeezing in forward four-wave mixing, including full atomic loss and noise effects.

The geometry of a forward degenerate four-wave mixer is as described by Kumar and Shapiro[21] (Fig. 5). One has two pump waves, amplitudes ϵ_1 and ϵ_2, with propagation vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively, and frequency ω. The waves copropagate at small angles $\pm\phi/2$ to the z axis. The weak fields α_3 and α_4 are described by propagation vectors $\mathbf{k}_3$ and $\mathbf{k}_4$ and are also of frequency ω. The geometry of the $\mathbf{k}_3$ and $\mathbf{k}_4$ vectors may be found by rotating the plane of $\mathbf{k}_1$ and $\mathbf{k}_2$ about the x axis. The choice of propagation vectors is such to satisfy the phase-matching condition (12).

As with the backward case, we aim to describe the slowly varying spatial amplitudes α_3 and α_4. With similar assumptions, the classical equations analogous to the Abrams and Lind equation (20) may be derived. Once again, to model the system quantum mechanically, we make the change of variable $z\leftrightarrow ct$ (for both α_3 and α_4) in the temporal differential equations. Our final equations are

$$\frac{d}{dz}\alpha_3(z)=-\alpha\alpha_3(z)+\bar{\chi}\alpha_4^\dagger(z)+G_1(z)\ ,$$
$$\frac{d}{dt}\alpha_4^\dagger(z)=-\alpha\alpha_4^\dagger(z)+\bar{\chi}^*\alpha_3(z)+G_2^\dagger(z)\ , \tag{42}$$

where

$$\langle G_1(z)G_2(z')\rangle=\bar{R}^*\delta(z-z')\ ,$$
$$\langle G_1^\dagger(z)G_2^\dagger(z')\rangle=\bar{R}\delta(z-z')\ ,$$
$$\langle G_1(z)G_1^\dagger(z')\rangle=\langle G_2(z)G_2^\dagger(z')\rangle=\bar{\Lambda}\delta(t-t')\ ,$$

and the definitions of α, $\bar{\chi}$, $\bar{R}$, and $\bar{\Lambda}$ have been modified slightly from those given in Sec. III to become

$$\alpha=\gamma_R/cn\cos(\phi/2),\quad \bar{\chi}=\chi/cn\cos(\phi/2)\ ,$$
$$\bar{\Lambda}=\Lambda/cn\cos(\phi/2)\ ,$$
$$\bar{R}=\bar{R}_R+i\bar{R}_I=R/cn\cos(\phi/2)\ .$$

The imaginary term relating to γ_I has contributed to an intensity-dependent refractive index n and does not [as may be seen by considering the rotations $\alpha_i'=\alpha_i\exp(i\gamma_I/c)z$] affect the squeezing attainable.

A solution to the linear equation (42) may be obtained using standard methods.[30] The input probe waves are classical so that the initial system is assumed to have the statistics of a coherent state. The drift $\underline{A}$ and diffusion $\underline{D}$ matrices are

$$\underline{A}=\begin{bmatrix}\alpha & 0 & 0 & -\bar{\chi}\\ 0 & \alpha & -\bar{\chi}^* & 0\\ 0 & -\bar{\chi} & \alpha & 0\\ -\bar{\chi}^* & 0 & 0 & \alpha\end{bmatrix},$$
$$\underline{D}=\begin{bmatrix}0 & \bar{\Lambda} & \bar{R}^* & 0\\ \bar{\Lambda} & 0 & 0 & \bar{R}\\ \bar{R}^* & 0 & 0 & \bar{\Lambda}\\ 0 & \bar{R} & \bar{\Lambda} & 0\end{bmatrix}. \tag{43}$$

One diagonalizes the drift

$$\underline{S}^{-1}\underline{A}\,\underline{S}=\mathrm{diag}(\lambda_1,\lambda_1,\lambda_2,\lambda_2)\ , \tag{44}$$

where

$$\lambda_{1,2}=\alpha\pm|\bar{\chi}|$$

and

$$\underline{S}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & a & 0 & -a\\ 0 & 1 & 0 & 1\\ a & 0 & -a & 0\end{bmatrix},\quad a=\frac{-|\bar{\chi}|}{\bar{\chi}}\ .$$

The covariance matrix in the new variables is

$$\underline{\pi}=\underline{S}^{-1}\underline{D}(\underline{S}^{-1})^T=\begin{bmatrix}0 & \pi_{12} & 0 & \pi_{23}\\ \pi_{12} & 0 & \pi_{23} & 0\\ 0 & \pi_{23} & 0 & \pi_{34}\\ \pi_{23} & 0 & \pi_{34} & 0\end{bmatrix}, \tag{45}$$

where

$$\pi_{12}=\frac{1}{2a^2}(2a\Lambda+R+a^2R^*)\ ,$$
$$\pi_{23}=\frac{1}{2a^2}(-R+a^2R^*)\ ,$$
$$\pi_{34}=\frac{1}{2a^2}(-2a\Lambda+R+a^2R^*)\ .$$

On transforming back to the original variables, the solution for the covariance matrix is as follows:

$$\underline{C}=(\langle\alpha_i\alpha_j\rangle)$$
$$=\begin{bmatrix}0 & C_{12} & C_{13} & 0\\ C_{12} & 0 & 0 & C_{13}^*\\ C_{13} & 0 & 0 & C_{12}\\ 0 & C_{13}^* & C_{12} & 0\end{bmatrix}, \tag{46}$$

where

$$C_{12}=\frac{a}{2}(\Sigma_{12}-\Sigma_{34})\ ,$$
$$C_{13}=\tfrac{1}{2}(\Sigma_{12}+2\Sigma_{32}+\Sigma_{34})=\mathrm{Re}C_{13}+i\,\mathrm{Im}C_{13}\ ,$$

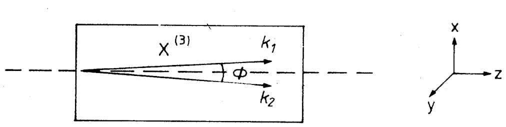

FIG. 5. Forward degenerate four-wave-mixing scheme.

and

$$\Sigma_{ij}=\frac{-\pi_{ij}}{\lambda_j+\lambda_j}[1-\exp(\lambda_i+\lambda_j)L] ,$$

where L is the length of the medium. If we consider the rotated mode

$$e=\frac{a_3+e^{i\theta}a_4}{\sqrt{2}}=X_1+iX_2 \tag{47}$$

the final result for squeezing in e is

$$\Delta X_{\frac{1}{2}}^2-\tfrac{1}{4}=\frac{1}{4}\left[\frac{(1-e^{-2\lambda_1 L})}{\lambda_1}[a\pi_{12}\pm(\cos\theta\,\mathrm{Re}\pi_{12}-\sin\theta\,\mathrm{Im}\pi_{12})]+\frac{(1-e^{-2\lambda_2 L})}{\lambda_2}[a\pi_{34}\pm(\cos\theta\,\mathrm{Re}\pi_{34}-\sin\theta\,\mathrm{Im}\pi_{34})]\pm 2\frac{(1-e^{-2\alpha L})}{\alpha}(\cos\theta\,\mathrm{Re}\pi_{23}-\sin\theta\,\mathrm{Im}\pi_{23})\right] . \tag{48}$$

Unlike the backward mixing which shows sinusoidal behavior, the forward solutions are of an exponential nature. For simplicity, we consider the large-L limit, for which $\exp(-2\lambda_1 L)\ll 1$. In this limit the optimal choice of ψ is to avoid divergence (and hence no squeezing) in the solution due to the last two terms dependent on λ_2 and α. One obtains as a final expression for squeezing [with $\cos\phi=-\overline{X}_R/|\overline{X}|$, $\sin\theta=\overline{X}_I/|\overline{X}|$ (Ref. 13)]

$$\Delta X_1^2-\tfrac{1}{4}=\frac{1}{4}\frac{A}{\alpha+|\overline{X}|} , \tag{49}$$

where

$$A=\overline{\Lambda}-\frac{\overline{R}_R\overline{X}_R-\overline{R}_I\overline{X}_I}{|\overline{X}|} .$$

The term A reflects the quantum atomic fluctuations, while α represents the atomic loss.

If one considers the ideal-noise limits described in Sec. III by Eq. (36), one can write

$$A=-|\overline{X}| . \tag{50}$$

This ideal-noise result is subject to the conditions (37) and corresponds to deriving the polarization from the Hamiltonian H_I of (39). If one additionally chooses to neglect atomic loss ($\alpha=0$), the solution indicates perfect squeezing ($\Delta X_1^2\to 0$) to be attainable. We now wish to examine the full atomic loss and noise effects on the squeezing.

To examine the effect of atomic loss α alone it is convenient to assume, in the first instance, the ideal-noise conditions while including α. As pointed out in Sec. II, this is equivalent to deriving the polarization from the phenomenological Hamiltonian (41). This was the procedure used by Kumar and Shapiro[21] to analyze the effect of loss on the squeezing in forward degenerate four-wave mixing, and our solutions agree with theirs in this limit. One finds for the ideal-noise case

$$\Delta X_1^2=\frac{1}{4}-\frac{1}{4\left[1+\dfrac{\alpha}{|\overline{X}|}\right]} \tag{51}$$

and we see that there is a reduction in the amount of squeezing present determined by the ratio

$$\frac{\alpha}{|\overline{X}|}=\frac{1+\dfrac{2I}{I_s}}{\dfrac{2I}{I_s}\sqrt{1+\delta^2}} , \tag{52}$$

which is of the order $\delta/2S$ if the ideal-noise requirements (37) are satisfied. Threshold occurs at $|\overline{X}|=\alpha$. Below threshold, $|\overline{X}|<\alpha$, the loss dominates and squeezing is minimal. At threshold, the squeezing is 50% ($\Delta X_1^2=\frac{1}{8}$). Further squeezing is obtained by increasing the pump intensity above the threshold value. In fact, in the ideal-noise limit, one could obtain arbitrarily large squeezing by appropriately increasing S. This has been pointed out by Kumar and Shapiro[21] and is why squeezing in the forward propagating model would seem to be less sensitive to loss than the backward system. The effect of loss α alone on the squeezing is shown in curve B, Fig. 6, which plots expression (51).

Unfortunately, as discussed in Sec. III for the backward example, as one increases the pump intensity S the conditions (in particular $10S^2/\delta^3\ll 1$) for ideal-noise break-

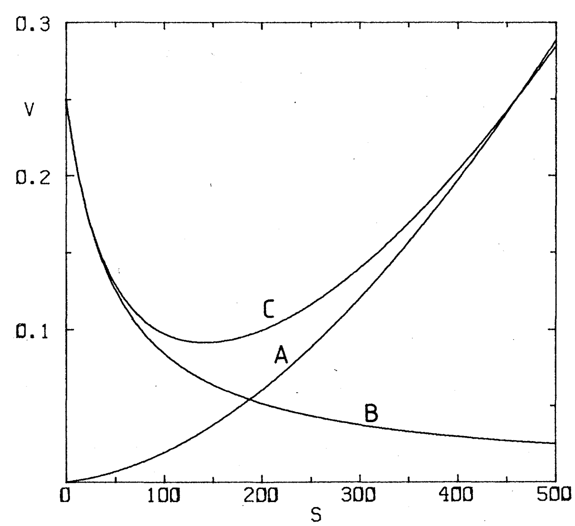

FIG. 6. Forward propagating degenerate four-wave mixing. $V=\Delta X_1^2$ vs $S=|\epsilon|^2/n_0$: $\delta=100$. A, loss ignored; B, loss included but ideal noise assumed; C, both loss and nonideal noise included.

down. For such values of S, the $\overline{\Lambda}$ term becomes significant and contributes a positive term to A, thus destroying squeezing. This effect is shown in curve A, Fig. 4, where the full expression for A is retained but loss ignored.

The total solution (curve C of Fig. 6) shows the combined result of the two desqueezing effects discussed above. Thus for very low values of normalized pump intensity satisfying $S \ll \delta$, the amount of squeezing is reduced because of the dominance of loss. For high values of S satisfying $10S^2 \gg \delta^3$, squeezing is destroyed because of atomic fluctuations. Thus for a given large detuning $\delta \gg 1$, there exists a middle range of saturation S, satisfying

$$S \gg \delta, \quad S \ll \delta^2, \quad 10S^2 \ll \delta^3 , \tag{53}$$

giving optimal squeezing. Figure 7 shows the full solutions for various values of detuning δ. In fact, the best squeezing attainable in the forward four-wave mixing is not significantly improved over the backward example. The values of normalized pump intensity S required for this minimum are of the same order in both the forward and backward cases, although the former is less sensitive to the attenuation coefficient $\alpha_0 L$.

However, we require $|\overline{\chi}|L$ large to attain good squeezing and there is thus an implicit lower limit on the attenuation coefficient given by $\alpha_0 L/\delta^2 \gg \delta/2S$ ($R \simeq 0.5$), as for the backward case [the value S already being determined by the loss and ideal-noise conditions (53)]. Because the lower limit implies very high atomic densities anyway and is likely to be the important factor limiting squeezing in an experiment, we conclude that the copropagating example does not offer any significant advantages over the counterpropagating configuration. However, because the squeezing in the forward case does not depend directly on the absolute value of loss ($\alpha_0 L/\delta^2$), there is no upper limit (as in the backward case) placed on the parameter $\alpha_0 L$ for good squeezing.

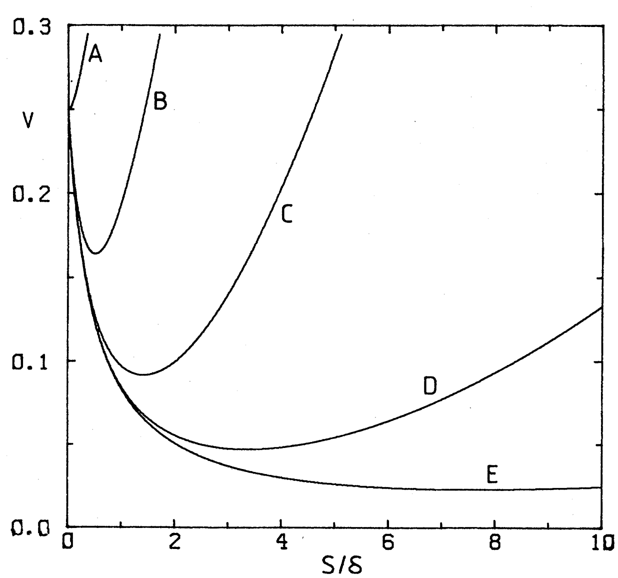

FIG. 7. Forward degenerate four-wave mixing: full loss and noise solutions. $V = \Delta X_1^2$ vs S/δ. A, $\delta=1$; B, $\delta=10$; C, $\delta=100$; D, $\delta=10^3$; E, $\delta=10^4$.

Finally, we point out that because in the forward-mixing case the best squeezing occurs above threshold, the results may be sensitive to pump depletion. Recently Scharf and Walls[31] have considered the effects of pump quantization and depletion in parametric amplification. Studies to investigate the effect of pump depletion in forward four-wave mixing are in progress.

V. FOUR-WAVE MIXING IN A SINGLE-ENDED CAVITY

The analyses in Secs. III and IV of the propagating configurations have indicated good squeezing to require a very high $g^2 N/\gamma_\perp$ value. One possible way to decrease the value of $g^2 N/\gamma_\perp$ required, for the same amount of squeezing, is to place the medium inside a high Q optical cavity. In this section we examine the squeezing in the output field for a four-wave-mixing system inside a single-ended cavity.

Consider the experimental scheme as illustrated in Fig. 8. We will use the methods of Collett and Gardiner[8,32] to analyze the output fields. One has two input fields $a_{3\text{in}}$ and $a_{4\text{in}}$, separated from the outputs by a circulator outside the cavity. The internal cavity modes are designated a_3 and a_4, respectively. In addition, there are the two pump waves, classically treated and assumed nondepleting, of amplitudes ϵ_1 and ϵ_2. These fields interact in the ring cavity with a medium in the manner described in Sec. II. The output modes $a_{3\text{OUT}}$ and $a_{4\text{OUT}}$ are combined, using a mirror scheme, to form the new mode

$$e = (a_{3\text{OUT}} + e^{i\phi} a_{4\text{OUT}})/\sqrt{2} = X_1 + iX_2 , \tag{54}$$

and we wish to examine the squeezing possible in this new mode.

If one describes the medium via a third-order nonlinear susceptibility, the idealized Hamiltonian for the internal modes is written

$$\begin{aligned} H &= H_I + H_R , \\ H_I &= h(\chi^* a_3 a_4 + \chi a_3^\dagger a_4^\dagger) , \\ H_c &= a_3 \Gamma^\dagger + a_3^\dagger \Gamma + a_4 \Gamma^\dagger + a_4^\dagger \Gamma . \end{aligned} \tag{55}$$

The amplitudes ϵ_1 and ϵ_2 have been absorbed into χ. This Hamiltonian corresponds to the zero atomic loss and ideal-noise limits of Eqs. (36). The reservoir Hamiltonian

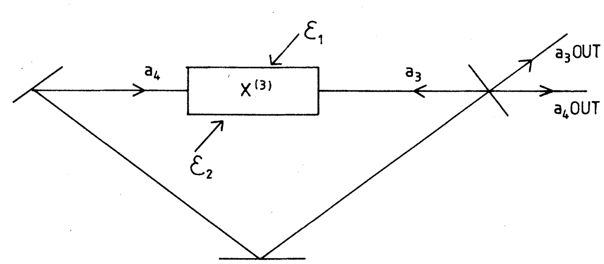

FIG. 8. Degenerate four-wave mixing in a cavity with a single input/output port.

H_c describes the coupling of the internal modes to the input (reservoir) modes outside the cavity, represented by the operators Γ. One may derive the master equation for the density operator ρ using standard techniques.[27] Upon expanding $\hat{\rho}$ in terms of a generalized P,[33] one can write the Fokker-Planck equation for the P function P. The equivalent Langevin equations are

$$\dot{\alpha}_3=-\gamma_c\alpha_3-i\,|\chi|\,\alpha_4^\dagger+\Gamma_3(t)\ ,$$
$$\dot{\alpha}_4=-\gamma_c\alpha_4-i\,|\chi|\,\alpha_3^\dagger+\Gamma_4(t)\ , \tag{56}$$

and the nonzero correlations are

$$\langle\Gamma_3(t)\Gamma_4(t')\rangle=-i\,|\chi|\,\delta(t-t')\ ,$$
$$\langle\Gamma_3^\dagger(t)\Gamma_4^\dagger(t')\rangle=i\,|\chi|\,\delta(t-t')\ ,$$

where we have ignored thermal noise. γ_c is the cavity damping due to loss through the cavity mirrors.

If one made the assumption that the output modes are the same as the internal modes, the final squeezing in mode e ($\cos\phi=\chi_R/|\chi|$ and $\sin\phi=-\chi_I/|\chi|$) in the steady state is identical to that derived for the degenerate parametric amplifier with classical pump field inside the cavity:

$$\Delta X_2^2-\tfrac{1}{4}=\frac{1}{4\left[1+\dfrac{\gamma_c}{|\chi|}\right]}\ . \tag{57}$$

The situation is almost analogous to the traveling-wave case with high loss (51). Below threshold ($|\chi|<\gamma_c$) loss γ_c dominates, a minimum variance $\Delta X_2^2=\frac{1}{8}$ being achieved at threshold ($|\chi|=\gamma_c$). Unfortunately, one cannot further improve squeezing by increasing $|\chi|$ above threshold, as would seem to be the case from solution (57), since the assumption of a classical nondepleting pump is no longer valid in this cavity case above threshold.[5]

Of more interest outside the cavity where one has traveling modes is the normally ordered spectrum of squeezing. To calculate this we use the methods developed by Gardiner and Collett[8] and Collett and Walls:[19]

$$:S_{\substack{1\\2}\mathrm{OUT}}(\omega):=\langle:X_{\substack{1\\2}\mathrm{OUT}}(\omega),X_{\substack{1\\2}\mathrm{OUT}}:\rangle$$
$$=2\gamma_c\tfrac{1}{4}\{S_{12}(\omega)+S_{21}(\omega)$$
$$\pm[e^{2i\phi}S_{22}(\omega)+e^{-2i\phi}S_{11}(\omega)]\}\ , \tag{58}$$

where $S_{ij}(\omega)$ are the matrix elements of the spectrum $\underline{S}$ of the stationary two-time correlation matrix $\underline{G}_s(\tau)$ in the P representation,[33]

$$\underline{S}(\omega)=\int_{-\infty}^{\infty}e^{-i\omega\tau}\underline{G}_s(\tau)d\tau\ . \tag{59}$$

The final result for the squeezing spectrum is

$$:S_{2\mathrm{OUT}}(\omega):=\frac{-2\gamma_c}{2}\frac{|\chi|}{(\gamma_c+|\chi|)^2+\omega^2}\ . \tag{60}$$

The solution is identical to that derived for the degenerate parametric amplifier[8] and predicts ideal squeezing [$:S_{2\mathrm{OUT}}(0):=-\frac{1}{4}$] to be possible, with appropriate filtering, in the resonant mode at threshold ($|\chi|=\gamma_c$).

We now extend the calculations to include the nonideal effects of the full atomic polarization (8). The Langevin equation (55) becomes

$$\dot{\alpha}_3=-(\gamma_R+\gamma_c)\alpha_3+\chi\alpha_4^\dagger+\Gamma_3\ ,$$
$$\dot{\alpha}_4=-(\gamma_R+\gamma_c)\alpha_4+\chi\alpha_3^\dagger+\Gamma_4\ , \tag{61}$$

where

$$\langle\Gamma_3(t)\Gamma_4(t')\rangle=R^*\delta(t-t')\ ,$$
$$\langle\Gamma_3^\dagger(t)\Gamma_4^\dagger(t')\rangle=R\delta(t-t')\ ,$$
$$\langle\Gamma_3(t)\Gamma_3^\dagger(t')\rangle=\langle\Gamma_4(t)\Gamma_4^\dagger(t')=\Lambda\delta(t-t')\ .$$

The final solution for the spectrum outside the cavity including full atomic effects (valid below threshold $|\chi'|<1+\gamma_R'$) gives for the resonant mode

$$:S_{2\mathrm{OUT}}(0):=\frac{A'}{(1+\gamma_R'+|\chi'|)^2}\ , \tag{62}$$

where

$$\gamma_R'=\frac{2C\left[1+\dfrac{2I}{I_s}\right]}{(1+\delta^2)\left[1+\dfrac{4I}{I_s}\right]^{3/2}}\ ,$$
$$\chi'=\frac{2C\dfrac{2I}{I_s}(1-i\delta)}{(1+\delta^2)\left[1+\dfrac{4I}{I_s}\right]^{3/2}}\ ,$$
$$A'=\frac{A}{\gamma_c}\ ,$$

and C, the cooperativity parameter, is $2C=g^2N/\gamma_1\gamma_c$.

To obtain best squeezing we will require the ideal noise and loss conditions of (53). In addition, there will be restrictions on $2C$. With the ideal-noise conditions (37), $A'=-|\chi'|$, and the best squeezing is attained for $|\chi'|$ as large as possible, that is, at threshold $|\chi'|=1+\gamma_R'$, where

$$:S_{2\mathrm{OUT}}(0):=-\frac{1}{4(1+\gamma_R')}\ . \tag{63}$$

Clearly the loss decreases the squeezing possible at threshold. To minimize this effect, we want

$$\gamma_R'\simeq\frac{2C}{\delta^2}\ll 1\ . \tag{64}$$

We also must ensure that the threshold value of $|\chi'|$ is possible without increasing S, the normalized pump intensity, to such an extent that the noise conditions are no longer satisfied. Since [with (64) holding] at threshold

$$|\chi'|=\frac{2C}{\delta^3}2S\simeq 1 \tag{65}$$

and we want $2S/\delta^2$ small, $2C/\delta$ will have to be large to

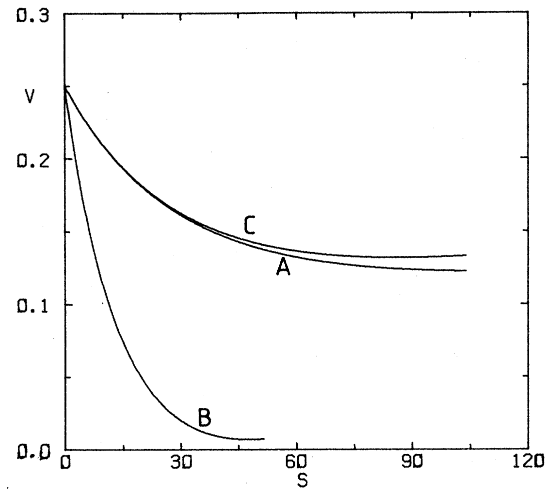

FIG. 9. Degenerate four-wave mixing in a single-ended cavity: squeezing in the external field. The high-loss case. $:S_{2\mathrm{OUT}}(0):+0.25$ vs S. $\delta=100$, $2C=10^4$. A, loss included but ideal noise assumed; B, loss ignored but ideal noise included; C, both loss and nonideal noise included.

allow this:

$$\frac{2C}{\delta} \gg 1 \; . \tag{66}$$

Thus there is a middle range of values $2C$ satisfying (64) and (66) for which an optimal squeezing is obtained. In fact, the conditions on $2C$ are directly analogous to those on $\alpha_0 L$ for the backward traveling case discussed in Sec. III.

The various limits are illustrated in Figs. 9 and 10. Figure 9 illustrates parameters for which loss is the sole

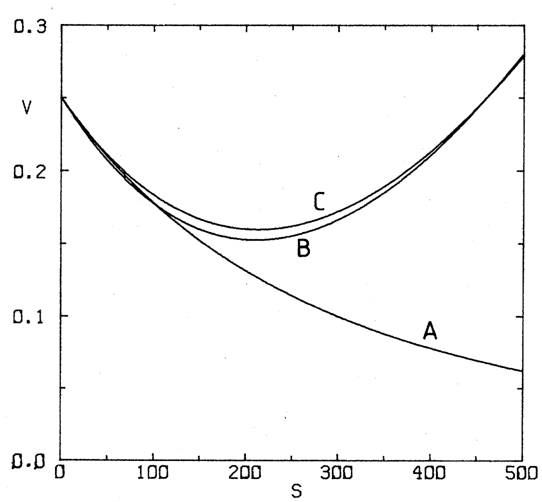

FIG. 10. Degenerate four-wave mixing in a single-ended cavity: squeezing in the external field. The low-loss case. $\delta=100$, $2C=500$. A, loss included but ideal noise assumed; B, loss ignored but nonideal noise included; C, both loss and nonideal noise included.

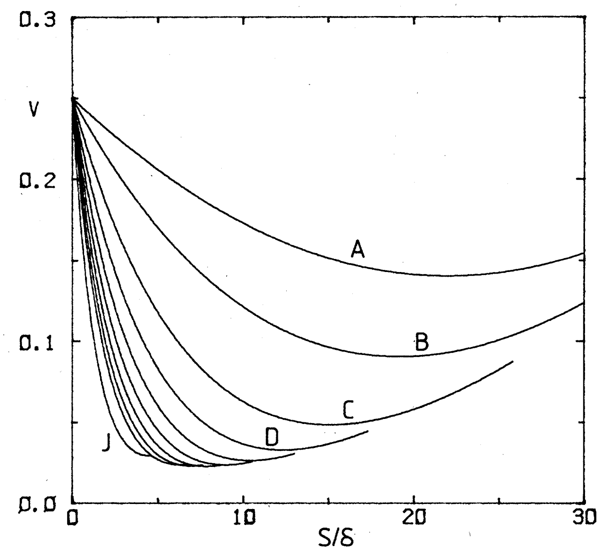

FIG. 11. Degenerate four-wave mixing in a single-ended cavity: squeezing in the external field. Full loss and noise solutions vs S/δ. $\delta=10^4$. A, $2C=50\times10^4$; B, $2C=100\times10^4$; C, $2C=200\times10^4$; D, $2C=300\times10^4$; E, $2C=400\times10^4$; F, $2C=500\times10^4$; G, $2C=600\times10^4$; H, $2C=700\times10^4$; I, $2C=800\times10^4$; J, $2C=1200\times10^4$.

reason for not attaining ideal squeezing. The value of $2C/\delta$ is large and the effect of nonideal noise is negligible, even at threshold. However, $2C/\delta^2 \sim 1$ and loss reduces at threshold the squeezing by 50%. At the other extreme, Fig. 10 illustrates parameters for which nonideal atomic fluctuations are the sole reason for destroying squeezing. The value $2C/\delta^2$ is small and atomic loss plays no significant role. However, $2C/\delta$ is not large and the effect of the nonideal noise is present even though threshold is not reached due to saturation.

The maximum squeezing achievable, for the optimal value of $2C$, is sensitive to the detuning δ. In fact, this maximum and the value of pump intensity S required coincide with those obtained in the propagation examples considered in Secs. III and IV (in particular Fig. 7). Figure 11 plots (62) for $\delta=10^4$ and shows good squeezing to be possible as one approaches threshold for a suitable choice of $2C\sim10^7$. For a sufficiently high Q cavity, this means a lower $g^2N/\gamma_\perp$ value is required for the medium than in the propagating examples.

VI. CONCLUSION

We have presented a treatment of the quantum statistics of degenerate four-wave mixing in which the quantization of the medium is taken into account. Our model thus enables study of the effect of both loss and spontaneous emission on the squeezing attainable.

The significance of this work is that we have shown the ideal nonlinear polarization to hold only in certain ideal-noise limits of normalized detuning δ and pump intensity S (given approximately by $\delta \gg 1$, $2S/\delta^2 \ll 1$, and $10S^2 \ll \delta^3$). Otherwise, spontaneous-emission terms will act to counter squeezing. Previous treatments by other authors have not included these effects.

Also countering squeezing is the effect of loss or absorption from the medium. One requires for good squeezing that the loss be small compared to the magnitude of the nonlinearity parameter. The result is to provide the additional requirement between S and δ ($S/\delta \gg 1$, in the ideal-noise limit). Thus while it is possible to ignore nonideal noise by (for sufficient large detuning) reducing S, this is in general where loss effects need consideration. The conditions for both small loss and ideal atomic fluctuations are compatible only in the very large normalized detuning limit ($\delta \sim 10^4$). The range of S required for maximum squeezing and the amount of squeezing attained has been illustrated for various detunings (for $S \sim 10^4$, $2S/\delta \sim 14$ is required).

We have analyzed three types of four-wave mixing separately, namely the counterpropagating, copropagating, and cavity examples. The maximum squeezing possible and corresponding value of S are approximately the same in each case, for a given detuning. The squeezing is additionally sensitive to the weak-field line-center attenuation coefficient parameter $\alpha_0 L$ and the cooperativity parameter $2C$ for the propagating and cavity examples, respectively. For good squeezing one must be able to attain a reasonable nonlinearity parameter [$|\bar{\chi}|L \sim (\alpha_0 L/\delta)(2S/\delta^2) \sim 1$ and $|\chi'| = (2C/\delta)(2S/\delta^2) \sim 1$ for the propagating and cavity cases, respectively], with the value of S already optimized according to the loss and ideal-noise conditions above. Thus we require a minimum $\alpha_0 L$ and $2C$ of order $\delta^3/2S$. The counterpropagating and cavity examples are additionally restricted by the requirement that the absolute value of loss be small ($\alpha_0 L/\delta^2, 2C/\delta^2 \ll 1$) and thus have an upper limit of $\alpha_0 L$ and $2C$ for good squeezing. In fact, for $\delta = 10^4$, best squeezing is attained at $\alpha_0 L, 2C \sim 10^7$. Because of the very high $\alpha_0 L$ and $2C$ values required (and consequently very high atomic densities) we do not see the latter upper limits required in the backward and cavity examples as particularly restricting to squeezing. Thus we do not see, because of the effect of atomic fluctuations, copropagating four-wave mixing as particularly advantageous over the backward configuration. Because a high value of $2C$ is more easily achievable (in a high Q cavity) than a high $\alpha_0 L$, the cavity example is more likely to be advantageous.

In conclusion, then, the effects of loss and atomic fluctuations in an atomic medium mean that a careful selection of atomic parameters is necessary if one is to achieve squeezing in degenerate four-wave mixing.

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