Action-angle variables and fluctuation-dissipation relations for a driven quantum oscillator

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For a classical oscillator subject to a time-dependent perturbation, the average change in the action variable $\langle \Delta J \rangle = \langle J - J_0 \rangle$ is related to its fluctuation $\langle (\Delta J)^2 \rangle$ by $\langle \Delta J \rangle = \frac{1}{2}\partial \langle (\Delta J)^2 \rangle / \partial J_0$, where J_0 is the initial value of J and the average is over the initial value θ_0 of the angle variable. In this paper, the quantum-mechanical generalization of this relation is derived, and we discuss the correspondence between our work and the usual treatment of the fluctuation-dissipation theorem for a system which initially is described by a canonical distribution.

I. INTRODUCTION

For a classical oscillator described by the Hamiltonian $H_0(p,q)$, the action variable,[1]

$$2\pi J = \oint p\, dq \; , \tag{1.1}$$

is a constant of the motion and the angular oscillation frequency is

$$\omega_0(J) = dH_0(J)/dJ \; . \tag{1.2}$$

If an external time-dependent perturbation is applied, the system is described by the Hamiltonian $H_0(p,q)+V(p,q,t)$, and the action variable is no longer constant. When calculated to second order in the interaction V, the average drift $\langle \Delta J \rangle = \langle J(t) - J_0 \rangle$ and fluctuation $\langle (\Delta J)^2 \rangle$ are found to be related by[2,3]

$$\langle \Delta J \rangle = \frac{1}{2}\frac{\partial}{\partial J_0}\langle (\Delta J)^2 \rangle \; , \tag{1.3}$$

where J_0 is the initial value of the action variable and the average is over the initial value θ_0 of the conjugate angle variable.

In this paper the quantum-mechanical generalization of Eq. (1.3) is derived. We consider a quantum oscillator described by the Hamiltonian

$$H = H_0 + V(t) \; , \tag{1.4}$$

and suppose H_0 to have a nondegenerate, discrete spectrum. The orthonormal complete set of eigenvectors of H_0 are denoted by $|n\rangle$, $n=0,1,2,\ldots,$ and we assume that initially at time $t=0$ the oscillator is in a coherent state[4]

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \; . \tag{1.5}$$

Writing

$$\alpha = \sqrt{J}e^{i\theta} \; , \tag{1.6}$$

the average of an operator O is defined to be

$$\langle O \rangle \equiv \int_0^{2\pi}\frac{d\theta}{2\pi}\langle \alpha | O | \alpha \rangle = \sum_n \rho_n(J)\langle n | O | n \rangle \; , \tag{1.7}$$

where

$$\rho_n(J) = \frac{J^n}{n!}e^{-J} \tag{1.8}$$

is the Poisson distribution. Corresponding to the action variable in classical mechanics is the number operator N_0 defined by:

$$N_0|n\rangle = n|n\rangle \; . \tag{1.9}$$

Working in the Heisenberg representation, we determine its time dependence to second order in V,

$$N_0(t) = N_0 + N_1(t) + N_2(t) + \cdots \; . \tag{1.10}$$

Then, we find (taking $\hbar=1$)

$$\langle N_1^2(t) \rangle = \sum_{m,n}(m-n)^2\rho_n(J)B_{mn}(t) \; , \tag{1.11}$$

$$\langle N_2(t) \rangle = \sum_{m,n}(m-n)\rho_n(J)B_{mn}(t) \; , \tag{1.12}$$

where $B_{mn}(t)$ is the transition probability between states m and n. Defining

$$T_p = \sum_n \rho_n(J)B_{n+p,n}(t) \; , \tag{1.13}$$

the above equations can be rewritten as

$$\langle N_1^2(t) \rangle = \sum_{p\,(>0)}\langle N_1^2(t)\rangle_p = \sum_{p\,(>0)} p^2(T_p + T_{-p}) \tag{1.14}$$

and

$$\langle N_2(t) \rangle = \sum_{p\,(>0)}\langle N_2(t)\rangle_p = \sum_{p\,(>0)} p(T_p - T_{-p}) \; . \tag{1.15}$$

The symmetry of the transition probability,

$$B_{mn}(t) = B_{nm}(t) \; , \tag{1.16}$$

together with the identity

$$D_p\rho_n(J) \equiv (1+\partial/\partial J)^p\rho_n(J) = \rho_{n-p}(J) \;\; (n \ge p) \tag{1.17}$$

imply

$$D_p T_{-p} = T_p \; , \tag{1.18}$$

which is the fluctuation-dissipation relation we are seeking. Applying Eq. (1.18) to Eqs. (1.14) and (1.15) results in

$$\langle N_2(t)\rangle_p=\left[\frac{1}{2}\frac{\partial}{\partial J}-\frac{1}{4}\frac{\partial^2}{\partial J^2}-\frac{p^2-4}{24}\frac{\partial^3}{\partial J^3}+\cdots\right]\langle N_1^2(t)\rangle_p \ , \tag{1.19}$$

the quantum-mechanical generalization of the classical result presented in Eq. (1.3).

In Sec. II we review the classical oscillator. We emphasize that the derivation of Eq. (1.3) rests on the same property of the Poisson bracket as does the fluctuation-dissipation theorem[5] often quoted for the canonical ensemble. We also review the connection between Eq. (1.3) and the Einstein relation in the classical theory of Brownian motion.[6]

The quantum oscillator is discussed in Sec. III, where Eqs. (1.18) and (1.19) are derived in detail. After this is accomplished, we consider the special case of a perturbing potential of the form $V(t)=V\eta(t)$, where V is a time-independent operator and $\eta(t)$ is a scalar function of time. In this case we are able to discuss the drift and fluctuation of the energy as well as of the occupation number, by use of Fourier transforms. Next we consider the unperturbed Hamiltonian to be harmonic,

$$H_0=\Omega(N_0+\tfrac{1}{2}) \ , \tag{1.20}$$

and show that in this case our results for an initial Poisson distribution are related by a Laplace transformation to the fluctuation-dissipation theorem which applies when the initial state is described by a canonical distribution.

Since in classical mechanics there is a precise correspondence[3] between the fluctuation-dissipation relation of Eq. (1.3) and the Madey gain-spread theorem[7] for the free-electron laser, it is reasonable to hope that the work presented in this paper (or a generalization of it) will be of relevance to the quantum-mechanical treatment of the free-electron laser.

II. CLASSICAL OSCILLATOR

We consider a driven oscillator described by the Hamiltonian

$$H=H_0(J)+V(J,\theta,t) \ , \tag{2.1}$$

where J and θ are the action angle variables.[1] The potential $V(J,\theta,t)$ is periodic in θ with period 2π, and the equations of motion are

$$\dot{J}=-\frac{\partial V}{\partial\theta} \ , \tag{2.2}$$

$$\dot{\theta}=\omega_0(J)+\frac{\partial V}{\partial J} \ , \tag{2.3}$$

where $\omega_0(J)=dH_0/dJ$ is the oscillation frequency of the unperturbed oscillator. The equations of motion can be solved by perturbation theory, and to second order in V the solution is

$$J(t)=J_0+J_1(t)+J_2(t) \ , \tag{2.4}$$

$$\theta(t)=\theta_0+\omega_0(J_0)t+\theta_1(t)+\theta_2(t) \ , \tag{2.5}$$

with

$$J_1(t)=-\int_0^t dt'\frac{\partial V_I(t')}{\partial\theta_0} \ , \tag{2.6}$$

$$J_2(t)=-\int_0^t dt'\int_0^{t'} dt''\left\{\frac{\partial V_I(t')}{\partial\theta_0},\ V_I(t'')\right\} . \tag{2.7}$$

We shall have no need for $\theta_1(t)$ and $\theta_2(t)$, so we shall not give their explicit forms. In the above Eqs. (2.6) and (2.7) we have used the simplified notation

$$V_I(t)\equiv V(J_0,\theta_0+\omega_0(J_0)t,t) \ , \tag{2.8}$$

and the Poisson bracket $\{A,B\}$ is

$$\{A,B\}=\frac{\partial A}{\partial\theta_0}\frac{\partial B}{\partial J_0}-\frac{\partial A}{\partial J_0}\frac{\partial B}{\partial\theta_0} \ . \tag{2.9}$$

The fluctuation-dissipation relations[5] are a consequence of the identity

$$\{A,B\}=\frac{-\partial}{\partial J_0}\left[A\frac{\partial B}{\partial\theta_0}\right]+\frac{\partial}{\partial\theta_0}\left[A\frac{\partial B}{\partial J_0}\right] . \tag{2.10}$$

Introducing the average $\langle\ \rangle$ defined by

$$\langle O\rangle=\int_0^{2\pi}\frac{d\theta_0}{2\pi}O(J_0,\theta_0,t) \ , \tag{2.11}$$

we see that when A and B are periodic in θ_0 with period 2π, then

$$\langle\{A,B\}\rangle=\frac{-\partial}{\partial J_0}\left\langle A\frac{\partial B}{\partial\theta_0}\right\rangle . \tag{2.12}$$

The fluctuation-dissipation relations follow directly from Eq. (2.12). Taking $A=\partial V_I(t')/\partial\theta_0$ and $B=V_I(t'')$, Eqs. (2.6) and (2.7) together with (2.12) show that

$$\langle J_2(t)\rangle=\frac{1}{2}\frac{\partial}{\partial J_0}\langle J_1^2(t)\rangle \ , \tag{2.13}$$

which is the desired result.

Instead of the average defined in Eq. (2.11), consider the average over the canonical ensemble,

$$\langle O\rangle_c=Z^{-1}\int_0^{\infty}dJ_0\int_0^{2\pi}\frac{d\theta_0}{2\pi}O(J_0,\theta_0,t)e^{-\beta H_0(J_0)} \ , \tag{2.14}$$

where Z is the partition function

$$Z=\int_0^{\infty}dJ\,e^{-\beta H_0(J)} \ . \tag{2.15}$$

The averages of Eqs. (2.11) and (2.14) are related by

$$\langle O\rangle_c=Z^{-1}\int_0^{\infty}dJ\,e^{-\beta H_0(J)}\langle O\rangle \ . \tag{2.16}$$

Assuming A and B to vanish at $J_0=0$, Eqs. (2.12) and (2.16) imply

$$\langle\{A,B\}\rangle_c=\left\langle-\beta\omega_0(J_0)A\frac{\partial B}{\partial\theta_0}\right\rangle_c \ . \tag{2.17}$$

Let us now restrict our attention to the special form of the perturbing potential,

$$V(J,\theta,t)=v(J,\theta)\eta(t) \ , \tag{2.18}$$

where $\eta(t)$ does not depend upon J and θ. Then

$V_I(t)=v_I(t)\eta(t)$, where

$$v_I(t)=v(J_0,\theta_0+\omega_0(J_0)t) . \tag{2.19}$$

Taking $A=v_I(t)$ and $B=v_I(t')$ in Eq. (2.17), we find

$$\langle\{v_I(t),v_I(t')\}\rangle_c=-\beta\langle v_I(t)\dot{v}_I(t')\rangle_c , \tag{2.20}$$

where $\dot{v}_I(t)=dv_I(t)/dt=\omega_0(J_0)\,\partial v_I(t)/\partial\theta$. Equation (2.20) is the fluctuation-dissipation relation as usually stated in the canonical ensemble.[5]

Before proceeding to the discussion of the quantum-mechanical oscillator in the next section, let us briefly review another aspect of the fluctuation-dissipation relation (2.13) which manifests itself in the discussion of stochastic processes using the Fokker-Planck equation.[6] If we include the effects of a frictional force, the equations of motion become

$$\dot{q}=\frac{\partial H_0}{\partial p}+\frac{\partial V}{\partial p} , \tag{2.21}$$

$$\dot{p}=-\frac{\partial H_0}{\partial q}-\frac{\partial V}{\partial q}-\gamma p , \tag{2.22}$$

or in terms of the action-angle variables,

$$\dot{J}=-\frac{\partial V}{\partial\theta}-\gamma p\frac{\partial q}{\partial\theta} , \tag{2.23}$$

$$\dot{\theta}=\omega_0(J)+\frac{\partial V}{\partial J}+\gamma p\frac{\partial q}{\partial J} , \tag{2.24}$$

where of course p and q are now expressed as functions of J and θ. If the potential V has the proper stochastic character then the average motion on a long time scale is described by the Fokker-Planck equation[2]

$$\frac{\partial\psi}{\partial t}=-\frac{\partial}{\partial J}(A_1\psi)+\frac{1}{2}\frac{\partial^2}{\partial J^2}(A_2\psi) ,$$

where $\psi(J,t)dJ$ is the probability of finding the action variable between J and $J+dJ$ at time t. The coefficients are given by

$$A_1=\langle\!\langle \Delta J/\Delta t\rangle\!\rangle , \tag{2.25}$$

$$A_2=\langle\!\langle (\Delta J)^2/\Delta t\rangle\!\rangle , \tag{2.26}$$

where the double bracket indicates an average over the initial value of θ and over the stochastic ensemble. In this case the fluctuation-dissipation relation becomes

$$A_1=\frac{1}{2}\frac{\partial A_2}{\partial J}-\gamma J , \tag{2.27}$$

so the Fokker-Planck equation reduces to

$$\frac{\partial\psi}{\partial t}=\frac{1}{2}\frac{\partial}{\partial J}\left[A_2\frac{\partial\psi}{\partial J}\right]+\gamma\frac{\partial}{\partial J}(J\psi) . \tag{2.28}$$

A sufficient condition for $\psi_{\rm eq}$ to be a time-independent solution of Eq. (2.28) is

$$\frac{1}{2}A_2\frac{\partial\psi_{\rm eq}}{\partial J}+\gamma J\psi=0 . \tag{2.29}$$

Let us write

$$\psi_{\rm eq}(J)=e^{-S(J)} , \tag{2.30}$$

then Eq. (2.29) implies

$$S(J)=\int\frac{2\gamma J dJ}{A_2(J)} . \tag{2.31}$$

In the special case when the external potential $V=-q\xi(t)$, where $\xi(t)$ is a stochastic function with a white-noise spectrum,

$$\langle\xi(t)\rangle=0 , \tag{2.32}$$

$$\langle\xi(t)\xi(t')\rangle=2D\delta(t-t') , \tag{2.33}$$

then

$$A_2(J)=2DJ/\omega_0(J) , \tag{2.34}$$

hence[8]

$$\psi_{\rm eq}(J)=e^{-\beta H_0(J)} , \tag{2.35}$$

with

$$\beta=\gamma/D . \tag{2.36}$$

If the equilibrium is described by a canonical ensemble with inverse temperature β, then Eq. (2.36) is the Einstein relation.[6]

III. QUANTUM OSCILLATOR

We consider a driven quantum-mechanical oscillator described by the Hamiltonian

$$H=H_0+V(t) . \tag{3.1}$$

The unperturbed Hamiltonian H_0 has a nondegenerate, discrete spectrum and its orthonormal complete set of eigenstates are denoted $|n\rangle$, $n=0,1,2,\ldots$. For ease of notation we set $\hbar=1$ and denote the eigenvalues of H_0 by $\omega_0<\omega_1<\omega_2<\cdots$,

$$H_0|n\rangle=\omega_n|n\rangle . \tag{3.2}$$

In the Heisenberg representation, the time evolution of H_0 is given to second order in $V(t)$ by

$$H_0(t)=H_0+H_1(t)+H_2(t)+\cdots \tag{3.3}$$

with

$$H_1(t)=-i\int_0^t dt'[H_0,V_I(t')] , \tag{3.4}$$

$$H_2(t)=-\int_0^t dt'\int_0^{t'}dt''[[H_0,V_I(t')],V_I(t'')] , \tag{3.5}$$

and

$$V_I(t)=e^{iH_0t}V(t)e^{-iH_0t} . \tag{3.6}$$

The commutator $[A,B]=AB-BA$.

Let us introduce coherent states[4]

$$|\alpha\rangle=e^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{\sqrt{n!}}|n\rangle , \tag{3.7}$$

then writing

$$\alpha=\sqrt{J}e^{i\theta} , \tag{3.8}$$

we define the average of an operator O by

$$\langle O\rangle=\int_0^{2\pi}\frac{d\theta}{2\pi}\langle\alpha\,|\,O\,|\,\alpha\rangle\ . \tag{3.9}$$

From Eq. (3.7) it is seen that

$$\langle O\rangle=\sum_n \rho_n(J)\langle n\,|\,O\,|\,n\rangle\ , \tag{3.10}$$

where

$$\rho_n(J)=\frac{J^n}{n!}e^{-J}$$

is the Poisson distribution. Corresponding to the action variable in classical mechanics is the number operator N_0 defined by

$$N_0\,|\,n\rangle=n\,|\,n\rangle\ , \tag{3.11}$$

and it is easily seen that

$$\langle N_0\rangle=J\ , \tag{3.12}$$

$$\langle(N_0-\langle N_0\rangle)^2\rangle=J\ , \tag{3.13}$$

and

$$\langle H_0\rangle=\sum_n \omega_n\rho_n(J)=E_0(J)\ . \tag{3.14}$$

The transition probability between states m and n is

$$B_{mn}(t)=\left|\int_0^t dt'\langle m\,|\,V_I(t')\,|\,n\rangle\right|^2\ , \tag{3.15}$$

and one finds the energy moments

$$\langle H_1^2(t)\rangle=\sum_{m,n}(\omega_m-\omega_n)^2\rho_n(J)B_{mn}(t)\ , \tag{3.16}$$

$$\langle H_2(t)\rangle=\sum_{m,n}(\omega_m-\omega_n)\rho_n(J)B_{mn}(t)\ . \tag{3.17}$$

The time dependence of the number operator can be determined in the same manner as that of H_0, and to second order in $V(t)$,

$$N_0(t)=N_0+N_1(t)+N_2(t)+\cdots\ , \tag{3.18}$$

$$\langle N_1^2(t)\rangle=\sum_{m,n}(m-n)^2\rho_n(J)B_{mn}(t)\ , \tag{3.19}$$

$$\langle N_2(t)\rangle=\sum_{m,n}(m-n)\rho_n(J)B_{mn}(t)\ . \tag{3.20}$$

Defining

$$T_p=\sum_n \rho_n(J)B_{n+p,n}(t)\ , \tag{3.21}$$

Eqs. (3.19) and (3.20) can be rewritten in the form

$$\langle N_1^2(t)\rangle=\sum_{p\,(>0)}\langle N_1^2(t)\rangle_p=\sum_{p\,(>0)}p^2(T_p+T_{-p})\ , \tag{3.22}$$

$$\langle N_2(t)\rangle=\sum_{p\,(>0)}\langle N_2(t)\rangle_p=\sum_{p\,(>0)}p(T_p-T_{-p})\ . \tag{3.23}$$

We do not explicitly state any restriction on the sum over p except $p>0$, but consider $B_{mn}(t)$ to vanish if either m or n becomes negative.

Introducing the differential operator

$$D_p=(1+\partial/\partial J)^p\ , \tag{3.24}$$

we note that

$$D_p\rho_n(J)=\rho_{n-p}(J),\quad n\geq p\ . \tag{3.25}$$

Equation (3.25) together with the symmetry of the transition probability,

$$B_{mn}(t)=B_{nm}(t)\ , \tag{3.26}$$

yields the fluctuation-dissipation relation

$$D_pT_{-p}=T_p\ . \tag{3.27}$$

Using Eq. (3.27) in Eqs. (3.22) and (3.23) shows that

$$\langle N_2(t)\rangle_p=\left[\frac{1}{2}\frac{\partial}{\partial J}-\frac{1}{4}\frac{\partial^2}{\partial J^2}-\frac{p^2-4}{24}\frac{\partial^3}{\partial J^3}+\cdots\right]\langle N_1^2(t)\rangle_p\ , \tag{3.28}$$

which is the generalization in quantum mechanics of the classical result of Eq. (2.13)

$$\langle J_2(t)\rangle=\frac{1}{2}\frac{\partial}{\partial J}\langle J_1^2(t)\rangle\ .$$

When $\langle N_1^2(t)\rangle_p$ are polynomials in J, the leading terms in the classical limit $J\rightarrow\infty$ satisfy

$$\langle N_2(t)\rangle_p=\frac{1}{2}\frac{\partial}{\partial J}\langle N_1^2(t)\rangle_p\ . \tag{3.29}$$

We note that the coefficient of the second-order derivative term in Eq. (3.28) is independent of p, but the coefficients of the third- and higher-order derivatives are different for different values of p.

In order to make contact with the literature on the fluctuation-dissipation theorem,[5] let us restrict our attention to a Hamiltonian of the form

$$H=H_0+V\eta(t)\ , \tag{3.30}$$

where V is now a time-independent operator and $\eta(t)$ is a scalar function of time. We define

$$V_I(t)=e^{iH_0t}Ve^{-iH_0t}\ , \tag{3.31}$$

and rewrite Eqs. (3.4) and (3.5) as

$$\langle H_1^2(t)\rangle=\int_0^t dt'\int_0^t dt''\,\eta(t')\eta(t'')\langle \dot V_I(t')\dot V_I(t'')\rangle \tag{3.32}$$

and

$$\langle H_2(t)\rangle=i\int_0^t dt'\int_0^{t'}dt''\,\eta(t')\eta(t'')\times\langle[\dot V_I(t'),V_I(t'')]\rangle\ . \tag{3.33}$$

Now we introduce the Fourier transform $S(\omega)$ by

$$\langle[V_I(t)-\langle V_I(t)\rangle][V_I(t')-\langle V_I(t')\rangle]\rangle=\int d\omega\,S(\omega)e^{-i\omega(t-t')}\ , \tag{3.34}$$

from which it follows that

$$\langle[V_I(t),V_I(t')]\rangle=\int d\omega[S(\omega)-S(-\omega)]e^{-i\omega(t-t')}\ . \tag{3.35}$$

Since

$$S(\omega)=\sum_p S_p(\omega) \tag{3.36}$$

with

$$S_p(\omega)=\sum_n \rho_n(J)\,|\langle n\,|\,V\,|\,n+p\rangle|^2\delta(\omega-\omega_{n+p}+\omega_n)\ , \tag{3.37}$$

we can derive

$$\langle H_1^2(t)\rangle=\int_0^\infty d\omega\, b(\omega,t)\omega^2 \sum_{p\,(>0)}[S_p(\omega)+S_{-p}(-\omega)]\ , \tag{3.38}$$

$$\langle H_2(t)\rangle=\int_0^\infty d\omega\, b(\omega,t)\omega \sum_{p\,(>0)}[S_p(\omega)-S_{-p}(-\omega)]\ , \tag{3.39}$$

where

$$b(\omega,t)=\left|\int_0^t dt'\,\eta(t')e^{-i\omega t'}\right|^2 . \tag{3.40}$$

As stated earlier following Eq. (3.23), we consider matrix elements $\langle m\,|\,V\,|\,n\rangle$ to vanish when m or n become negative.

From the definition of $S_p(\omega)$ in Eq. (3.37) and the definition of D_p in Eq. (3.24), we derive the fluctuation-dissipation relation

$$D_pS_{-p}(-\omega)=S_p(\omega)\ . \tag{3.41}$$

Equation (3.41) implies

$$S_p(\omega)-S_{-p}(-\omega)=\left[\frac{p}{2}\frac{\partial}{\partial J}-\frac{p}{4}\frac{\partial^2}{\partial J^2}-\frac{p(p^2-4)}{24}\frac{\partial^3}{\partial J^3}+\cdots\right][S_p(\omega)+S_{-p}(-\omega)]\ , \tag{3.42}$$

relating $S_p(\omega)-S_{-p}(-\omega)$ which determines the dissipation $\langle H_2(t)\rangle$ to $S_p(\omega)+S_{-p}(-\omega)$ which determines the fluctuation $\langle H_1^2(t)\rangle$. The spectral relations for the number operator are

$$\langle N_1^2(t)\rangle=\int_0^\infty d\omega\, b(\omega,t)\sum_{p\,(>0)} p^2[S_p(\omega)+S_{-p}(-\omega)]\ , \tag{3.43}$$

$$\langle N_2(t)\rangle=\int_0^\infty d\omega\, b(\omega,t)\sum_{p\,(>0)} p[S_p(\omega)-S_{-p}(-\omega)]\ . \tag{3.44}$$

These equations are, of course, consistent with Eqs. (3.22) and (3.23) discussed earlier.

Additional insight is obtained by considering the special case when the unperturbed system is an harmonic oscillator,

$$H_0=\Omega(N_0+\tfrac{1}{2})\ . \tag{3.45}$$

In this example, $\omega_{n+p}-\omega_n=p\Omega$, so

$$S_p(\omega)=S_p\delta(\omega-p\Omega)\ , \tag{3.46}$$

with

$$S_p=\sum_n \rho_n(J)\,|\langle n\,|\,V\,|\,n+p\rangle|^2\ . \tag{3.47}$$

Then

$$\langle N_1^2(t)\rangle=\sum_p b(p\Omega,t)p^2(S_p+S_{-p})\ , \tag{3.48}$$

$$\langle N_2(t)\rangle=\sum_p b(p\Omega,t)p(S_p-S_{-p})\ , \tag{3.49}$$

and

$$D_pS_{-p}=S_p\ . \tag{3.50}$$

For a dipole interaction, $V=a+a^\dagger$, with $a^\dagger$ and a being the usual raising and lowering operators satisfying $[a,a^\dagger]=1$ we find

$$\langle N_1^2(t)\rangle=b(\Omega,t)(2J+1)\ , \tag{3.51}$$

$$\langle N_2(t)\rangle=b(\Omega,t)\ . \tag{3.52}$$

Since $\langle N_1^2(t)\rangle$ is linear in J, the higher derivative terms in Eq. (3.28) vanish, so the classical relation (3.29) is valid.

When $V=(a+a^\dagger)^2$, then

$$\langle N_1^2(t)\rangle=b(2\Omega,t)(8J^2+16J+8)\ , \tag{3.53}$$

$$\langle N_2(t)\rangle=b(2\Omega,t)(8J+4)\ . \tag{3.54}$$

Since $\langle N_1^2(t)\rangle$ is quadratic in J, the third- and higher-order derivatives in Eq. (3.28) vanish, so

$$\langle N_2(t)\rangle=\left[\frac{1}{2}\frac{\partial}{\partial J}-\frac{1}{4}\frac{\partial^2}{\partial J^2}\right]\langle N_1^2(t)\rangle\ . \tag{3.55}$$

Finally, when $V=(a+a^\dagger)^3$, then

$$\langle N_1^2(t)\rangle= b(\Omega,t)9(2J^3+9J^2+8J+1) +b(3\Omega,t)9(2J^3+9J^2+18J+6)\ , \tag{3.56}$$

$$\langle N_2(t)\rangle= b(\Omega,t)9(3J^2+6J+1) +b(3\Omega,t)9(3J^2+6J+2)\ . \tag{3.57}$$

In this case the third derivative term in Eq. (3.28) must be kept, and since the coefficient of $\partial^3/\partial J^3$ is not simply proportional to p, distinct differential relationships hold for $p=1$ and $p=3$.

We shall now conclude this section by demonstrating the equivalence of Eq. (3.50) to the fluctuation-dissipation relation as usually stated for the canonical ensemble. The expectation value $\langle\ \rangle_c$ in the canonical ensemble is

$$\langle V_I(t)V_I(t')\rangle_c = Z^{-1}\mathrm{Tr}[e^{-\beta H_0}V_I(t)V_I(t')] , \quad (3.58)$$

where Z is the partition function and $\beta=1/kT$. In this ensemble the fluctuation-dissipation relation is[5]

$$\langle V_I(t)V_I(t')\rangle_c = \langle V_I(t')V_I(t+i\beta)\rangle_c . \quad (3.59)$$

Introducing the Fourier transform $F(\omega)$ by

$$\langle [V_I(t)-\langle V_I(t)\rangle][V_I(t')-\langle V_I(t')\rangle]\rangle_c = \int d\omega\, F(\omega)e^{-i\omega(t-t')} , \quad (3.60)$$

we see that Eq. (3.59) implies

$$F(\omega)=e^{\beta\omega}F(-\omega) . \quad (3.61)$$

When H_0 is the harmonic oscillator Hamiltonian of Eq. (3.32), then

$$F(\omega)=F_p\delta(\omega-p\Omega) , \quad (3.62)$$

where

$$F_p=e^{\beta\Omega p}F_{-p} . \quad (3.63)$$

Using the diagonal representation,[4] we can write

$$\frac{1}{Z}e^{-\beta H_0} = \int \frac{d^2\alpha}{\pi}P(J)\,|\alpha\rangle\langle\alpha| , \quad (3.64)$$

where the coherent states $|\alpha\rangle$ were defined in Eq. (3.7), and then it follows that

$$\frac{1}{Z}e^{-\beta\omega_n} = \int_0^\infty dJ\, P(J)\rho_n(J) . \quad (3.65)$$

Therefore, the canonical average defined in Eq. (3.58) is related to the Poisson average of Eq. (3.10) by

$$\langle O\rangle_c = \int_0^\infty dJ\, P(J)\langle O\rangle , \quad (3.66)$$

which is the quantum-mechanical generalization of Eq. (2.16). For the harmonic oscillator Hamiltonian of Eq. (3.45), one knows[4]

$$P(J)=ye^{-yJ} \quad (3.67)$$

with

$$y=e^{\beta\Omega}-1 . \quad (3.68)$$

Therefore,

$$F_p=\int_0^\infty dJ\, ye^{-yJ}S_p , \quad (3.69)$$

so Eq. (3.63) becomes

$$\int_0^\infty dJ\, e^{-yJ}S_p = \int_0^\infty dJ(1+y)^p e^{-yJ}S_{-p} . \quad (3.70)$$

From its definition in Eq. (3.47), it is clear that

$$\left.\frac{\partial^k S_{-p}}{\partial J^k}\right|_{J=0} = 0 \text{ for } 0\le k<p , \quad (3.71)$$

so Eq. (3.70) is simply the Laplace transform of the relation $S_p=D_pS_{-p}$ of Eq. (3.50).

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[1]See, e.g., H. Goldstein, *Classical Mechanics* (Addison-Wesley, Boston, 1950).

[2]Equation (1.3) has recently been discussed by D. Boussard, G. Dome, and C. Graziani, in Proceedings of the XIth International Conference on High Energy Accelerators, Geneva, 1980, p. 620 and by S. Krinsky and J. M. Wang, Part. Accel. **12**, 107 (1982).

[3]The relationship between Eq. (1.3) and Madey's gain-spread theorem for the free-electron laser has been derived by S. Krinsky, J. M. Wang, and P. Luchini, J. Appl. Phys. **53**, 5453 (1982).

[4]R. J. Glauber, Phys. Rev. **131**, 2766 (1963); P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968); J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968); J. R. Klauder, Phys. Rev. A **29**, 2036 (1984).

[5]An introduction to the fluctuation-dissipation theorem in classical and quantum mechanics is given in P. C. Martin, *Measurements and Correlation Functions* (Gordon and Breach, New York, 1968).

[6]See, e.g., M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945).

[7]J. M. J. Madey, Nuovo Cimento **B50**, 64 (1979); N. M. Kroll, *Physics of Quantum Electronics* (Addison-Wesley, Boston, 1982), Vol. 8, p. 281.

[8]Another approach to this problem is given by H. G. Hereward, Stanford Linear Accelerator PEP Summer Study, Note 53 (1973) (unpublished).