Quantum theory of the free-electron laser: Large gain, saturation, and photon statistics

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Starting from the general QED Hamiltonian for the free-electron laser (FEL), a quantum-mechanical, many-particle theory of the FEL in the laboratory frame is presented. When suitable variables are introduced, the Hamiltonian is seen to a good approximation to be formally identical to the nonrelativistic Hamiltonian in the resonant (Bambini-Renieri) frame used in other treatments. The derivation is given for a general multimode laser field, although only the simpler case of single-mode operation is discussed in detail. It is shown how the large-gain evolution equations for the field, in the small-signal regime, may be obtained from the quantum theory. Then, from fourth-order perturbation theory, the change in the first two moments of the photon-number distribution in a single pass through the FEL is computed. Large-gain and saturation terms are obtained, for arbitrary values of the quantum recoil (i.e., both the classical and quantum-mechanical regimes are included). The evolution of the photon statistics over many cavity round trips is discussed. In the small-signal regime, the variance of the photon-number distribution is shown to correspond, to a good approximation, to that of thermal, or "chaotic," radiation. At saturation, a significant reduction of the fluctuations is expected, but no conclusions can be drawn from the perturbation-theory approach. A numerical calculation of the buildup of the field from vacuum (through spontaneous emission) is presented. For simplicity, a uniform, circularly polarized, static wiggler is assumed in the text; however, the theory may be generalized (along lines shown) to deal with nonuniform, linearly polarized and/or traveling electromagnetic-wave wigglers.

I. INTRODUCTION

Most of the interesting properties of the free-electron laser (FEL) can be studied within the framework of an entirely classical theory (see, for instance, Refs. 1 and 2); in fact, the vast majority of calculations being carried out these days that deal with actual (or projected) devices are based on the classical evolution equations for the electromagnetic (EM) field and the electrons. Nevertheless, since the very beginning of FEL theories, attempts have been made at developing alternate, quantum-mechanical treatments. The reason for this is not just one of personal taste; on the contrary, it seemed only natural that, once the classical theory of the FEL was reasonably well established, one should start asking the same kind of questions that were asked about conventional lasers at that point; questions involving the photon statistics and the quantum coherence of the field radiated by an FEL (Ref. 3). Obviously, a fully quantum-mechanical theory (in which both the electrons and the field were quantized) was desirable for this.

In the second place, the theory of the start-up of the free-electron laser, and the evolution of coherence from the initial noise in the oscillator, appear also to call for a quantum-mechanical treatment that would include spontaneous emission in a natural way.

In addition to these, there may be a third reason to look at the FEL from a quantum-mechanical point of view. The classical theory is, after all, only an approximation, although an excellent one for most existing and projected devices; but, in an attempt to extend the range of FEL operation into the x-ray region, some devices have been considered which would be essentially quantum-mechanical in their behavior. These schemes involve the (stimulated) Compton backscattering of a high-intensity laser pulse by an electron beam of comparatively low energy. Gain calculations have been presented in Ref. 4 that show that the potential exists to obtain a gain per pass on the order of 100% at a wavelength λ_s of 100 Å, or the smaller value of 40% at $\lambda_s = 5.6$ Å. In these devices, the Compton recoil of the electron is of the same order of magnitude as the homogeneous (transit-time) broadening of the emission line (which means that once an electron has emitted a photon, the probability of its emitting another one is considerably decreased); this makes a quantum-mechanical treatment of the electrons necessary. (Such a "semiclassical" theory of the FEL has been succinctly presented in Ref. 5.)

Most quantum-mechanical treatments of the FEL to date have avoided the complications arising from the relativistic motion of the electrons by performing a Lorentz transformation to a moving frame [the Bambini-Renieri frame (Ref. 6)] where ordinary quantum mechanics (as opposed to QED) may be used (Refs. 7 and 8). This has not always been the case, however, and substantial work has also been done in the laboratory frame (Refs. 9—11). In the single-mode, cw-operation limit, the moving frame is just as good as the laboratory frame (see the comments on the Hamiltonian [Eq. (36)] in Sec. II below); the necessity of performing Lorentz transformations, however, tends to obscure the physics in the moving frame when one wants to deal with multimode fields, pulse-propagation problems, or nonuniform wigglers. Recent quantum-mechanical treatments have dealt with mode competition in nonuniform wigglers (Ref. 10), photon statistics in the small-signal regime (Refs. 8 and 12), and the photon statistics at saturation (Ref. 13), the latter only in the context of a single-particle theory. Genuinely quantum-mechanical effects such as antibunching and squeezing have been investigated in the papers listed in Ref. 8.

In this paper, the interaction Hamiltonian for the evolution of the electrons-photons system in the laboratory frame will be derived from quantum electrodynamics in a fairly general way. It is shown how different kinds of wigglers may be considered, and the theory is from the outset a many-particle theory. This last point is particularly important, since many-particle effects (e.g., terms proportional to N_e^2 in the small-signal gain formula, where N_e is the number of electrons) are essential to the high-gain regime (Ref. 14) in which some high-current, short-wavelength FEL's now envisioned are expected to operate. Amplified spontaneous emission [Ref. 8(c)] is also a many-particle effect, and many-particle contributions have been shown (Ref. 12) to completely modify the single-particle photon statistics at start-up. A multimode radiation field is explicitly considered in the derivation, although the rest of the paper deals only with the simpler case of single-mode operation. The more general multimode Hamiltonian [Eq. (34) below] is necessary to study the development of (temporal) coherence, as well as pulse-propagation effects; this will be done in a later publication.

In Sec. III, the FEL as amplifier is studied; that is, the evolution of the radiation field in a single pass through the device is considered. In the linear (small-signal) regime, an evolution equation for the electric field amplitude in the high-gain case is derived, which, for the large-quantum-recoil regime, was first presented in Ref. 4. Saturation (nonlinear) terms are also derived, which had only been obtained before in the small-gain, small-quantum-recoil limit (Ref. 15), and their dependence on the detuning and quantum recoil is presented. Formulas are given for the change (in a single pass) in the first two moments of the photon number distribution.

These formulas are then used, in Sec. IV, to study the oscillator problem, that is, the time evolution of the radiation field in the laser cavity over many round trips, starting from vacuum (through spontaneous emission). The photon statistics in the linear regime and at saturation are discussed, and a numerical calculation of the buildup of the field is given.

Finally, Appendix A contains some lengthy mathematical expressions left out of the main text for clarity.

II. INTERACTION HAMILTONIAN IN THE LABORATORY FRAME

The QED interaction Hamiltonian for charged particles and radiation is given by

$$H_I = e \int (j \cdot A_s) d^3x \ . \tag{1}$$

To facilitate the comparison with the usual (classical) FEL theory, MKS units (instead of the more usual relativistic units) are used throughout.

In Eq. (1), j is the four-vector operator current density, and A_s (also a four-vector) is the vector potential operator for the laser (or "scattered," whence the subindex s) radiation field. The wiggler field is considered as an external, prescribed, classical field (this is clearly justified when the wiggler is a static, permanent magnet; when it is an electromagnetic pump pulse, this amounts to neglecting its depletion); it does not appear explicitly in the Hamiltonian (1), but enters it through the electron states (which will be "dressed" states, see below).

The particle-current density j in Eq. (1) is given by

$$j = \overline{\Psi} \gamma \Psi \ , \tag{2}$$

where the γ^μ are the Dirac matrices, the bar means the Dirac conjugate, and the field operators Ψ are defined in terms of a set $\{\psi_p\}$ of solutions of the Dirac equation for the electron in the field of the wiggler alone:

$$\begin{aligned} \Psi &= \sum_p \psi_p(x) b_p \ , \\ \overline{\Psi} &= \sum_{p'} \overline{\psi}_{p'}(x) b_{p'}^\dagger \ . \end{aligned} \tag{3}$$

Here x is the four-vector position, the $b_p, b_p^\dagger$ are fermion annihilation and creation operators, and negative-energy solutions have been ignored. The functions ψ_p (to be calculated below) are chosen in such a way that outside the interaction region they reduce to ordinary plane waves of four-momentum p. The electrons are taken to be initially traveling along the z axis, i.e., transverse momentum is neglected.

The wiggler vector potential will be written as the four-vector

$$A_i{}^\mu(\phi) = (0, \mathbf{A}_i) \ . \tag{4}$$

The vector $\mathbf{A}_i$ is assumed to depend only on the phase $\phi = k \cdot x$, where the dot denotes the four-dimensional dot product of the position vector x with the "wave vector" k given by

$$k^\mu = (0,0,0,-k_q) \tag{5a}$$

(for a static wiggler) or

$$k = (\omega_i/c, 0, 0, -k_i) \tag{5b}$$

(for an electromagnetic-pulse wiggler). Hence $\phi = k_q z$ in the first case, and $\phi = \omega_i t + k_i z$ in the second. Inasmuch as the exact dependence of A_i on ϕ is not specified, nonuniform wigglers are included in this treatment.

The electron states in the wiggler field are calculated as follows. If a solution of the form

$$\psi_p = e^{-ipx/\hbar} F(\phi) \tag{6}$$

is assumed, it may be shown (Ref. 16) that the Dirac equation for ψ_p reduces to the following second-order equation for the function F:

$$\begin{aligned} &-\hbar^2 k^2 F'' + 2i\hbar p \cdot k F' \\ &\quad + \{ -e^2 \mathbf{A}_i^2 - i\hbar e (k_\mu \gamma^\mu)[(A_i')_\mu \gamma^\mu] \} F = 0 \ , \end{aligned} \tag{7}$$

(the primes on F and A_i denote derivatives with respect to ϕ). Note that $p \cdot A_i = 0$ has been used; if this does not hold, an additional term would appear in Eq. (7).

In the case of an EM-pulse wiggler, when k is given by Eq. (5a), one has $k^2=0$ and Eq. (7) is a first-order equation that may be solved exactly. One has then (Ref. 16)

$$F=\exp\left[-\frac{i}{\hbar}\int_0^{kx}\frac{e^2}{2k\cdot p}\mathbf{A}_i{}^2 d\phi + \frac{e(k_\mu\gamma^\mu)[(A_i)_\mu\gamma^\mu]}{2k\cdot p}\right]u_p \,, \quad (8)$$

where u_p is some constant bispinor determined by the normalization condition.

If the second exponential in Eq. (8) is expanded, making use again of the fact that $k^2=0$, one arrives at the Volkov solution for an electron in the field of an EM plane wave:

$$\psi_p=\left[1+\frac{e}{2p\cdot k}(k_\mu\gamma^\mu)[(A_i)_\mu\gamma^\mu]\right]u_p e^{-ipx/\hbar} \times\exp\left[-i\int_0^{kx}\frac{e^2\mathbf{A}_i{}^2}{2\hbar p\cdot k}d\phi\right] . \quad (9)$$

A solution like Eq. (9) may be expected to hold approximately true for the case of a static wiggler (where $k^2\neq 0$) provided that the term with the second derivative of F in Eq. (7) may be neglected versus the others. The condition for this to be so is to have

$$e^2\mathbf{A}_i{}^2 \ll \mathbf{p}^2 \quad (10)$$

which is well satisfied for relativistic electrons. It is in fact not hard to see that Eq. (9) is correct, as an approximation to the case of a static wiggler, up to terms of order $(e^2\mathbf{A}_i{}^2/\mathbf{p}^2)$. Since it is exact for an electromagnetic-pulse wiggler, its use for a static wiggler is equivalent to the Weizsäcker-Williams approximation frequently used in FEL theory. [The neglect of the second derivative in Eq. (7) is also analogous to the WKB approximation of Ref. 10.] Note also that, outside the wiggler, Eq. (9) reduces to a plane wave of momentum p, which determines the normalization of the bispinor u_p.

One should mention at this point that the Dirac equation can be solved exactly for an electron in the field of a circularly polarized static wiggler. Some effects appear then (which are discussed in Ref. 11) which we are neglecting here by using the approximate solution (9); their magnitude is, however, extremely small.

When Eq. (9) is used in Eqs. (2) and (3) one finds

$$\begin{aligned} j\cdot A_s &= \overline{\Psi}\gamma^\mu\Psi(A_s)_\mu \\ &= \sum_{p,p'}\bar{u}_{p'}\left[1+\frac{e}{2p'\cdot k}[(A_i)_\mu\gamma^\mu](k_\mu\gamma^\mu)\right][\gamma^\mu(A_s)_\mu]\left[1+\frac{e}{2p\cdot k}(k_\mu\gamma^\mu)[(A_i)_\mu\gamma^\mu]\right]u_p e^{-i(p-p')x/\hbar} \\ &\quad\times\exp\left[-i\int_0^{kx}\frac{e^2\mathbf{A}_i^2}{2\hbar}\left[\frac{1}{k\cdot p}-\frac{1}{k\cdot p'}\right]d\phi\right]b_{p'}^\dagger b_p \,. \end{aligned} \quad (11)$$

The next natural approximation is to neglect the difference between p and p' in the bispinors $u_p,\bar{u}_p$ and the denominators $1/p\cdot k$, $1/p'\cdot k$ (but *not* in the phase factor, however). This is reasonable inasmuch as they are merely multiplicative factors and it will be seen later [Eq. (22)] that the difference between p and p' is of the order of the momentum of a laser photon, which is a very small fraction of p.

Under this assumption, some algebra using the properties of the Dirac matrices shows that Eq. (11) breaks up into the scalar product of $\bar{u}_p\gamma^\mu u_p$ with terms proportional to $(A_i)_\mu$, $(A_s)_\mu$, and k_μ. Now, u_p is just the bispinor corresponding to a plane wave of momentum p, so that $\bar{u}_p\gamma^\mu u_p$ is just the free-particle current density,

$$\bar{u}_p\gamma^\mu u_p = c^2p^\mu/E_pV \quad (12)$$

(where V is a quantization volume and $E_p=\gamma mc^2$ is the electron energy corresponding to the momentum p). The products $p\cdot A_i$ and $p\cdot A_s$ are equal to zero if the electrons are assumed to be injected with zero transverse momentum, and the only surviving term is

$$j\cdot A_s=\frac{e\mathbf{A}_i\cdot\mathbf{A}_s}{\gamma mV}\sum_{p,p'}e^{-i(p-p')x/\hbar}\exp\left[-i\int_0^{kx}\frac{e^2\mathbf{A}_i^2}{2\hbar}\left[\frac{1}{k\cdot p}-\frac{1}{k\cdot p'}\right]d\phi\right]b_{p'}^\dagger b_p \,. \quad (13)$$

Use of Eq. (13) in Eq. (1) specifies the Hamiltonian for our problem.

The four-momentum p has been taken to be of the form

$$p=(E_p/c,0,0,p_z) \,. \quad (14)$$

Then the phase factor in Eq. (13) is approximately given by

$$\exp\left[-i\int_0^{kx}\frac{e^2\mathbf{A}_i^2}{2\hbar}\left[\frac{1}{k\cdot p}-\frac{1}{k\cdot p'}\right]d\phi\right] \simeq\exp\left[i\frac{e^2}{2\hbar k_q p_z^2}(p_z-p_z')\int_0^{k_q z}\mathbf{A}_i{}^2(\phi)d\phi\right] \quad (15a)$$

$$\simeq\exp\left[i\frac{e^2}{4\hbar k_i p_z^2}(p_z-p_z')\int_0^{\omega_i t+k_i z}\mathbf{A}_i{}^2(\phi)d\phi\right], \quad (15b)$$

where Eq. (15a) holds for a static wiggler and Eq. (15b) for an EM-pulse wiggler. For the simplest case of a circularly polarized, uniform static wiggler, the integral in Eq. (15a) is just a linear function of z, and the phase factor may be identified as a "mass-shift" term. For the case of a linearly polarized wiggler, it leads to the well-known difference of Bessel functions in the small-signal gain formula [and, in a multimode analysis, to harmonic generation (Ref. 17)]. For the case of an EM-pulse wiggler consisting of a focused Gaussian beam from a high-power laser, the mass-shift effect (15b) leads to corrections to the gain formula that have been discussed in Ref. 4. In general, Eq. (15) allows one to consider a variety of nonuniform wigglers. We shall not dwell on this here, and, for the remainder of the paper, shall assume a uniform, circularly polarized, static wiggler, of the form

$$\mathbf{A}_i = i\left[\frac{\hat{x}+i\hat{y}}{2}\right]A_q e^{-i\phi} + \text{c.c.} ,\qquad \phi = k_q z \tag{16}$$

(for the convention adopted for the normalization of circularly polarized light, see Ref. 18).

Then Eq. (15a) reduces to

$$\exp\left[i\frac{K^2}{2\gamma^2}\frac{(p_z-p_z')z}{\hbar}\right] , \tag{17}$$

where the wiggler parameter

$$K = \frac{eA_q}{mc} \tag{18}$$

has been introduced, and the relativistic factor γ is given by

$$\gamma = [1+(p_z/mc)^2]^{1/2} . \tag{19}$$

The laser radiation field may then be written [in accordance with Eq. (16)] in the form

$$\mathbf{A}_s = i\left[\frac{\hat{x}+i\hat{y}}{2}\right]\left[\frac{\hbar}{\epsilon_0 V\omega_s}\right]^{1/2} \times \sum_k e^{-i(\omega t - kz)} a_k + \text{H.c.} , \tag{20}$$

which is a general superposition of plane-wave modes (all traveling in the positive z direction). The operator a_k is the usual annihilation operator for photons of frequency $\omega = ck$, and V is the quantization volume. A central frequency $\omega_s = ck_s$ will be arbitrarily chosen, and the condition $|\omega-\omega_s| \ll \omega_s$ will be assumed to hold for all the modes in Eq. (20). (Harmonic generation is therefore ignored.)

The Hamiltonian (1) then reads [using Eqs. (13), (17), and (20)]

$$H_I = \frac{e^2}{2\gamma mL}\left[\frac{\hbar}{\epsilon_0 V\omega_s}\right]^{1/2} A_q \sum_k \sum_{p,p'} e^{-i[(E_p-E_{p'})/\hbar + ck]t} \int_0^L dz \exp\left\{i\left[\left[1+\frac{K^2}{2\gamma^2}\right]\frac{p-p'}{\hbar} + k + k_q\right]z\right\} b_{p'}^\dagger b_p a_k + \text{H.c.} \tag{21}$$

Here and in what follows four-dimensional notation has been dropped and p,p' stand for p_z,p_z', respectively. L is the length of the wiggler, and the integral over the transverse coordinates is assumed to give just the cross section of the electron beam (which is assumed in turn to equal that of the laser mode, for simplicity). The integral over z may be well approximated by a δ function, which results in the condition

$$p' = p + \frac{(k+k_q)\hbar}{1+K^2/2\gamma^2} \simeq p + \hbar(k+k_q)(1-K^2/2\gamma^2) . \tag{22}$$

At this point, it is useful to introduce the "detuning parameter" μ as

$$E_p = \gamma_p mc^2 = \gamma_0 mc^2(1+\gamma_0^2\mu/k_s) \tag{23}$$

(see Ref. 5) so that there is a one-to-one correspondence between μ and p (or E_p). The energy γ_0 is defined in terms of λ_s, λ_q, and K, through the "resonance condition"

$$\lambda_s = \frac{\lambda_q}{2\gamma_0^2}(1+K^2) . \tag{24}$$

When Eqs. (22), (23), and (24) are used, one finds that, to first order in the (typically) small quantity $\gamma_0^2\mu/k_s$,

$$(E_p - E_{p'})/\hbar \simeq -\omega + \mu c + c(k-k_s)(1+K^2)/2\gamma_0^2 - cq/(1+K^2) , \tag{25}$$

where the "quantum recoil" parameter q is defined as[4]

$$q = \frac{\hbar k_s k_q}{mc\gamma_0} . \tag{26}$$

From Eq. (25) itself, and the definitions (23) and (26), it may be seen that when p changes from p to p', the corresponding value of μ changes, to first order, from μ to $\mu + 2qk/k_s$ [this is given by the first term on the right-hand side (rhs) of Eq. (25), which is much larger than all the others]. The Hamiltonian (21) may then be written as

$$H_I = \frac{e^2}{2\gamma_0 m}\left[\frac{\hbar}{\epsilon_0 V\omega_s}\right] A_q \sum_k \sum_\mu \exp\left[i\left[\mu - (k-k_s)\frac{1+K^2}{2\gamma_0^2} + \frac{q}{1+K^2}\right]ct\right] b_{\mu+2qk/k_s}^\dagger b_\mu a_k , \tag{27}$$

where the fermion creation and annihilation operators are now labeled by the variable μ instead of p.

To proceed, it is convenient to consider the degeneracy of the system of electrons. If the electrons are confined within a region of the order of L_e (where L_e is the length of the electron pulse) in the z direction, then the density of allowed states for the longitudinal momentum is of the order of $L_e/\pi\hbar$. If one has N_e electrons, all of them having different longitudinal momenta (as would seem to be required by the exclusion principle), the total spread in momentum of the electrons could not be less than $\pi\hbar N_e/L_e$. It is important, however, to realize that a small transverse momentum is in practice inevitable. If this is allowed for, and the number of allowed states of transverse momentum are counted (with the restriction that the transverse spreading of the electron beam during the interaction, due to this transverse momentum, be still negligible in the calculations), one finds[19] that, for all existing and proposed devices of which we are aware, the assumption of a very small longitudinal momentum spread does not conflict with the exclusion principle. In particular, there is no problem in assuming the spread to be small enough to treat the beam as monoenergetic (homogeneously broadened regime) in the calculations presented in Secs. III and IV below.

It is convenient, as a matter of fact, to consider all the electrons as initially having different values of the transverse momentum. Since this is a constant of the motion, this amounts to, in effect, labeling all the particles, which thereby become distinguishable. That this is a reasonable assumption may be seen more explicitly as follows. If the cross section of the electron beam is A_0, the density of transverse momentum states is $A_0/(\pi\hbar)^2$. If there are N_e electrons, all with different transverse momenta, they will fill up all the states up to a maximum transverse momentum on the order of

$$P_{\perp_{\max}}=\pi\hbar(N_e/A_0)^{1/2} . \tag{28}$$

For a pulse of $N_e \simeq 10^8$ electrons, and a beam of cross section $A_0 \simeq 10^{-7}$ m^2, one has $P_{\perp_{\max}} \simeq 10^{-4}mc$, which is small enough to be neglected everywhere in our calculations. The transverse velocity corresponding to this momentum is $P_{\perp_{\max}}/\gamma m$, which in most cases will be small enough to ensure negligible spread of the electron beam.

In this way, we may think of the operators $b_p^\dagger, b_p$ as having an additional pair of indices for the transverse components of the momentum [a sum over which is then implicit in Eq. (27)]; or, equivalently, we may number all the particles, and adopt a first-quantization formalism. A state in which particle 1 has (longitudinal) momentum μ_1, particle 2 has momentum μ_2, etc., may be written as

$$|\mu_1\rangle\,|\mu_2\rangle \cdots |\mu_{N_e}\rangle \tag{29}$$

and the operator $\sum_\mu e^{i\mu ct} b^\dagger_{\mu+2qk/k_s} b_\mu$ (with the sum over the transverse momentum implicit) may be replaced by a sum of operators acting on the individual particles

$$\sum_\mu e^{i\mu ct} b^\dagger_{\mu+2qk/k_s} b_\mu \rightarrow \sum_{i=1}^{N_e} \sum_\mu (\,|\mu+2qk/k_s\rangle\langle\mu|\,)_i e^{i\hat{\mu}_i ct} , \tag{30}$$

where $\hat{\mu}_i$ is defined by $\hat{\mu}_i\,|\mu\rangle_i=\mu\,|\mu\rangle_i$. A "canonically conjugate" operator may be defined by the relationship

$$e^{i\hat{\theta}_i}=\sum_\mu (\,|\mu+2q\rangle\langle\mu|\,)_i \tag{31}$$

(it is easy to see that the rhs of Eq. (31) is a unitary operator, so that $\hat{\theta}_i$ is Hermitian). It follows from Eq. (31) that

$$[\hat{\mu}_j, e^{i\hat{\theta}_i}]=2qe^{i\hat{\theta}_i}\delta_{ij} , \tag{32}$$

which is equivalent to the relationship

$$[\hat{\theta}_i,\hat{\mu}_j]=2iq\delta_{ij} . \tag{33}$$

The interaction Hamiltonian (27) may then be written in the form

$$H_I=\hbar g \sum_k \sum_{i=1}^{N_e} e^{ik\hat{\theta}_i/k_s} \times e^{i[\hat{\mu}_i-(k-k_s)(1+K^2)/2\gamma_0^2+q]ct} a_k + \text{H.c.} \tag{34}$$

with the "coupling constant" g defined as

$$g=\frac{e^2A_q}{2\gamma_0 m}\left[\frac{1}{\epsilon_0\omega_s V\hbar}\right]^{1/2} . \tag{35}$$

Note that the term $q/(1+K^2)$ in the exponent of Eq. (27) has been replaced by simply q in Eq. (34). This has been done for convenience, since it leads to more symmetric forms of the equations. The difference, which is $qK^2/(1+K^2)$, may be considered to be absorbed in the operators $\hat{\mu}_i$. Such a redefinition clearly does not change the commutation relations, and in practice will be negligible in all cases: in the classical regime, K^2 may be of the order of unity, but q is much smaller then typical values of μ, whereas in the quantum regime, where q is large, K^2 is typically very small.

When only one mode of the radiation field is considered, it is natural to let $k=k_s$. One then obtains

$$H_I=\hbar g \sum_{i=1}^{N_e} e^{i\hat{\theta}_i} e^{i(\hat{\mu}_i+q)ct} a + \text{H.c.} \tag{36}$$

This is of precisely the same form as the many-particle Hamiltonian used in Ref. 8(c), which was derived in the moving (Bambini-Renieri) frame: one has only to identify $\hat{\theta}_i$ with kz_i, $\hat{\mu}_i c$ with kp_i/m, and q with $\hbar k^2/2mc$ in Eq. (5) of Ref. 8(c). The commutation relation (33) readily follows (z_i and p_i are the position and momentum of electron i in the moving frame). Hence both Hamiltonians describe essentially the same physics. A Hamiltonian similar to Eq. (36) was introduced in Ref. 5 to study the "semiclassical" theory of the FEL (the radiation field was not quantized there, and the variable z was used in place of ct); there it may be seen that the operator $\hat{\theta}_i$ corresponds to the phase of the ith electron in the ponderomotive potential.

The great advantage of the Hamiltonian (34), of course, is that it allows one to use ordinary quantum mechanics (instead of QED) in the laboratory frame. This will be exploited in the following sections. A single mode of the radiation field will be considered for the remainder of this paper; multimode calculations will be presented elsewhere.

III. SINGLE-PASS EVOLUTION EQUATIONS

A. The large-gain evolution equations

In this section we shall make use of a density matrix to describe the system "electrons + photons" in the FEL. We assume that no correlations exist between the two kinds of particles at the initial time $t=0$, and also that no correlations exist between the electrons themselves at that time, so that one has

$$\rho(0)=\rho_{\rm el}(0)\otimes\rho_{\rm ph}(0)\ , \tag{37}$$

where $\rho_{\rm el}(0)$ is a diagonal operator in the basis of electron's momentum eigenstates $|\mu_1\rangle\cdots|\mu_{N_e}\rangle$.

The density matrix obeys the equation

$$\frac{d\rho}{dt}=-\frac{i}{\hbar}[H_I(t),\rho(0)]-\frac{1}{\hbar^2}\int_0^t [H_I(t),[H_I(t'),\rho(t')]]dt' \tag{38}$$

which is obtained by iterating the equation of motion for $\rho(t)$ once. It is possible to obtain from Eq. (38) an equation of motion for the reduced density matrix $\rho_{\rm ph}$ (for the photons alone) under the following assumptions. First, one may notice that for as long as the laser field intensity is not too high, the probability of any one electron emitting a photon is very small, which means that a sizable gain results only from considering a large number of electrons. Under these conditions, it seems reasonable to use a "reservoir"-like approximation[20] for the system of electrons and write for the total density matrix at time t'

$$\rho(t')\simeq\rho_{\rm el}(0)\otimes\rho_{\rm ph}(t') \tag{39}$$

in the rhs of Eq. (38). Note that, since that equation has been obtained by iterating the equation of motion for $\rho(t)$ once, the approximation (39) does *not* amount to neglecting altogether the change in the electron's momentum, but, rather, to taking it into account to first order in the interaction only (the "classical" version of the large-gain equation is derived from the pendulum equations under the same kind of approximation).

When Eq. (39) is substituted into the rhs of Eq. (38), one obtains an equation which, after tracing over the electronic variables, gives

$$\frac{d\rho_{\rm ph}}{dt}=-\frac{1}{\hbar^2}\int_0^t {\rm Tr}_{\rm el}[H_I(t),[H_I(t'),\rho_{\rm el}(0)\otimes\rho_{\rm ph}(t')]]dt'\ , \tag{40}$$

where $\rho_{\rm ph}(t)\equiv{\rm Tr}_{\rm el}\,\rho(t)$ [in obvious agreement with (39)] and the relationship

$${\rm Tr}_{\rm el}[H_I(t),\rho(0)]=0 \tag{41}$$

has been used [Eq. (41) is a trivial consequence of the fact that $\rho_{\rm el}(0)$ was assumed to be diagonal in the basis $|\mu_1\rangle\cdots|\mu_{N_e}\rangle$, while $H_I(t)$ has no diagonal elements in that basis]. When the Hamiltonian (36) is used in Eq. (40) one gets

$$\begin{aligned}\frac{d\rho_{\rm ph}}{dt}=-g^2\int_0^t dt'\Big\langle\sum_i e^{i(\hat{\mu}_i-q)c(t-t')}\Big\rangle&\times[aa^\dagger\rho_{\rm ph}(t')-a^\dagger\rho_{\rm ph}(t')a]\\&+\Big\langle\sum_i e^{-i(\hat{\mu}_i+q)c(t-t')}\Big\rangle\\&\times[a^\dagger a\rho_{\rm ph}(t')-a\rho_{\rm ph}(t')a^\dagger]+{\rm H.c.}\ ,\end{aligned} \tag{42}$$

where the angle brackets denote an average over the initial (many-particle) electron density matrix. We shall assume that each electron has initially a well-defined value of the momentum μ (i.e., it is in a plane-wave state) and that these values are distributed according to some function $f_0(\mu)$. One will then have

$$\Big\langle\sum_{i=1}^{N_e} e^{i(\hat{\mu}_i\pm q)c(t-t')}\Big\rangle=N_e\int d\mu\, f_0(\mu)e^{i(\mu\pm q)c(t-t')}\ . \tag{43}$$

When the reduced density matrix $\rho_{\rm ph}$ is used to calculate the expectation value of the annihilation operator a the following equation is obtained:

$$\frac{d\langle a\rangle}{dt}=g^2N_e\int_0^t dt'\langle e^{-i\mu c(t-t')}\rangle\times(e^{iqc(t-t')}-e^{-iqc(t-t')})\langle a\rangle(t')\ . \tag{44}$$

This is identical to the large-gain equation presented in Ref. 4, except that the variable ct is used instead of z (position along the wiggler) and $\langle a\rangle$ replaces the electric field amplitude E. In the classical limit, that is, when $\hbar\to0$ [which implies $q\to0$, $g\to\infty$, $g^2q<\infty$, cf. Eqs. (26) and (35)], it reduces to the classical large-gain equation (Ref. 14).

B. Saturation terms

The most straightforward extension of the linear theory is higher-order perturbation theory. The applicability of perturbation theory to the FEL has been questioned before (see, for instance, Ref. 9). The main criticism may be made clear by noticing the fact that, near saturation, the average number of photons emitted (or absorbed) by *each* electron is of the order of $1/(qcT)$, where q is the quantum recoil and T is the interaction time. In devices operating in the infrared, this number may be of the order of 10^5 of larger. Hence it would seem that (say) fourth-order perturbation theory would be completely inappropriate to describe the behavior of the FEL at saturation.

One may, however, look at the problem from a different angle. Let us say that we are interested in some physical quantity, such as the increase $\Delta\langle n\rangle$ in the average number of photons per pass. This will be a function (in general unknown) of a large number of variables, such as the initial values of $\langle n\rangle$, $\langle n^2\rangle$, etc., and also the interaction constant g^2 [that g should enter as g^2 is due to the symmetry reasons mentioned in connection with Eq. (41)]. Now, one may consider a Taylor expansion of $\Delta\langle n\rangle$ in powers of g,

$$\Delta\langle n\rangle = g^2 f_2 + (g^2)^2 f_4 + \cdots , \tag{45}$$

where $f_1, f_2, \ldots$ will be functions of the remaining variables. Now it is clear that (i) if the perturbation series converges at all, the functions $f_1, f_2, \ldots$ are going to coincide with those obtained from second-, fourth-, etc. order perturbation theory, respectively (since the perturbation series is formally a series in powers of g), (ii) the question, then, of how good nth order perturbation theory is *for the particular quantity that we are interested in* ($\Delta\langle n\rangle$ in this example) amounts, in practice, to asking just how well that function ($\Delta\langle n\rangle$) can be approximated by the first $n/2$ terms in its Taylor expansion in powers of g^2. As it turns out, in the classical, small-gain case, the first two terms in (45) describe the gain (and saturation) surprisingly well [see Ref. 15(a)], at least for a laser operating not too far above threshold. This fact will be used as an empirical justification for the use of perturbation theory in this section, as well as an estimate on the possible limits of validity of the new results here derived (whenever these results may be compared to former, more "exact" treatments, they also are seen to be good approximations; see, for instance, Fig. 4).

The first nonvanishing correction to the linear equation (40) is

$$\frac{d\rho_{\rm ph}}{dt} = -\frac{1}{\hbar^2}\int_0^t dt'\,{\rm Tr}_{\rm el}[H_I(t),[H_I(t'),\rho_{\rm el}(0)\otimes\rho_{\rm ph}(0)]]$$
$$+\frac{1}{\hbar^4}\int_0^t dt'\int_0^{t'} dt''\int_0^{t''} dt'''\,{\rm Tr}_{\rm el}[H_I(t),[H_I(t'),[H_I(t''),[H_I(t'''),\rho_{\rm el}(0)\otimes\rho_{\rm ph}(t''')]]]] . \tag{46}$$

Equation (46) may be used to derive evolution equations for the moments $\langle n^k\rangle$ of the photon distribution function: these will be integro-differential equations [analogous to Eq. (44)] since $\langle n^k\rangle$ in the last integral will be a function of t'''. If this is replaced by its initial value, $\langle n^k\rangle(0)$, the result is of course fully equivalent to fourth-order perturbation theory.

When this is done, the integration of the single-pass evolution equations is straightforward, though long and tedious. In the cold-beam (homogeneously broadened) limit, in which the initial momentum distribution function $f_0(\mu)$ may be approximated by a δ function [i.e., $f_0(\mu)\simeq\delta(\mu-\mu_0)$], the result, for the first two moments $\langle n\rangle$ and $\langle n^2\rangle$, is

$$\begin{aligned}\Delta\langle n\rangle &\equiv \langle n\rangle(T)-\langle n\rangle(0)\\ &= g^2N_eT^2(A_1\langle n\rangle_0+A_2)\\ &\quad + g^4N_eT^4[B_1\langle n^2\rangle_0+(B_2N_e+B_3)\langle n\rangle_0\\ &\qquad + B_4N_e+B_5] ,\end{aligned} \tag{47}$$

$$\begin{aligned}\Delta\langle n^2\rangle &\equiv \langle n^2\rangle(T)-\langle n^2\rangle(0)\\ &= g^2N_eT^2(C_1\langle n^2\rangle_0+C_2\langle n\rangle_0+C_3)\\ &\quad + g^4N_eT^4[D_1\langle n^3\rangle_0+(D_2N_e+D_3)\langle n^2\rangle_0\\ &\qquad + (D_4N_e+D_5)\langle n\rangle_0+D_6N_e+D_7] ,\end{aligned} \tag{48}$$

where the coefficients $A_1,\ldots,D_7$ are functions of the initial momentum μ_0 and the quantum recoil q. Expressions for them are found in Appendix A.

The linear gain, in the small-gain limit, is given by the first term in Eq. (47), that is,

$$(\Delta\langle n\rangle)_{\rm linear\ gain} = g^2N_eT^2A_1(\mu_0,q) . \tag{49}$$

It is immediately seen from the formulas in Appendix A that in the classical limit ($q\to 0$) Eq. (49) becomes

$$(\Delta\langle n\rangle)_{\rm linear\ gain} = G\left.\left[-\frac{d}{dx}\frac{\sin^2 x}{x^2}\right]\right|_{x=\mu_0L/2} , \tag{50}$$

where

$$G = g^2N_eT^2qL , \tag{51}$$

and $L=cT$ is the interaction length. Equation (50) is the familiar antisymmetric gain formula (illustrated by the solid curve in Fig. 1). As seen from Fig. 1, it is a very good approximation to the actual small-signal, small-gain function A_1, for as long as the quantum recoil is small (that is, $qL<1$). The condition $qL>1$ defines the quantum regime (see Ref. 4); the solid line in Fig. 2 shows the behavior of A_1 in this regime (for the value $qL=10$, which is close to those that would obtain in some of the devices proposed in Ref. 4, in which the "wiggler" would be a high-power pulse of infrared radiation from a Nd:glass laser).

The first of the fourth-order terms in (47), that is, B_1, is the first nonlinear correction to the gain. In the classical limit it goes as $(qL)^3$; the function $B_1/(qL)^3$ in this limit is shown in Fig. 3, and it coincides with the result derived, in the same limit, by Lindberg and Stenholm [Ref. 15(b)]. Again, for as long as $qL<1$, the classical limit is a very good approximation (deviations from the solid curve in Fig. 3 are unnoticeable for $qL<0.1$), but, as shown by the dashed curve in Fig. 3, a significant depar-

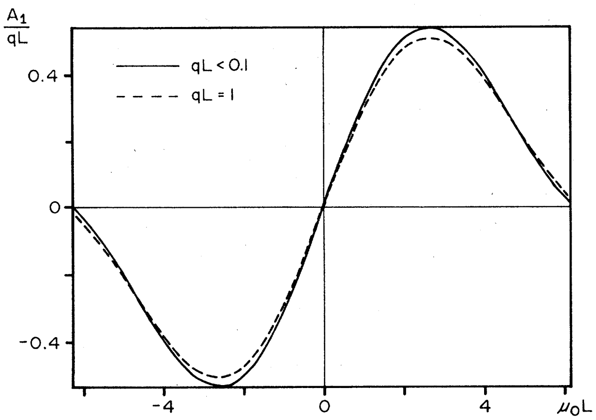

FIG. 1. The small-signal gain function in the classical limit (solid line) and the beginning of the quantum regime (dashed line, corresponding to $qL=1$), as functions of the detuning parameter $\mu_0 L$.

ture from the classical limit is already obtained for $qL=1$. The shape of B_1 well within the quantum regime ($qL=10$) is shown in Fig. 2; it may be seen that in this regime its sign is (almost) always opposite that of the linear gain term A_1, which makes it appropriate to interpret it as a saturation term; in the classical regime, however, there are values of μ_0 for which both A_1 and B_1 are positive, which means that no saturation (rather, a gain enhancement) is obtained in this order for those values of μ_0 [that this effect is real has been shown, by comparison with the result of the pendulum equations, by Bambini *et al.*, Ref. 15(a)]. We will, accordingly, restrict our time-evolution calculations (Sec. IV) to the region where $B_1<0$, since only in this case may a steady state be reached (to this order in perturbation theory, that is).

The term proportional to B_2 in Eq. (47) is of fourth order in the coupling constant g, which is to say of second order in the wiggler field intensity $|A_q|^2$, yet it is linear in the laser field intensity $\langle n\rangle$. Such a term obviously cannot be obtained in the perturbative treatments that make use of the small-gain assumption that treats the laser field as constant during the interaction, since in these treatments the laser and wiggler field intensities enter together in the coupling constant (so that they would always appear raised to the same power). Hence B_2 is a *large-gain* correction to the small-gain formula [Eq. (49)]. This is a many-particle correction (since it is proportional to the square of the number of electrons N_e^2); in the classical limit it goes as $(qL)^2$, which means that in that limit its magnitude is proportional to G^2 [see Eq. (51)], while the usual gain obtained in the small-gain approximation [Eq. (50)] is proportional to G.

To determine the accuracy and validity of this correction, one should compare it to the solution to the large-gain equation [Eq. (44)] which may be calculated exactly: for this, one has to solve a third-degree equation (see Ref. 4) for the eigenvalues of Eq. (44), and then determine the coefficients in the linear superposition of the three eigenmodes that will constitute the solution. In the classical regime [compare Ref. 14(b)] one obtains then a gain curve such as the one shown in Fig. 4 (solid line) for the value $G=10$: the small-gain formula Eq. (50), also plotted in that figure, would predict a peak gain of ~ 5.4 for this case, whereas the exact large-gain equation [Eq. (44)] gives a peak value >14 (that is, 1400% gain per pass, or 14 photons emitted per incident photon). Even for such a large value of the gain, the approximation represented by the sum of the A_1 and B_2 terms in Eq. (47) is remarkably good (see Fig. 4, dashed line). Hence Eq. (47) takes into account large-gain effects quite accurately, up to values of the gain much higher than almost any now envisioned (see, however, the "long-wiggler" example proposed in Ref. 4). (Large-gain results are not presented here for the quantum regime since it is unlikely that devices operating in that regime could easily have gains greater than unity, and under those circumstances the correction B_2 would typically be small.)

Finally, the remaining terms in Eq. (47) are B_4, which corresponds to amplified spontaneous emission [cf. Ref. 8(c)], and B_3 and B_5, which represent small, higher-order corrections to the stimulated and spontaneous emission terms, respectively. (Spontaneous emission is given to first order by the term A_2.)

As regards the equation (48) for the change in the second moment $\langle n^2\rangle$, the leading coefficients in it are related to those in Eq. (47) by the following relationships:

$$C_1=2A_1\ , \tag{52a}$$

$$C_2=-A_1+4A_2\ , \tag{52b}$$

$$C_3=A_2\ , \tag{52c}$$

$$D_1=2B_1\ , \tag{52d}$$

$$D_2=A_1^2+2B_2\ . \tag{52e}$$

Relations analogous to Eqs. (52a)–(52d) also hold in ordinary laser theory,[21] the main difference being (as pointed out in Ref. 13) that there one also has $A_1=A_2$. Relation (52e) does not have a parallel in ordinary laser theory;

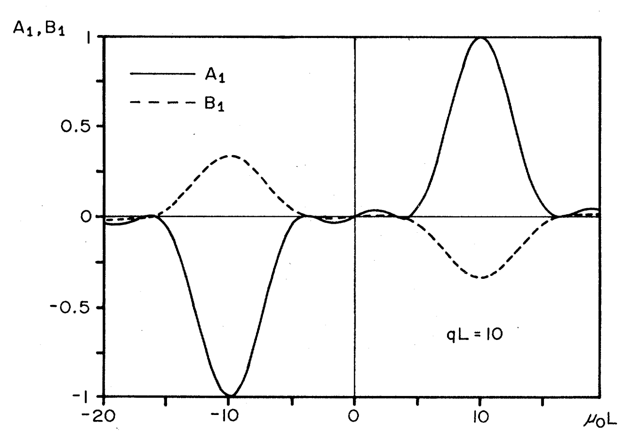

FIG. 2. The small-signal gain function (solid line) and the first nonlinear correction to the gain (dashed line) well within the quantum regime ($qL=10$). Gain is peaked around $\mu_0 L=qL$; the nonlinear term is negative there, corresponding to saturation, and maximum in magnitude.

both D_2 and B_2 are many-particle terms, describing collective effects which in atomic lasers are usually neglected (see, however, Ref. 22).

IV. OSCILLATOR THEORY

A. Evolution equations for the FEL oscillator

Equations (47) and (48) may be used to follow the evolution of the photon statistics in a FEL over many cavity round trips. To do this, one has to introduce the losses at the mirrors. This may be done as in Ref. 23, where the following equation is derived for the change, in one round trip, in the radiation density matrix, due to cavity losses:

$$\delta\rho_{n,n'} = -\tfrac{1}{2}l(n+n')\rho_{n,n'} + l[(n+1)(n'+1)]^{1/2}\rho_{n+1,n'+1} \; . \tag{53}$$

Here l is the round-trip fractional loss, and ρ stands for the reduced density matrix ρ_{ph} of Sec. III. From Eq. (53) one can derive the change in the moments $\langle n\rangle$, $\langle n^2\rangle$, etc., due to the losses. This has then to be added to the change due to the interaction with the electrons, as given by Eqs. (47) and (48), to obtain the total change per round trip.

Let $\langle n\rangle_i$, $\langle n^2\rangle_i$, etc., be the values of $\langle n\rangle$, $\langle n^2\rangle$, etc., after i round trips. Their values after the $(i+1)$th round trip will then be

$$\langle n\rangle_{i+1} = \langle n\rangle_i + \left[\frac{G}{qL}A_1 - l\right]\langle n\rangle_i + \frac{G}{qL}A_2 + \frac{G^2}{(qL)^2}[B_1\langle n^2\rangle_i/N_e + (B_2+B_3/N_e)\langle n\rangle_i + B_4 + B_5/N_e] \; , \tag{54}$$

$$\langle n^2\rangle_{i+1} = \langle n^2\rangle_i + \left[\frac{G}{qL}C_1 - 2l\right]\langle n^2\rangle_i + \left[\frac{G}{qL}C_2 + l\right]\langle n\rangle_i + \frac{G}{qL}C_3 + \frac{G^2}{(qL)^2}[D_1\langle n^3\rangle_i/N_e + (D_2+D_3/N_e)\langle n^2\rangle_i + (D_4+D_5/N_e)\langle n\rangle_i + D_6 + D_7/N_e] \; , \tag{55}$$

where the definition (51) has been used, and losses have been included as discussed.

Equations (54) and (55) are a system of coupled difference equations, which could be integrated immediately to get the time evolution if it were not for the presence of $\langle n^3\rangle$ in Eq. (55). In general, of course, the moment equations will be an infinite hierarchy of coupled equations, which has to be truncated at some point. A possible truncation scheme will be considered later; before this, however, it is useful to look at some limiting cases: specifically, the linear regime and the steady state (saturation).

B. The linear regime

The linear (or small-signal) regime is that period of FEL operation during which the intensity of the radiation field inside the cavity is not yet large enough for the nonlinear (saturation) terms in Eqs. (54) and (55) to be appreciable. These terms are those proportional to B_1 and D_1, respectively, and just when they become important depends of course on the value of these coefficients, but it is easy to obtain an order-of-magnitude estimate as follows.

In the classical regime, as was said in Sec. III, B_1 and D_1 are proportional to $(qL)^3$. This means that, in this limit, the nonlinear terms will be negligible for as long as

$$\langle n\rangle_i \ll N_e/qL \tag{56}$$

(this is assuming that $\langle n^2\rangle$ and $\langle n^3\rangle$ are of the order of magnitude of $\langle n\rangle^2$ and $\langle n\rangle^3$, respectively). This result may be understood by recalling that, when an electron emits a photon, its energy (measured in units of μ) changes by an amount $2q$. In the same units, the width of the gain curve is of the order of $1/L$. In the classical limit, $qL \ll 1$, and therefore an electron may emit on the order of $1/qL$ photons before its energy is shifted out of resonance, i.e., before saturation becomes appreciable. The right-hand side of Eq. (56), accordingly, is the order of magnitude of the field when all N_e electrons have emitted on the order of $1/qL$ photons.

In the quantum regime, on the other hand, B_1 and D_1 are roughly of the same order of magnitude as the other coefficients, and the small-signal gain is proportional to $G/(qL)$, so that the condition (56) is replaced by

$$\langle n\rangle_i \ll N_e \; . \tag{57}$$

This may again be understood by noticing that, in the quantum regime, $qL \gg 1$, and therefore the emission of a single photon is enough to drive an electron out of resonance; hence nonlinear, or saturation, effects become important when the number of photons emitted is of the order of magnitude of the total number of electrons, which is the right-hand side of Eq. (57).

In addition to neglecting the nonlinear terms, one might also consider the case of very small gain per pass, and neglect all the large-gain terms (terms proportional to G^2) in Eqs. (54) and (55). This yields the simplest set of equations (directly obtainable from second-order perturbation theory):

$$\langle n\rangle_{i+1} = \langle n\rangle_i + \left[\frac{G}{qL}A_1 - l\right]\langle n\rangle_i + \frac{G}{qL}A_2 \; , \tag{58}$$

$$\langle n^2\rangle_{i+1} = \langle n^2\rangle_i + 2\left[\frac{G}{qL}A_1 - l\right]\langle n^2\rangle_i + \left[\frac{G}{qL}(4A_2 - A_1) + l\right]\langle n\rangle_i + \frac{G}{qL}A_2 \tag{59}$$

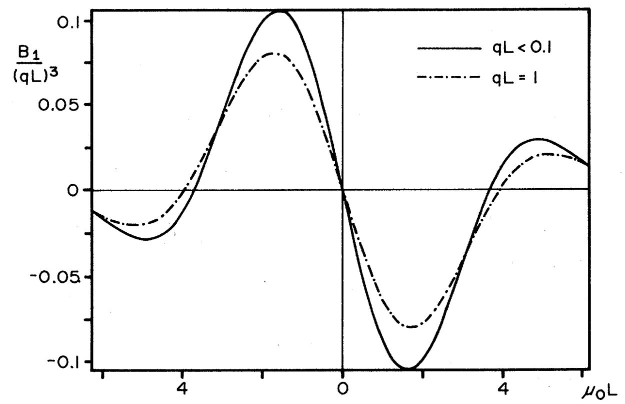

FIG. 3. The first nonlinear correction to the gain in the classical regime (solid line). As long as $qL<0.1$, deviations form the asymptotic form are negligible. When $qL=1$, however (the beginning of the quantum regime), an appreciable change may be seen (dashed line). The classical curve changes sign at $\mu_0 L=3.69$, resulting in gain enhancement instead of saturation [Ref. 15(b)].

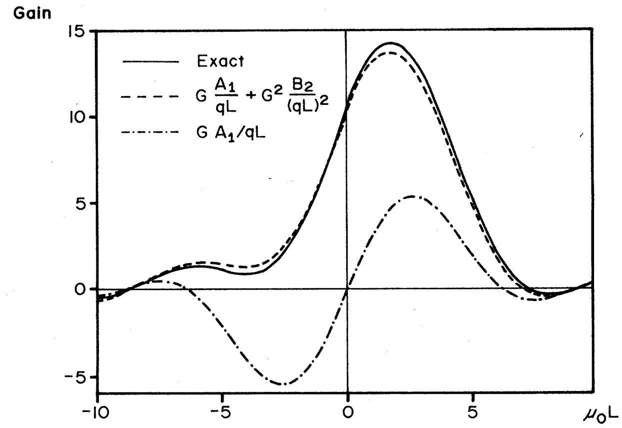

FIG. 4. The linear gain in the large-gain regime (solid line) compared to the small-gain result (dashed-dotted line) and to the approximation represented by the linear terms in Eq. (47) (dashed line). The results are classical, i.e., the exact curve has been calculated from the classical theory and spontaneous emission [as well as the small term B_3 in Eq. (47)] has been ignored in the calculation of the dashed curve. The value of G is 10.

[note that the relationships (52a)—(52c) have been used].

Equations (58) and (59) can of course be easily solved as they stand, that is, as difference equations. One might, however, choose to approximate them by a system of differential equations, by setting

$$\frac{\langle n\rangle_{i+1}-\langle n\rangle_i}{\Delta t}\simeq\frac{d\langle n\rangle}{dt} \tag{60}$$

(and likewise for $\langle n^2\rangle$), where Δt is the duration of one round trip (and, therefore, $t=i\Delta t$). If this is done, one obtains the system

$$\frac{d\langle n\rangle}{dt}=\alpha\langle n\rangle+\sigma\ , \tag{61}$$

$$\frac{d\langle n^2\rangle}{dt}=2\alpha\langle n^2\rangle+(4\sigma-\alpha)\langle n\rangle+\sigma \tag{62}$$

with

$$\alpha\equiv\left[\frac{G}{qL}A_1-l\right]\Big/\Delta t\ , \tag{63}$$

$$\sigma\equiv\left[\frac{G}{qL}A_2\right]\Big/\Delta t\ . \tag{64}$$

It may be immediately verified (by direct substitution) that the solution to Eqs. (61) and (62) that starts from vacuum $[\langle n\rangle(0)=\langle n^2\rangle(0)=0]$ satisfies at all times

$$\langle n^2\rangle=2\langle n\rangle^2+\langle n\rangle\ . \tag{65}$$

This relationship is characteristic of Gaussian (thermal) photon statistics. Becker and McIver (Ref. 12) have shown that the spontaneous radiation (emitted in a single pass, starting from vacuum) in an FEL obeys thermal statistics. Equation (65) seems to imply that this is also true for the field at later times, i.e., as it is amplified in successive passes, as long as one stays within the linear regime.

This derivation is, however, not entirely satisfactory, since, after all, the differential equations (61) and (62) are only an approximation to the more exact difference equations (58) and (59). As a matter of fact, if Eq. (58) and (59) are solved exactly, it is found that the solution does not satisfy Eq. (65). It clearly does not satisfy it after the first pass already, since for $i=0$ Eq. (58) gives

$$\langle n\rangle_1=\frac{G}{qL}A_2 \tag{66}$$

while Eq. (59) gives

$$\langle n^2\rangle_1=\frac{G}{qL}A_2\ , \tag{67}$$

which clearly do not satisfy Eq. (65). The difference, however, is of second order in the small-signal gain G, and this is an indication that the terms in G^2 in Eqs. (54) and (55) cannot be neglected if one wants to test the validity of Eq. (65) accurately. In other words, one has to consider now the full equations (54) and (55), with only the nonlinear terms (those in B_1 and D_1) missing. They form a system that may be written as

$$\langle n\rangle_{i+1}=(1+a)\langle n\rangle_i+s\ , \tag{68}$$

$$\langle n^2\rangle_{i+1}=(1+a')\langle n^2\rangle_i+b\langle n\rangle_i+s'\ , \tag{69}$$

where the various coefficients a,a',b,s,s' can be read immediately off Eqs. (54) and (55); all of them have a part that is proportional to G and a part that is proportional to G^2. (In what follows, powers of the loss parameter l are counted as powers of G.)

The exact solution to Eqs. (68) and (69) is not hard to obtain: after i round trips, starting from vacuum, the values of $\langle n\rangle$ and $\langle n^2\rangle$ are

$$\langle n\rangle_i=\frac{s}{a}[(1+a)^i-1]\ , \tag{70}$$

$$\langle n^2\rangle_i = \frac{bs}{a}\left[\frac{(1+a')^i-(1+a)^i}{a'-a}\right] + \left[\frac{s'}{a'}-\frac{bs}{aa'}\right][(1+a')^i-1] \,. \tag{71}$$

Note, however, that this result can only be accurate to second order in G, since, in principle, higher-order perturbation theory would yield other linear terms proportional to G^2, G^3, etc., in Eqs. (54) and (55), that we have no way to account for. This means that only second-order terms in a and a' need be kept in Eqs. (70) and (71). The approximate result is

$$\langle n\rangle_i \simeq is + \tfrac{1}{2}i(i-1)as \,, \tag{72}$$

$$\langle n^2\rangle_i \simeq is' + \tfrac{1}{2}i(i-1)(bs+a's') \,. \tag{73}$$

In order for this result to satisfy Eq. (65) (again, to second order in G only) one must then have

$$i[s'-\tfrac{1}{2}(bs+a's')]+\tfrac{1}{2}i^2(bs+a's') = i(s-\tfrac{1}{2}as)+\tfrac{1}{2}i^2(as+4s^2) \tag{74}$$

[which results from substituting (72) and (73) into Eq. (65)]. When the coefficients $a, a', \ldots$ etc. are expressed in terms of $A_1, \ldots, D_7$, and powers of G higher than the second are neglected, one may see that Eq. (74) will be satisfied (for all values of i) if the following relationship holds:

$$B_4+B_5/N_e = D_6+D_7/N_e-2A_2^2 \,. \tag{75}$$

Using the expressions given in the Appendix for these coefficients, it is possible to show that

$$B_4 = D_6-2A_2^2 \tag{76}$$

does indeed hold, whereas in general B_5 will be different from D_7. Hence, Eq. (65) is "almost exactly" satisfied to this order, since the terms that violate it are proportional to the very small quantity $1/N_e$ (the number, N_e, of electrons in one pulse may be of the order of 10^7–10^8). This agrees with the results of Ref. 12, which show that thermal statistics for the spontaneous radiation results from having a very large number of electrons. The same appears to be the case here, for the field emitted and (linearly) amplified over several passes. Note that this is also consistent with the results presented in Ref. 24 for the single-pass situation (amplified spontaneous emission).

In conclusion, one may say that, to second order in the small-signal gain parameter G, the radiation field that evolves in an FEL in the linear regime has intensity fluctuations characteristic of "chaotic" or thermal radiation, aside from small corrections, the relative size of which is $1/N_e$. Note that this result holds both in the classical and the quantum-mechanical regimes.

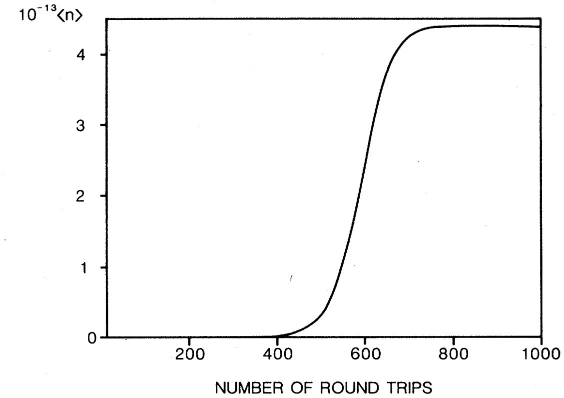

FIG. 5. The buildup of the laser field from vacuum, and into its steady-state value, as a function of time (expressed in units of cavity round trips). The small-signal gain was chosen to equal 0.054, and the losses per round trip to be 2%; $qL = 10^{-5}$ (classical regime) and $N_e = 10^7$.

C. Saturation and steady state

It is tempting to use Eqs. (54) and (55) to obtain information on the photon statistics at saturation also. In this case, the number of photons in the field is of the order of magnitude of the right-hand side of Eq. (56) (in the classical regime) or Eq. (57) (in the quantum regime). For definiteness, the classical regime will be explicitly assumed in what follows, although a similar analysis may be carried out for the quantum regime.

In turns out to be convenient, in the classical regime, to introduce a new variable x, defined by

$$x = \frac{n}{N_e}qL \tag{77}$$

which will be (roughly) of order of magnitude unity at saturation, as was discussed previously. One may think of x as a stochastic variable, whose moments are trivially related to the expectation values $\langle n^k\rangle$. In fact, it is proportional to the energy in the laser field (i.e., its "intensity") in a way that is independent of $\hbar$, and may therefore be considered a completely classical variable.

In terms of the variable x, Eqs. (54) and (55) may be rewritten as

$$\langle x\rangle_{i+1}-\langle x\rangle_i = [G(A_1/qL)-l]\langle x\rangle_i + GA_2/N_e + G^2\{[B_1/(qL)^3]\langle x^2\rangle_i + [(B_2+B_3/N_e)/(qL)^2]\langle x\rangle_i + [(B_4+B_5/N_e)/qL]/N_e\} \,, \tag{78}$$

$$\langle x^2\rangle_{i+1}-\langle x^2\rangle_i = 2[G(A_1^2/qL)-l]\langle x^2\rangle_i + \{4GA_2-[G(A_1/qL)-l]qL\}\langle x\rangle_i/N_e + GA_2qL/N_e + G^2\{2[B_1/(qL)^3]\langle x^3\rangle_i + [(A_1^2+2B_2+D_3/N_e)/(qL)^2]\langle x^2\rangle_i + [(D_4+D_5/N_e)/qL]\langle x\rangle_i/N_e + (D_6+D_7/N_e)/N_e^2\} \,. \tag{79}$$

In Eqs. (78) and (79), all the terms in square brackets [such as, e.g., $B_2/(qL)^2$, D_4/qL, etc.] have a finite limit when qL goes to zero. This means that at saturation, when $\langle x\rangle$ is of order unity or larger, the relative importance of the different terms is determined by the factors of $1/N_e$, $1/N_e^2$, etc. The dominant terms are then

$$\langle x\rangle_{i+1}-\langle x\rangle_i \simeq \{G(A_1/qL)-l+G^2[B_2/(qL)^2]\}\langle x\rangle_i + G^2[B_1/(qL)^3]\langle x^2\rangle_i \ , \tag{80}$$

$$\langle x^2\rangle_{i+1}-\langle x^2\rangle_i \simeq [2G(A_1/qL)-2l + G^2(A_1^2+2B_2)/(qL)^2]\langle x^2\rangle_i + G^2[2B_1/(qL)^3]\langle x^3\rangle_i \ . \tag{81}$$

When a steady-state field configuration is reached, the change in the photon number distribution (or, equivalently, its moments) over one round trip should be zero. One would then solve for the steady-state values of $\langle x\rangle$ and $\langle x^2\rangle$, by setting the right-hand sides of Eqs. (78) and (79) equal to zero. Of course, as was mentioned before, Eq. (79) involves $\langle x^3\rangle$, so that the system is not closed, and some approximation to $\langle x^3\rangle$ in terms of $\langle x\rangle$ and $\langle x^2\rangle$ has to be used. On the other hand, if one looks only at the dominant terms [as in Eqs. (80) and (81)] it turns out to be possible to derive, from fourth-order perturbation theory, a general expression for the kth moment of the photon number distribution, which reads

$$\langle x^k\rangle_{i+1}-\langle x^k\rangle_i \simeq k[GA_1/qL-l+G^2B_2/(qL)^2]\langle x^k\rangle_i + \tfrac{1}{2}k(k-1)G^2[A_1^2/(qL)^2]\langle x^k\rangle_i + kG^2[B_1/(qL)^3]\langle x^{k+1}\rangle_i \ . \tag{82}$$

This gives Eqs. (80) and (81) for $k=1$ and 2, respectively, and neglects terms of order $1/N_e$, as was done there.

When the right-hand side of Eq. (82) is set equal to zero, one obtains an infinite hierarchy of equations for the moments of x (or n), which may be seen to correspond to a gamma distribution for x. This result, however, is not reliable, since higher-order terms in perturbation theory might substantially alter it. To show this, consider the simple case in which the losses are set equal to zero (that is, $l=0$). Equation (82) predicts that the photon statistics are given by a gamma distribution with appreciable intensity fluctuations (proportional, in fact, to the small-signal gain). On the other hand, the same Eq. (82) with $l=0$ may be thought of as an approximation (to second order in G) to an equation of the form

$$\langle x^k\rangle_{i+1}=\langle x^k\{1+GA_1/qL+G^2B_2/(qL)^2 + G^2[B_1/(qL)^3]x\}^k\rangle_i \tag{83}$$

and the steady-state solution to this equation is a number state, i.e., a state with no intensity fluctuations at all. To see this, notice that for a number state $\langle x^k\rangle=\langle x\rangle^k$ for all k, and that

$$\langle x\rangle=-\frac{GA_1/qL+G^2B_2/(qL)^2}{G^2B_1/(qL)^3} \ , \tag{84}$$

when substituted into Eq. (83), yields the steady-state condition $\langle x^k\rangle_{i+1}=\langle x^k\rangle_i$.

Clearly it is impossible to decide between Eqs. (82) and (83) on the basis of fourth-order perturbation theory alone, since the difference between the two is of the order of G^3, and terms proportional to G^3 appear only in sixth-order perturbation theory. Since they yield very different statistics, it follows that the photon statistics at steady-state cannot, for the FEL, be reliably derived from perturbation theory. (One should mention here that, even though the above discussion has assumed zero losses, similar results may be obtained for closer-to-threshold situations; hence this conclusion is quite general.)

The fact that Eq. (82) coincides with the first few terms in the expansion of Eq. (83), however, is suggestive, and one may conjecture that (at least for zero losses) the dominant terms in the field evolution equations would indeed lead to a fluctuationless state (a number state), and that fluctuations in the steady-state distribution would arise from the terms proportional to $1/N_e$ in Eqs. (78) and (79) (in much the same way as the deviations from thermal statistics in the linear regime were due to terms proportional to $1/N_e$). If this were true, the relative size of the intensity fluctuations might be of the order of

$$\frac{(\langle\Delta^2 n\rangle)^{1/2}}{\langle n\rangle}\sim\frac{1}{\sqrt{N_e}} \tag{85}$$

(at least, for zero losses). This is still larger than the fluctuations associated with a Poisson distribution, in the classical regime, since there $\langle n\rangle$ is not proportional to just N_e, but to N_e/qL, so that $\langle\Delta^2 n\rangle$ [if given by Eq. (85)] would be proportional to $\langle n\rangle/qL$ instead of $\langle n\rangle$. At present, however, this is only a conjecture that would have to be proved or disproved by some other means.

The difference between this approach and that of Ref. 13 is that there all the fourth-order terms were ignored except for B_1 and $D_1=2B_1$. This leads to steady-state fluctuations of the order of magnitude hypothesized above; yet, neglecting all those terms cannot be considered satisfactory.

In conclusion, in order to investigate the second-order coherence of the free-electron laser at steady state, a nonperturbative treatment would be necessary. This is not an easy task, but it is important to realize that, in the context of the present theory, it could be done using the techniques of ordinary (i.e., nonrelativistic) quantum mechanics, instead of QED. This is particularly interesting, since the free-electron laser is a quantum-relativistic system, and coherent processes in such systems have not yet received very much attention. Possibly a comparison between the results of the perturbative and non-perturbative treatments would be illuminating in this context.

D. Field evolution: a numerical example

Finally, as was mentioned before, it is possible to integrate Eqs. (54) and (55) numerically, to follow the time evolution of $\langle n\rangle$ and $\langle n^2\rangle$ over many round trips (starting from the vacuum) and the approach to the steady state. As was also mentioned above, some approximation to the value of $\langle n^3\rangle$, which appears in Eq. (55), in terms of $\langle n\rangle$ and $\langle n^2\rangle$ must be used. The following approximation comes from a truncation procedure suggested by

Sargent, Scully, and Lamb[21] for the ordinary laser:

$$\langle n^3 \rangle \simeq (2\langle n^2 \rangle^{1/2} - \langle n \rangle)^3 . \tag{86}$$

Equation (86) is exact if the field is in a number state. It is correct for a coherent state up to (but not including) terms of the order of $\langle n \rangle$; and is less good for thermal statistics, for which $\langle n^3 \rangle = 6\langle n \rangle^3 + 6\langle n \rangle^2 + \langle n \rangle$ while the rhs of Eqs. (86) gives $6.11\langle n \rangle^3 + 7.09\langle n \rangle^2 + O(\langle n \rangle)$.

The result of one such calculation is shown in Fig. 5. The parameters were chosen as follows: $G=0.1$, $l=0.02$, $qL=10^{-5}$ (classical regime), $N_e=10^7$, $\mu_0 L=2.5$ (giving a small-signal gain $GA_1/qL=0.054$, or 5.4%). Here N_e has been chosen to be of the order of magnitude of the number of electrons within a micropulse. The actual value of qL does not essentially affect the time evolution (as long as one stays within the classical regime); it merely changes the vertical scale in Fig. 5 (since the number of photons at saturation is of the order of N_e/qL, as was discussed previously). The spontaneous emission term in Eq. (54) (the one proportional to A_2) is, of course, modified when qL changes, but this is not visible in the figure, since it is always many orders of magnitude smaller than the steady-state intensity.

The steady-state intensity, in turn, may be calculated to a good approximation by setting the rhs of Eq. (54) equal to zero, then keeping only the dominant terms (in powers of $1/N_e$) and making the fluctuationless approximation $\langle n^2 \rangle \simeq \langle n \rangle^2$. The result is

$$\langle n \rangle_{\rm st} = \frac{N_e}{qL} \frac{G(A_1/qL) + G^2[B_2/(qL)^2] - l}{G^2[B_1/(qL)^3]} \tag{87a}$$

and is here written in a form appropriate for the classical regime (as before, the terms in square brackets all tend to finite limits when $q \to 0$). A form more suitable for the quantum regime would be

$$\langle n \rangle_{\rm st} = N_e \frac{(G/qL)A_1 + (G/qL)^2 B_2 - l}{(G/qL)^2 B_1} \tag{87b}$$

since in that regime the small-signal gain is proportional to G/qL.

The intensity fluctuations can of course be obtained from the numerical integration, but, as has been discussed above, the values obtained cease to be reliable the moment that saturation terms become important, and even in the linear regime they eventually become unreliable due to the cumulative effect of powers of G higher than the second. For this particular calculation, Eq. (65) was found to hold, within a few percent, for the first 60 round trips or so. If higher-order terms in G were available, it would probably be found to hold for much longer.

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APPENDIX

The coefficients $A_1, \ldots, D_7$ in Eqs. (47) and (48) are derived from Eq. (46). The A's and C's come from the first term, the B's and D's from the second one. To obtain the latter, the nest of four commutators has to be written out, and the trace taken over the electron variables assuming an initially monoenergetic beam; this results in an integro-differential equation for $\rho_{\rm ph}(t)$. From this, one may obtain the corresponding equations for $\langle n \rangle(t)$ and $\langle n^2 \rangle(t)$; to evaluate the change in these two moments, one treats all the $\langle n \rangle$, $\langle n^2 \rangle$, $\langle n^3 \rangle$, etc., on the right-hand side of those equations as constants (equal to their initial values), and performs all the time integrations (four of them are needed to obtain the increments $\Delta\langle n \rangle$ and $\Delta\langle n^2 \rangle$). The result[25] is of the form (47) and (48), where the coefficients $A_1, \ldots, D_7$ are functions of μ_0 (the initial electron energy) and $L=cT$ (where T is the interaction time, and L the interaction length).

Let

$$a = (\mu_0 + q)L , \tag{A1a}$$

$$b = (\mu_0 - q)L , \tag{A1b}$$

$$c = (\mu_0 + 3q)L , \tag{A1c}$$

$$d = (\mu_0 - 3q)L . \tag{A1d}$$

Then,

$$A_1 = \left[\frac{\sin(b/2)}{b/2} \right]^2 - \left[\frac{\sin(a/2)}{a/2} \right]^2 , \tag{A2}$$

$$A_2 = \left[\frac{\sin(b/2)}{b/2} \right]^2 , \tag{A3}$$

and [see Eqs. (52)]

$$C_1 = 2A_1 , \tag{A4}$$

$$C_2 = -A_1 + 4A_2 , \tag{A5}$$

$$C_3 = A_2 . \tag{A6}$$

To express the B and D coefficients, it is useful to introduce the following auxiliary functions:

$$\begin{aligned} F_1(x,y) = & -\frac{2}{x^2 y(y-x)}(1-\cos x) \\ & -\frac{2}{xy(x+y)^2}[1-\cos(x+y)] \\ & +\frac{2}{y^2 x(y-x)}(1-\cos y) , \end{aligned} \tag{A7}$$

$$\begin{aligned} F_2(x,y) = & -\frac{4}{x^2(x^2-y^2)}(1-\cos x) \\ & +\frac{4}{y^2(x^2-y^2)}(1-\cos y) , \end{aligned} \tag{A8}$$

$$F_3(x,y)=2\frac{x-2y}{x^3y^2}(1-\cos x)+\frac{2\sin x}{x^2y}-\frac{2}{y^2(x+y)^2}[1-\cos(x+y)]\ , \quad \text{(A9)}$$

$$F_4(x)=\frac{4}{x^4}(1-\cos x)-\frac{2}{x^3}\sin x\ . \quad \text{(A10)}$$

Then one has[25]

$$B_1=F_3(a,c)-F_1(c,a)+F_3(a,b)+F_1(a,b)-F_2(c,a)-F_2(a,b)+2F_4(a)-F_3(b,d)+F_1(d,b)-F_3(b,a)-F_1(b,a)+F_2(d,b)+F_2(b,a)-2F_4(b)\ , \quad \text{(A11)}$$

$$B_2=2F_4(a)-F_2(a,b)-F_1(b,-a)-F_3(b,-a)-2F_4(b)+F_2(b,a)+F_1(a,-b)+F_3(a,-b)\ , \quad \text{(A12)}$$

$$B_4=2F_4(b)-F_3(b,-a)-F_1(a,-b)-F_2(a,b)\ , \quad \text{(A13)}$$

$$D_1=2B_1\ , \quad \text{(A14)}$$

$$D_2=A_1^2+2B_2\ , \quad \text{(A15)}$$

$$D_4=-6F_4(a)+26F_4(b)-4F_3(a,a)+12F_3(b,b)-2[F_1(a,b)+F_3(b,a)+F_3(a,b)+F_1(b,a)]-[F_1(b,-a)+F_2(b,a)+F_3(a,-b)]-10[F_1(a,-b)+F_2(a,b)+F_3(b,-a)]\ , \quad \text{(A16)}$$

$$D_6=10F_4(b)+13F_3(b,b)-F_3(b,-a)-F_1(a,-b)-F_2(a,b)\ . \quad \text{(A17)}$$

The coefficients B_3, B_5, D_3, D_5, and D_7 multiply terms that are negligible versus the terms containing B_2, B_4, D_2, D_4, and D_6, respectively [as may be seen from Eqs. (47) and (48)], provided that the number of electrons N_e is large, as is normally the case. The expressions for them are, however, long and complicated; only those for B_5 and D_7 will therefore be given here, since they are relevant to Eq. (75) in the text:

$$B_5=-4F_4(b)+2F_2(d,b)+2F_1(d,b)-2F_3(b,d)+F_3(b,-a)+F_1(a,-b)+F_2(a,b)\ , \quad \text{(A18)}$$

$$D_7=-12F_4(b)+6F_2(d,b)+6F_1(d,b)-2F_3(b,d)-8F_3(b,b)+F_3(b,-a)+F_1(a,-b)+F_2(a,b)\ . \quad \text{(A19)}$$

The classical limit of B_1 has been evaluated before [Ref. 15(b)]: it is given by

$$B_1=8(qL)^3[(53x-x^3)\sin x+(48-13x^2)\cos x+11x\sin(2x)+(18-2x^2)\cos(2x)-66]/x^7\ , \quad \text{(A20)}$$

where $x=\mu_0 L$.

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