PHYSICAL REVIEW A VOLUME 31, NUMBER 3 MARCH 1985

Diffractive effects in pulse propagation through a resonant medium

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(Received 17 September 1984)

The propagation of a weak optical pulse through a resonant two-level atomic medium is discussed in the paraxial approximation. Transverse propagation effects within that approximation are treated exactly by development of the method of Fresnel transforms. In particular, for a temporally short pulse with an amplitude of arbitrary transverse spatial dependence it is shown that longitudinal propagation is not affected by diffractive spreading. Thus diffraction does not attenuate the characteristic ringing of one-dimensional pulse propagation in resonant media.

I. INTRODUCTION

The propagation of coherent pulses through resonant media has received considerable study in recent years.[1–8] Several interesting phenomena, such as self-induced transparency[2] and photon echoes,[3] have been predicted theoretically and then observed experimentally. But in almost all analytical calculations of these effects to date the pulse has been assumed to propagate in a strictly one-dimensional fashion. The assumption of strict one-dimensionality is questionable, however, because several effects arising from the finite transverse dimensions of the systems are not accounted for. The most obvious of these effects is the diffractive spreading that is always present in propagation of actual laser pulses used to study such phenomena. These systems are thus at best quasi-one-dimensional.

The study of the passage of a weak optical pulse through a semi-infinite medium of resonant two-level atoms includes a particular class of propagation problems that is of much interest. What we treat in this paper is the problem of diffractive spreading of the pulse cross section and the manner in which such spreading affects the interaction of the pulse with the resonant medium.

The transverse spreading, which arises from a spatial nonuniformity of the amplitude distribution of the pulse over its cross section, is assumed to be mild enough that the propagation may still be approximately described by a single wave vector $\mathbf{k}_0=k_0\hat{\mathbf{z}}$. More precisely, we assume that we can make the classical Fresnel-diffraction, or paraxial, approximation in the wave equation. In addition, the pulse is taken to be weak enough that its passage causes only a negligible population transfer between the two atomic levels. This assumption amounts to an approximation of linearity in the equations that describe pulse propagation.

In order to illustrate the physical nature of such propagation we will first consider a temporally short pulse, the amplitude of which may in the limit be taken to be a δ function in time. One of the earliest calculations of this kind was carried out by Burnham and Chiao[4] for a one-dimensional problem. They showed that in the absence of incoherent relaxation processes the resonance fluorescence excited by a uniform plane-wave δ-function pulse should exhibit damped oscillations, or ringing, of the Bessel-function form, $J_1(\alpha\sqrt{\tau})/\tau$, where τ is the retarded time and α is a constant that depends on the length and density of the sample. An input pulse with a spatially nonuniform amplitude over its cross section, on the other hand, will spread by diffraction, and lead to different distributions of excitations in different transverse planes. That complicates the problem, leaving it no longer clear what kind of time dependence will obtain. To study this problem we develop an integral transform method based on a kernel function which describes Fresnel diffraction. We will show, by using this transform, that within the paraxial approximation the effects of transverse diffraction decouple exactly from the one-dimensional, longitudinal process of energy exchange which is responsible for the simple Bessel-function pattern. Thus, the Bessel-function ringing obtains once again for this more general case.

We will then generalize the calculations further to include incoherent relaxation processes of the kinds which lead both to homogeneous and inhomogeneous broadening. Also, the restriction that the exciting pulse be temporally short will be relaxed. We will show that diffraction can in that more general case still be easily included. These results extend some calculations of Crisp on one-dimensional pulse propagation[5] to those three-dimensional situations in which the forward-propagating pulse has an arbitrary spatial dependence.

II. EQUATIONS OF MOTION

The material medium is assumed to consist of identical atoms each having only two states, $|+\rangle$ and $|-\rangle$, with distinct energy eigenvalues. The behavior of an individual atom will be described in terms of its 2×2 density matrix ρ in the Schrödinger picture. The effect of incoherent relaxation of the atomic variables caused, e.g., by collisions and by the interaction of the atoms with the radiation field reservoir (spontaneous emission) can be represented schematically by introducing phenomenological relaxation times T_1 and T_2'. The quantity T_1 is the characteristic time for the decay of the population and T_2' that for the relaxation of the dipole moment of the atom.

The optical pulse will be taken to be nearly resonant, i.e., its center frequency ω_0 is nearly equal to the atomic frequency ω that separates the two states $|+\rangle$ and $|-\rangle$. The magnitude of the detuning Δ given by $\Delta \equiv \omega - \omega_0$ is thus assumed to be much smaller than ω_0. We let the electric field $\mathbf{E}(\mathbf{r},t)$ of the optical pulse, which travels in the $\mathbf{z}$ direction, be linearly polarized along the $\mathbf{x}$ axis,

$$\mathbf{E}(\mathbf{r},t)=\frac{\hat{\mathbf{x}}}{2}[\mathscr{E}(\mathbf{r},t)\exp(ik_0z-i\omega_0t)+\text{c.c.}] , \tag{1}$$

where $\hat{\mathbf{x}}$ is a unit vector in the $\mathbf{x}$ direction. The dipole moment induced in the atoms by the field then also points in the $\mathbf{x}$ direction.

In the electric dipole approximation the expectation value of the atomic dipole moment operator $\hat{\mu}$ is given by

$$\langle \hat{\mu} \rangle = \mu(\rho_{+-}+\rho_{-+}) , \tag{2}$$

where $\mu = e\langle + | \mathbf{r}\cdot\hat{\mathbf{x}} | - \rangle$ is the projection of the dipole matrix element along $\mathbf{E}$, and ρ_{+-} and ρ_{-+} are the off-diagonal elements of ρ. We have taken μ to be real by fixing appropriately the relative phases of the states $|+\rangle$ and $|-\rangle$.

Since ρ_{+-} is expected to have dominantly the time dependence $\exp(-i\omega_0t)$ we can define a pair of more slowly varying, real amplitudes $\mathscr{U}_\Delta$ and $\mathscr{V}_\Delta$ via the relations

$$\begin{aligned}\rho_{+-} &= \tfrac{1}{2}(\mathscr{U}_\Delta + i\mathscr{V}_\Delta)\exp(ik_0z-i\omega_0t) \\ &= \rho_{-+}^{*}\end{aligned} \tag{3}$$

($k_0=\omega_0/c$). The factor $\exp(ik_0z)$ is inserted to account for the spatial variation of the traveling fields.

The time evolution of ρ_{ij} ($i,j=+,-$), or equivalently, of $(\mathscr{U}_\Delta \pm i\mathscr{V}_\Delta)$ is easily found with the aid of the density matrix equation[9]

$$i\hbar\frac{d\rho}{dt}=[H,\rho]+\cdots , \tag{4}$$

where the ellipsis represents unspecified phenomenological relaxation terms. In this equation, the Hamiltonian H is the sum of the free atomic Hamitonian H_0 and the dipole interaction $-e\mathbf{r}\cdot\mathbf{E}$. The equation of motion for the polarization amplitude $(\mathscr{U}_\Delta + i\mathscr{V}_\Delta)$, which accounts suitably for transverse relaxation processes and neglects antiresonant terms oscillating at frequency $2\omega_0$, may then be shown[9] to be

$$\left[\frac{\partial}{\partial t}+i\Delta+\frac{1}{T_2'}\right](\mathscr{U}_\Delta+i\mathscr{V}_\Delta)=-\frac{i\mu}{\hbar}\mathscr{E}(\mathbf{r},t)\mathscr{W}_\Delta , \tag{5}$$

where we have written $\mathscr{W}_\Delta=\rho_{++}-\rho_{--}$ for the population inversion. The inversion furthermore obeys the equation

$$\begin{aligned}\frac{\partial}{\partial t}\mathscr{W}_\Delta = {} & \frac{i\mu}{2\hbar}[\mathscr{E}(\mathbf{r},t)(\mathscr{U}_\Delta - i\mathscr{V}_\Delta)-\mathscr{E}^*(\mathbf{r},t)(\mathscr{U}_\Delta+i\mathscr{V}_\Delta)] \\ & -(\mathscr{W}_\Delta-\mathscr{W}_\Delta^{(0)})/T_1\end{aligned} \tag{6}$$

in which $\mathscr{W}_\Delta^{(0)}$ is the steady-state value to which $\mathscr{W}_\Delta$ relaxes. We shall assume in fact that the system is at zero temperature so that $\mathscr{W}_\Delta^{(0)}=-1$.

In any actual sample of similar atoms the transition frequencies are inhomogeneously broadened (e.g., by Doppler broadening). In other words, there is a distribution of the detunings Δ, which we shall call $g(\Delta)$, normalized so that $\int_{-\infty}^{\infty} g(\Delta)d\Delta=1$. For such a medium we see from Eq. (2) that if n_0 is the number of atoms per unit volume, the expectation value of the polarization density $P(\mathbf{r},t)$ is given by

$$\begin{aligned}P(\mathbf{r},t) &= \frac{n_0\mu}{2}\left[\int_{-\infty}^{\infty}[\mathscr{U}_\Delta(\mathbf{r},t)+i\mathscr{V}_\Delta(\mathbf{r},t)]g(\Delta)d\Delta \right. \\ &\qquad \left. \times\exp(ik_0z-i\omega_0t)+\text{c.c.}\right] \\ &= \tfrac{1}{2}[\mathscr{P}(\mathbf{r},t)\exp(ik_0z-i\omega_0t)+\text{c.c.}] ,\end{aligned} \tag{7}$$

where

$$\mathscr{P}(\mathbf{r},t)=n_0\mu\int_{-\infty}^{\infty}[\mathscr{U}_\Delta(\mathbf{r},t)+i\mathscr{V}_\Delta(\mathbf{r},t)]g(\Delta)d\Delta . \tag{8}$$

We now assume that $\mathscr{P}(\mathbf{r},t)$ and $\mathscr{E}(\mathbf{r},t)$ vary much more slowly than the exponential factor $\exp(ik_0z-i\omega_0t)$, so that we may make the following slowly varying amplitude approximation:

$$\begin{aligned}&|\partial^2\mathscr{P}/\partial z^2| \ll k_0|\partial\mathscr{P}/\partial z| \ll k_0^2|\mathscr{P}| , \\ &|\partial^2\mathscr{P}/\partial t^2| \ll \omega_0|\partial\mathscr{P}/\partial t| \ll \omega_0^2|\mathscr{P}| ,\end{aligned} \tag{9}$$

with a similar approximation for $\mathscr{E}$. Then on substituting expressions (1), (7), and (8) into the wave equation,

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]E(\mathbf{r},t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}P(\mathbf{r},t) , \tag{10}$$

we obtain

$$\begin{aligned}&\left[\nabla_T^2+2ik_0\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right]\right]\mathscr{E}(\mathbf{r},t) \\ &\quad \simeq -k_0^2\mathscr{P}(\mathbf{r},t) \\ &\quad = -n_0\mu k_0^2\int_{-\infty}^{\infty}[\mathscr{U}_\Delta(\mathbf{r},t)+i\mathscr{V}_\Delta(\mathbf{r},t)]g(\Delta)d\Delta ,\end{aligned} \tag{11}$$

where

$$\nabla_T^2 \equiv (\partial^2/\partial x^2+\partial^2/\partial y^2) \tag{12}$$

is the transverse Laplacian operator.

We now make the weak-field approximation which assumes $\mathscr{E}(\mathbf{r},t)$ weak enough than the population inversion $\mathscr{W}_\Delta$ does not change appreciably from its steady-state value $\mathscr{W}_\Delta^{(0)}=-1$. This amounts to requiring[10] that the "pulse area" be small,

$$\frac{\mu}{\hbar}\int_{-\infty}^{\infty}|\mathscr{E}(\mathbf{r},t)|dt \ll 1 . \tag{13}$$

With this approximation we can replace $\mathscr{W}_\Delta$ by -1 in Eq. (5) which reduces the equation to the linear form:

$$\left[\frac{\partial}{\partial t}+i\Delta+\frac{1}{T_2'}\right][\mathscr{U}_\Delta(\mathbf{r},t)+i\mathscr{V}_\Delta(\mathbf{r},t)]=\frac{i\mu}{\hbar}\mathscr{E}(\mathbf{r},t)\ . \tag{14}$$

Since we shall always deal with traveling-wave problems it is expedient to transform the time variable t to the retarded time $\tau=(t-z/c)$. Under this transformation we have

$$\left[\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right]\to\frac{\partial}{\partial z},\quad \frac{\partial}{\partial t}\to\frac{\partial}{\partial \tau}\ . \tag{15}$$

We may therefore write Eqs. (11) and (14) as

$$\left[\nabla_T^2+2ik_0\frac{\partial}{\partial z}\right]\mathscr{E}(\mathbf{r},\tau)\simeq -n_0\mu k_0^2\int_{-\infty}^{\infty}[\mathscr{U}_\Delta(\mathbf{r},\tau)+i\mathscr{V}_\Delta(\mathbf{r},\tau)]g(\Delta)d\Delta \tag{16}$$

and

$$\left[\frac{\partial}{\partial \tau}+i\Delta+\frac{1}{T_2'}\right][\mathscr{U}_\Delta(\mathbf{r},\tau)+i\mathscr{V}_\Delta(\mathbf{r},\tau)]=\frac{i\mu}{\hbar}\mathscr{E}(\mathbf{r},\tau)\ . \tag{17}$$

These linear equations, when supplemented with the proper initial and boundary conditions, describe the most general propagation of an optical field (which is not too intense) through a resonant medium of two-level atoms.

III. PULSE PROPAGATION IN A SEMI-INFINITE MEDIUM

In order to illustrate the essential physics underlying pulse propagation in semi-infinite resonant media we assume in this section that incoherent relaxation processes are absent and that all atoms are on resonance at the incident frequency ω_0, $g(\Delta)=\delta(\Delta)$.

A. Purely one-dimensional propagation (no transverse effects)

The atomic medium extends to the right of the plane $z=0$. The driving input δ-function pulse has a uniform amplitude in its plane cross section and is incident at $z=0$ at time $t=0$. It travels in the positive sense along the z axis. Since the problem is then one-dimensional, ∇_T^2 can be dropped and Eqs. (16) and (17) with the foregoing assumptions simplify to

$$\frac{\partial}{\partial z}\mathscr{E}(z,\tau)\simeq\frac{ik_0n_0\mu}{2}[\mathscr{U}_0(z,\tau)+i\mathscr{V}_0(z,\tau)] \tag{18}$$

and

$$\frac{\partial}{\partial \tau}[\mathscr{U}_0(z,\tau)+i\mathscr{V}_0(z,\tau)]=\frac{i\mu}{\hbar}\mathscr{E}(z,\tau)\ . \tag{19}$$

By differentiating (18) with respect to τ we can reach a second-order differential equation for $\mathscr{E}(z,\tau)$ with real coefficients. We can then consistently assume $\mathscr{E}$ to be real for all z and τ since the input is real. (This cannot be done in general when transverse effects are allowed for, for the transverse flow of energy causes radial variation of the phase of the field.[11]) It follows then from Eq. (18) that $\mathscr{U}_0=0$ and Eqs. (18) and (19) can be reduced to the form

$$\frac{\partial}{\partial \tau}\mathscr{V}_0(z,\tau)=\frac{\mu}{\hbar}\mathscr{E}(z,\tau) \tag{20}$$

and

$$\frac{\partial^2}{\partial z\partial \tau}\mathscr{V}_0(z,\tau)+\frac{\omega_p^2}{4c}\mathscr{V}_0(z,\tau)=0\ , \tag{21}$$

where

$$\omega_p^2=\frac{2\omega_0n_0\mu^2}{\hbar}\ . \tag{22}$$

Now since the fields propagate in the forward direction the electric field, $\mathscr{E}(z,\tau)$, must vanish at $z=0$ after the passage of the input pulse, i.e., for all $\tau>0$. From Eq. (20) it then follows that

$$\mathscr{V}_0(0,\tau)=\mathscr{V}_0(0,0+)\ . \tag{23}$$

Furthermore if we assume that the input pulse does not attenuate noticeably as it propagates in the medium, we find from Eq. (20)

$$\int_{0-}^{0+}\frac{\partial}{\partial \tau}\mathscr{V}_0(z,\tau)d\tau=\frac{\mu}{\hbar}\int_{0-}^{0+}\mathscr{E}(z,\tau)d\tau\ ,$$

i.e.,

$$\mathscr{V}_0(z,0+)-\mathscr{V}_0(z,0-)=\frac{\mu\mathscr{E}_0}{\hbar}\ , \tag{24}$$

where $\mathscr{E}_0\delta(\tau)$ is the input-pulse amplitude.

If the medium has no polarization prior to the arrival of the input pulse ($\tau<0$), we have $\mathscr{V}_0(z,0-)=0$ and thus

$$\mathscr{V}_0(z,0+)=\frac{\mu\mathscr{E}_0}{\hbar}\ , \tag{25}$$

and according to Eq. (23)

$$\mathscr{V}_0(0,\tau)=\mathscr{V}_0(z,0+)=\frac{\mu\mathscr{E}_0}{\hbar}\ . \tag{26}$$

Equations (20) and (21) can be solved in conjunction with the boundary and initial conditions specified by Eqs. (26). The solution is the familiar result of Burnham and Chiao[4] that follows from the observation that $\mathscr{V}_0(z,\tau)$ can only depend on the product variable $z\tau$ for z and $\tau>0$:

$$\mathscr{V}_0(z,\tau)=\frac{\mu\mathscr{E}_0}{\hbar}J_0((\omega_p/c)\sqrt{cz\tau})\Theta(z)\Theta(\tau)\ . \tag{27}$$

The accompanying solution for the electric field is

$$\mathscr{E}(z,\tau)=\mathscr{E}_0\left[\delta(\tau)-\frac{\omega_p}{2c}\left[\frac{cz}{\tau}\right]^{1/2}\times J_1((\omega_p/c)\sqrt{cz\tau})\Theta(z)\Theta(\tau)\right]\ . \tag{28}$$

In these equations

$$\Theta(x)=\begin{cases}0, & x<0\\ 1, & x>0\end{cases}$$

is the usual Heaviside unit-step function and J_0, J_1 are Bessel functions of the first kind. The oscillations embodied in Eqs. (27) and (28) describe an alternating exchange of energy between the atomic medium and the electromagnetic field that takes place in the wake of the optical pulse.

B. Transversely nonuniform input pulse

The situation to be considered here is the same as the preceding one except that the δ-function input pulse now has a nonuniform x-y dependence as is invariably the case in actual experiments. This complicates the problem considerably since the pulse then undergoes transverse diffractive spreading as it propagates. Thus atoms at different positions experience altogether different driving fields. We can, therefore, no longer drop the ∇_T^2 term from Eq. (16). However, the method of Fresnel transforms[12] can be employed to solve the problem exactly. This method of solution also simplifies the physical discussion of the fluorescent response of the atomic system.

1. Fresnel transforms and the solution procedure

In order to motivate the definition and use of the Fresnel transforms we consider the propagation of electromagnetic fields in vacuum. A Maxwell field $E_0(\mathbf{r},t)$ propagating in vacuum obeys approximately the following integral equation:[13]

$$E_0(\mathbf{r},t)=\int_S \frac{d^2S'}{2\pi c\,|\mathbf{r}-\mathbf{r}'|}\times\frac{d}{dt}E_0(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)\ , \tag{29}$$

where d^2S' is an area element of the plane S given by $z'=\text{const}$. This relation is equivalent to the Rayleigh-Sommerfeld formulation of Huygens's diffraction principle. It is accurate as long as the point $\mathbf{r}$ is many wavelengths distant from the plane.

When Eq. (1) is used to express E_0 in terms of the amplitude $\mathscr{E}_0$ and the latter is assumed to be a slowly varying function of time, Eq. (29) simplifies to

$$\mathscr{E}_0(\mathbf{r},t)\simeq\frac{k_0}{2\pi i}\int\frac{d^2S'}{|\mathbf{r}-\mathbf{r}'|}\mathscr{E}_0(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)\times\exp[ik_0(|\mathbf{r}-\mathbf{r}'|-z)]\ . \tag{30}$$

The integration here is carried out over the plane $z'=0$.

The Fresnel approximation which describes near-forward propagation amounts to

$$|\mathbf{r}-\mathbf{r}'|\simeq\begin{cases}z+\dfrac{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}{2z} & \text{in the phase factor}\\ z & \text{elsewhere}\end{cases}$$

where $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are projections of $\mathbf{r}$ and $\mathbf{r}'$ on the $\mathbf{x}$-$\mathbf{y}$ plane. [For the problem at hand, the Fresnel approximation amounts exactly to the envelope approximation specified, for example, by Eq. (9) that enabled us to get Eq. (11) from Eq. (10). This has been shown in Ref. 14.] With this approximation, we obtain

$$\mathscr{E}_0(\mathbf{r},t)\simeq\frac{k_0}{2\pi iz}\int d^2\rho'\,\mathscr{E}_0(\mathbf{r}',t-z/c)\times\exp\left[\frac{ik_0}{2z}(\boldsymbol{\rho}-\boldsymbol{\rho}')^2\right]\ , \tag{31}$$

where $\mathbf{r}=(\boldsymbol{\rho},z)$, and $\mathbf{r}'=(\boldsymbol{\rho}',0)$.

On making the coordinate transformation, $(t,z)\rightarrow(\tau=t-z/c,\,z)$, Eq. (31) reduces to

$$\mathscr{E}_0(\boldsymbol{\rho},z,\tau)\simeq\frac{k_0}{2\pi iz}\int d^2\rho'\,\mathscr{E}_0(\boldsymbol{\rho}',0,\tau)\exp\left[\frac{ik_0}{2z}(\boldsymbol{\rho}-\boldsymbol{\rho}')^2\right]\ . \tag{32}$$

This expression determines the amplitude of the electric field on any plane z in terms of the amplitude on the plane $z=0$ at the same τ. The relation (32) which thus solves the problem of field propagation in vacuum suggests a technique for solving the same problem in a resonant medium. Quite analogously, corresponding to an arbitrary function $\mathscr{X}(\boldsymbol{\rho}',z,\tau)$ we define a function $\underline{X}(\boldsymbol{\rho}',z,\tau)$ by the relationship

$$\mathscr{X}(\boldsymbol{\rho},z,\tau)=\frac{k_0}{2\pi iz}\int\underline{X}(\boldsymbol{\rho}',z,\tau)\exp\left[\frac{ik_0}{2z}(\boldsymbol{\rho}-\boldsymbol{\rho}')^2\right]d^2\boldsymbol{\rho}'\ . \tag{33}$$

Thus $\underline{X}(\boldsymbol{\rho}',z,\tau)$ which we shall call the Fresnel transform of $\mathscr{X}(\boldsymbol{\rho},z,\tau)$ can be physically interpreted as that field distribution at the input face $z=0$ which, on propagation through a distance z in vacuum, will result in the field $\mathscr{X}(\boldsymbol{\rho},z,\tau)$ under the Fresnel approximation.

It is convenient as an abbreviation to introduce the propagation kernel

$$G(\boldsymbol{\rho}-\boldsymbol{\rho}',z)=\frac{k_0}{2\pi iz}\exp\left[\frac{ik_0(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{2z}\right] \tag{34}$$

in terms of which the Frensel transformation Eq. (33) becomes

$$\mathscr{X}(\boldsymbol{\rho},z,\tau)=\int\underline{X}(\boldsymbol{\rho}',z,\tau)G(\boldsymbol{\rho}-\boldsymbol{\rho}',z)d^2\rho'\ . \tag{35}$$

The feature of the kernel G that makes the transform useful comes from the relation[15]

$$\left[\frac{\partial}{\partial z}+\frac{1}{2ik_0}\nabla_T^2\right]G(\boldsymbol{\rho}-\boldsymbol{\rho}',z)=0\ . \tag{36}$$

When we apply this relation to Eq. (33) we find

$$\left[\frac{\partial}{\partial z}+\frac{1}{2ik_0}\nabla_T^2\right]\mathscr{X}(\boldsymbol{\rho},z,\tau)=\int[\partial\underline{X}(\boldsymbol{\rho}',z,\tau)/\partial z]G(\boldsymbol{\rho}-\boldsymbol{\rho}',z)d^2\rho'\ , \tag{37}$$

which means that the Fresnel transform function $\underline{X}$ is not subject to transverse spreading. We can use this property of the transform to remove the effects of the ∇_T^2 term while solving Eq. (16). As an example we may consider the input field $\mathscr{E}_0$ which by definition propagates in a vacuum, the response of the matter field representing in effect a scattered field. Equation (32) then shows that in the transform space the input field $\mathscr{E}_0$ does not spread at all laterally, i.e., it does not depend explicitly on z. In other words, in the transform space the atoms in all transverse planes see the same exciting field distribution as was present in the input plane $z=0$. It is that simplification which renders the problem soluble.

2. *Details of solution*

Some useful properties of the Fresnel transform are worth noting. First, we observe that the inversion of Eq. (33) is easily accomplished. We may do this by noting that $\mathscr{X}(\boldsymbol{\rho},z,\tau)\exp[-(ik_0/2z)\rho^2]$ is simply the Fourier transform of $\underline{X}(\boldsymbol{\rho}',z,\tau)\exp[(ik_0/2z)\rho'^2]$. After a bit of algebra we obtain easily

$$\underline{X}(\boldsymbol{\rho}',z,\tau)=-\frac{k_0}{2\pi iz}\int \mathscr{X}(\boldsymbol{\rho},z,\tau)\times\exp\left[-\frac{ik_0}{2z}(\boldsymbol{\rho}-\boldsymbol{\rho}')^2\right]d^2\rho$$
$$=\int \mathscr{X}(\boldsymbol{\rho},z,\tau)G(\boldsymbol{\rho}'-\boldsymbol{\rho},-z)d^2\rho\ . \tag{38}$$

Equation (38) differs from Eq. (33) in form only by the sign of z in the kernel, a property which leads to a natural interpretation of the relation. It shows that $\underline{X}(\boldsymbol{\rho}',z,\tau)$ is the field distribution obtained from the distribution $\mathscr{X}(\boldsymbol{\rho},z,\tau)$ by propagating it backward in time in vacuum and through a distance z. There is in other words a certain reciprocity between $\underline{X}(\boldsymbol{\rho}',z,\tau)$ and $\mathscr{X}(\boldsymbol{\rho},z,\tau)$.

Equations (35) and (38) are together equivalent to a completeness relation

$$\int G(\boldsymbol{\rho}-\boldsymbol{\rho}'',z)G(\boldsymbol{\rho}''-\boldsymbol{\rho}',-z)d^2\rho''=\delta^{(2)}(\boldsymbol{\rho}-\boldsymbol{\rho}')\ . \tag{39}$$

We note furthermore that in the limit $z\to 0$ the kernel G becomes in effect a δ function,

$$\lim_{z\to 0} G(\boldsymbol{\rho}-\boldsymbol{\rho}',z)=\delta^{(2)}(\boldsymbol{\rho}-\boldsymbol{\rho}')\ , \tag{40}$$

since propagation through a distance $z=0$ entails no spreading.

We can make another interesting observation about the Fresnel transformation in connection with the problem at hand. A comparison of Eqs. (32) and (33) shows that the z dependence of the transform $\underline{X}(\boldsymbol{\rho}',z,\tau)$ results from the fact that the propagation is taking place through a medium different from the vacuum. We now proceed with the solution.

Since $g(\Delta)=\delta(\Delta)$, $T_2'=\infty$, we can write Eqs. (16) and (17) in the following simplified form:

$$\left[\frac{\partial}{\partial z}+\frac{1}{2ik_0}\nabla_T^2\right]\mathscr{E}(\boldsymbol{\rho},z,\tau)=\frac{ik_0n_0\mu}{2}\mathscr{X}(\boldsymbol{\rho},z,\tau) \tag{41}$$

and

$$\frac{\partial}{\partial\tau}\mathscr{X}(\boldsymbol{\rho},z,\tau)=\frac{i\mu}{\hbar}\mathscr{E}(\boldsymbol{\rho},z,\tau)\ , \tag{42}$$

where we have defined $\mathscr{X}$ by

$$\mathscr{X}(\boldsymbol{\rho},z,\tau)=\mathscr{U}_0(\boldsymbol{\rho},z,\tau)+i\mathscr{V}_0(\boldsymbol{\rho},z,\tau)\ . \tag{43}$$

On substituting Eq. (42) into Eq. (41) we find the single equation for $\mathscr{X}$

$$\frac{\partial}{\partial\tau}\left[\frac{\partial}{\partial z}+\frac{1}{2ik_0}\nabla_T^2\right]\mathscr{X}(\boldsymbol{\rho},z,\tau)+\frac{\omega_p^2}{4c}\mathscr{X}(\boldsymbol{\rho},z,\tau)=0\ . \tag{44}$$

The Fresnel transform of this equation is obtained by multiplying it by $G(\boldsymbol{\rho}'-\boldsymbol{\rho},-z)$ and integrating it over $\boldsymbol{\rho}$. On integrating by parts over $\boldsymbol{\rho}$ the ∇_T^2 term in the equation and recalling Eq. (36) we see that the transform simply obeys the Burnham-Chiao equation (21),

$$\frac{\partial^2}{\partial z\partial\tau}\underline{X}(\boldsymbol{\rho}',z,\tau)+\frac{\omega_p^2}{4c}\underline{X}(\boldsymbol{\rho}',z,\tau)=0\ . \tag{45}$$

Next we must investigate the transformation of the initial condition and the boundary condition. These, as we shall see, simplify too. If

$$\mathscr{E}_0(\boldsymbol{\rho},z,\tau)=\mathscr{E}_0(\boldsymbol{\rho},z)\delta(\tau)$$

is the electric field of the driving pulse its amplitude as we see from Eq. (32) must satisfy the integral equation

$$\mathscr{E}_0(\boldsymbol{\rho},z)=\int \mathscr{E}_0(\boldsymbol{\rho}',0)G(\boldsymbol{\rho}-\boldsymbol{\rho}',z)d^2\rho'\ . \tag{46}$$

By integrating Eq. (42) over τ therefore from $0-$ to $0+$ we find

$$\mathscr{X}(\boldsymbol{\rho},z,0+)-\mathscr{X}(\boldsymbol{\rho},z,0-)=\frac{i\mu}{\hbar}\mathscr{E}_0(\boldsymbol{\rho},z)\ . \tag{47}$$

But $\mathscr{X}(\boldsymbol{\rho},z,0-)=0$ since the medium is quiescent before the arrival of the δ-function pulse. We now multiply Eq. (47) by $G(\boldsymbol{\rho}'-\boldsymbol{\rho},-z)$ and integrate over $\boldsymbol{\rho}$ to find the Fresnel transform of both sides. By recalling the completeness relation (39) we then find the initial condition

$$\underline{X}(\boldsymbol{\rho}',z,0+)=\frac{i\mu}{\hbar}\mathscr{E}_0(\boldsymbol{\rho}',0) \tag{48}$$

which is indeed independent of z as required.

The boundary condition, on the other hand, expresses the fact that at $z=0$ the electric field vanishes for all times $\tau>0$, i.e., no backward-propagating waves are present. With the aid of Eq. (42) we see thus that $\mathscr{X}(\boldsymbol{\rho},0,\tau)$ is then constant,

$$\mathscr{X}(\boldsymbol{\rho},0,\tau)=\mathscr{X}(\boldsymbol{\rho},0,0+)\quad\text{for } \tau>0\ . \tag{49}$$

But since from Eq. (35) and (40)

$$\lim_{z\to 0}\mathscr{X}(\boldsymbol{\rho},z,\tau)=\underline{X}(\boldsymbol{\rho},0,\tau)\ ,$$

Eq. (49) transforms to

$$\underline{X}(\boldsymbol{\rho},0,\tau)=\underline{X}(\boldsymbol{\rho},0,0+)\ . \tag{50}$$

Since $\underline{X}(\boldsymbol{\rho}',z,0+)$ is independent of z according to Eq. (48) we then have

$$\underline{X}(\rho',0,\tau)=\underline{X}(\rho',z,0+)=\frac{i\mu}{\hbar}\mathscr{E}_0(\rho',0)\ . \tag{51}$$

The differential equation (45) together with the conditions (51) constitute the formulation of the propagation problem for the Fresnel-transform function $\underline{X}$. The same analysis as leads to the Burnham-Chiao solution then leads to the more general result

$$\underline{X}(\rho',z,\tau)=\frac{i\mu}{\hbar}\mathscr{E}_0(\rho',0)J_0((\omega_p/c)\sqrt{cz\tau})\Theta(z)\Theta(\tau)\ . \tag{52}$$

The complex polarization $\mathscr{X}(\rho,z,\tau)$ is immediately obtained then by substituting this expression in Eq. (33). The effect is simply to turn $\mathscr{E}_0(\rho',0)$ into $\mathscr{E}_0(\rho,z)$ so that we have finally just

$$\mathscr{X}(\rho,z,\tau)=\frac{i\mu}{\hbar}\mathscr{E}_0(\rho,z)J_0((\omega_0/c)\sqrt{cz\tau})\Theta(z)\Theta(\tau)\ . \tag{53}$$

The expression for the electric field amplitude may then be obtained from Eq. (42). The function $\mathscr{E}_0(\rho,z)$ which appears in Eq. (53) is just the transverse amplitude distribution of the actual propagating input pulse. The diffractive spreading of fields has thus decoupled exactly from the ringing oscillations, and the Bessel-function ringing is once again obtained for this more general case.

3. *Discussion of results*

The propagation problem we have considered may also be approached as a multiple-scattering problem. An incident field $\mathscr{E}_0$ impinges on the medium and induces an oscillating polarization. That polarization radiates a secondary field which in turn induces a secondary polarization, and so on. Although a complicated diffraction process is taking place, the successive higher-order fields in fact maintain a simple relationship to the inducing field $\mathscr{E}_0$.

The first-order polarization at any point (ρ,z) is proportional, for example, to the amplitude $\mathscr{E}_0(\rho,z)$ which is the amplitude the incident pulse had when passing the point (ρ,z). The secondary field radiated by the polarization is the sum of the fields radiated by a succession of slices of the medium lying at smaller values of z. A slice extending from $z'-\Delta z$ to z', for example, radiates a field which at the point (ρ',z') is proportional to the polarization amplitude at the point or in turn to $\mathscr{E}_0(\rho',z')$. In the single-scattering approximation this field propagates in vacuum to the point (ρ,z). The spatial evolution of all of the fields radiated by the different slices is therefore identical to the spatial evolution of the incident pulse. All slices of the medium together contribute a reradiated field at (ρ,z) which is proportional to $\mathscr{E}_0(\rho,z)$. This simple relationship evidently holds to all orders of approximation and explains the decoupling of diffractive spreading from longitudinal propagation which is present in Eq. (53).

IV. PROPAGATION OF ARBITRARY PULSE FIELDS IN THE PRESENCE OF INCOHERENT RELAXATION

We now briefly turn our attention to the more general case in which the atomic dipoles can incoherently dephase by both inhomogeneous and homogeneous broadening processes $[g(\Delta)\neq\delta(\Delta),\ T_2'<\infty]$. Once again we use the Fresnel-transform method to render the problem effectively one dimensional. It can then be solved by Fourier analysis.

We need not in fact take the input pulse field to be a δ-function pulse. We shall still take it to be sufficiently weak, however, to preserve the accuracy of our linearized Eqs. (16) and (17).

We once again define the Fresnel transforms $\underline{E}(\rho',z,\tau)$ and $\underline{X}_\Delta(\rho',z,\tau)$ of $\mathscr{E}(\rho,z,\tau)$ and $\mathscr{X}_\Delta(\rho,z,\tau)\equiv[\mathscr{U}_\Delta(\rho,z,\tau)+i\mathscr{V}_\Delta(\rho,z,\tau)]$, respectively, by the relations

$$\mathscr{E}(\rho,z,\tau)=\int \underline{E}(\rho',z,\tau)G(\rho-\rho',z)d^2\rho' \tag{54}$$

and

$$\mathscr{X}_\Delta(\rho,z,\tau)=\int \underline{X}_\Delta(\rho',z,\tau)G(\rho-\rho',z)d^2\rho'\ . \tag{55}$$

The Fresnel transform of Eq. (16) is quite analogous to that of Eq. (44) and leads to the relation

$$2ik_0\frac{\partial}{\partial z}\underline{E}(\rho',z,\tau)\simeq -n_0\mu k_0^2\int_{-\infty}^{\infty}\underline{X}_\Delta(\rho',z,\tau)g(\Delta)d\Delta\ . \tag{56}$$

The transform of Eq. (17) is likewise analogous to that of Eq. (42) and leads to

$$\left[\frac{\partial}{\partial\tau}+i\Delta+\frac{1}{T_2'}\right]\underline{X}_\Delta(\rho',z,\tau)=\frac{i\mu}{\hbar}\underline{E}(\rho',z,\tau)\ . \tag{57}$$

The Fresnel transform thus reduces the problem to one involving only a single spatial dimension, z. An analysis of the one-dimensional propagation problem in a resonant medium has already been carried out by Crisp.[5] As we shall see, the results of this section represent a simple generalization of his.

We note from Eq. (56) that the field $\underline{E}(\rho',z,\tau)$ in the transform space does not depend on the z coordinate in the absence of the resonant medium $(n_0=0)$. We therefore write

$$\underline{E}(\rho',z,\tau)=\underline{E}_0(\rho',\tau)+\underline{E}_r(\rho',z,\tau)\ , \tag{58}$$

where the z-independent part $\underline{E}_0(\rho',\tau)$ represents the input field (in the absence of the medium) and $\underline{E}_r(\rho',z,\tau)$ is the part of the field radiated by the medium. Since we have assumed no backward-traveling waves are present, the boundary condition on $\underline{E}_r$ must be

$$\underline{E}_r(\rho',0,\tau)=0\ . \tag{59}$$

We may now solve Eqs. (56) and (57) by Fourier analysis. We define the Fourier transform and its inverse for a function $f(\tau)$ such as $\underline{X}_\Delta$ or $\underline{E}$ as

$$f(\tau)=\int_{-\infty}^{\infty}\tilde{f}(\Omega)\exp(-i\Omega\tau)d\Omega\ , \tag{60}$$

$$\tilde{f}(\Omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\tau)\exp(i\Omega\tau)d\tau\ . \tag{61}$$

It is then clear that Eqs. (56) and (57) are equivalent to

$$\frac{\partial}{\partial z}\underline{\tilde{E}}_r(\boldsymbol{\rho}',z,\Omega)=\frac{ik_0n_0\mu}{2}\int_{-\infty}^{\infty}\underline{\tilde{X}}_\Delta(\boldsymbol{\rho}',z,\Omega)g(\Delta)d\Delta \tag{62}$$

and

$$\left[-i(\Omega-\Delta)+\frac{1}{T_2'}\right]\underline{\tilde{X}}_\Delta(\boldsymbol{\rho}',z,\Omega)=\frac{i\mu}{\hbar}[\underline{\tilde{E}}_0(\boldsymbol{\rho}',\Omega)+\underline{\tilde{E}}_r(\boldsymbol{\rho}',z,\Omega)]\ , \tag{63}$$

in which we have introduced the decomposition given by Eq. (58). We can now eliminate $\underline{\tilde{X}}_\Delta(\boldsymbol{\rho}',z,\Omega)$ between Eqs. (62) and (63) to obtain

$$\frac{\partial}{\partial z}\underline{\tilde{E}}_r(\boldsymbol{\rho}',z,\Omega)=-\frac{\omega_p^2}{4c}[\underline{\tilde{E}}_0(\boldsymbol{\rho}',\Omega)+\underline{\tilde{E}}_r(\boldsymbol{\rho}',z,\Omega)]\tilde{A}(\Omega)\ , \tag{64}$$

where the function $\tilde{A}(\Omega)$ is given by

$$\tilde{A}(\Omega)=\int_{-\infty}^{\infty}g(\Delta)\left[\frac{1}{T_2'}+i(\Delta-\Omega)\right]^{-1}d\Delta\ . \tag{65}$$

With aid of Eq. (58) we may rewrite Eq. (64) as

$$\frac{\partial}{\partial z}\ln\underline{\tilde{E}}(\boldsymbol{\rho}',z,\Omega)=-\frac{\omega_p^2}{4c}\tilde{A}(\Omega)\ .$$

The solution which satisfies the boundary condition (59) is then

$$\underline{\tilde{E}}(\boldsymbol{\rho}',z,\Omega)=\underline{\tilde{E}}_0(\boldsymbol{\rho}',\Omega)\exp\left[-\frac{\omega_p^2}{4c}\tilde{A}(\Omega)z\right]\ . \tag{66}$$

On Fourier inversion, therefore, we have

$$\underline{E}(\boldsymbol{\rho}',z,\tau)=\int_{-\infty}^{\infty}\underline{\tilde{E}}(\boldsymbol{\rho}',\Omega)\exp\left[-i\Omega\tau-\frac{\omega_p^2}{4c}\tilde{A}(\Omega)z\right]d\Omega\ . \tag{67}$$

The final result for the electric field amplitude can then be found by an inverse Fresnel transform according to Eq. (54) and is given by

$$\mathscr{E}(\boldsymbol{\rho},z,\tau)=\int_{-\infty}^{\infty}\tilde{\mathscr{E}}_0(\boldsymbol{\rho},z,\Omega)\exp\left[-i\Omega\tau-\frac{\omega_p^2}{4c}\tilde{A}(\Omega)z\right]d\Omega\ , \tag{68}$$

where

$$\tilde{\mathscr{E}}_0(\boldsymbol{\rho},z,\Omega)=\int\underline{\tilde{E}}_0(\boldsymbol{\rho}',\Omega)G(\boldsymbol{\rho}-\boldsymbol{\rho}',z)d^2\rho' \tag{69}$$

is the Fourier transform of the amplitude $\mathscr{E}_0(\boldsymbol{\rho},z,\tau)$ of the propagating input field. The exact decoupling of diffraction and the longitudinal propagation thus holds even in this more general setting.

It is convenient, as the structure of Eq. (68) indicates, to define a function

$$F(z,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[-i\Omega\tau-\frac{\omega_p^2}{4c}\tilde{A}(\Omega)z\right]d\Omega\ , \tag{70}$$

so that by using the convolution theorem in Eq. (68) the general solution for the field amplitude $\mathscr{E}(\boldsymbol{\rho},z,\tau)$ may be written in terms of the input field $\mathscr{E}_0$ as

$$\mathscr{E}(\boldsymbol{\rho},z,\tau)=\int_{-\infty}^{\infty}\mathscr{E}_0(\boldsymbol{\rho},z,\tau')F(z,\tau-\tau')d\tau'\ . \tag{71}$$

For the special case of a Lorentzian distribution of atomic frequencies given by

$$g(\Delta)=\frac{(\pi T_2^*)^{-1}}{[\Delta^2+(T_2^*)^{-2}]}$$

the function $\tilde{A}(\Omega)$ reduces to

$$\tilde{A}(\Omega)=\frac{i}{(\Omega+iT_2^{-1})}\ ,$$

where

$$T_2^{-1}=(T_2')^{-1}+(T_2^*)^{-1}\ .$$

The function $F(z,\tau)$ may then be evaluated by shifting the integration contour in the Ω plane by iT_2^{-1} and is given by

$$\begin{aligned}F(z,\tau)&=\frac{1}{2\pi}e^{-\tau/T_2}\int_{-\infty+iT_2^{-1}}^{\infty+iT_2^{-1}}\exp\left[-i\nu\tau-\frac{i\omega_p^2z}{4c\nu}\right]d\nu\\&=e^{-\tau/T_2}\frac{\partial}{\partial\tau}\left[\frac{i}{2\pi}\int_{-\infty+iT_2^{-1}}^{\infty+iT_2^{-1}}\exp\left[-i\nu\tau-\frac{i\omega_p^2}{4c\nu}z\right]\frac{d\nu}{\nu}\right]\ .\end{aligned}$$

The latter integral is a representation[16] of the Bessel function $J_0((\omega_p/c)\sqrt{cz\tau})$ for $\tau>0$ and vanishes for $\tau<0$, so that

$$\begin{aligned}F(z,\tau)&=e^{-\tau/T_2}\frac{\partial}{\partial\tau}[J_0((\omega_p/c)\sqrt{cz\tau})\Theta(\tau)]\\&=\delta(\tau)-\frac{\omega_p}{2c}\left[\frac{cz}{\tau}\right]^{1/2}J_1((\omega_p/c)\sqrt{cz\tau})e^{-\tau/T_2}\Theta(\tau)\ .\end{aligned} \tag{72}$$

For the Lorentzian detuning distribution therefore, the final solution for the electric field amplitude is

$$\mathscr{E}(\boldsymbol{\rho},z,\tau)=\mathscr{E}_0(\boldsymbol{\rho},z,\tau)-\frac{\omega_p}{2c}\sqrt{cz}\int_{-\infty}^{\tau}\mathscr{E}_0(\boldsymbol{\rho},z,\tau')\frac{J_1((\omega_p/c)[cz(\tau-\tau')]^{1/2})}{(\tau-\tau')^{1/2}}\exp\left[-\frac{(\tau-\tau')}{T_2}\right]d\tau'\ . \tag{73}$$

Clearly, the second term above represents the response of the atomic medium to the input field $\mathscr{E}_0(\boldsymbol{\rho},z,\tau)$.

ACKNOWLEDGMENTS

This article is based on the dissertation submitted by S. P. in partial fulfillment of the requirements for the Ph.D. degree, Department of Physics, Harvard University, May 1983. This work was supported in part by the U.S. Department of Energy under Contract No. DE-AC02-76ER03064, and by the U.S. Air Force Office of Scientific Research.

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