Inner-vertex contribution to the decay rate of orthopositronium

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In this paper the order-α contribution of the inner-vertex graph to the decay rate of orthopositronium is obtained in analytic form.

The order-α corrections to the decay rate of parapositronium have been known analytically since the work of Harris and Brown in 1957.[1] Analytic results for orthopositronium are much harder to calculate, and even though the orthopositronium decay rate is experimentally more accessible than the parapositronium rate,[2] analytic results for orthopositronium have only recently been obtained. The self-energy contribution was evaluated in analytic form by Stroscio,[3] and later the outer-vertex result was obtained by the present author.[4] In this paper I describe a calculation of the inner-vertex contribution.

The orthopositronium decay rate has the form[5,6]

$$\Gamma=\frac{m}{2^7\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_3\frac{1}{3!}\langle\,|M|^2\rangle\ , \tag{1}$$

where $\langle\,|M|^2\rangle$ is the invariant matrix element squared, summed over final polarizations, and averaged over the three orthopositronium spin states. The two-dimensional phase space is parametrized by the normalized photon energies $x_i=\omega_i/m$. The invariant matrix element has contributions from each of the graphs of Fig. 1:

$$M=M_{\rm LO}+M_{\rm IV}+\cdots\ . \tag{2}$$

The lowest-order decay rate, which involves $|M_{\rm LO}|^2$, was first calculated by Ore and Powell.[7] It is

$$\Gamma_{\rm LO}=\frac{2}{9\pi}(\pi^2-9)m\alpha^6\ . \tag{3}$$

Order-α corrections come from cross terms in the square of M. The $O(\alpha)$ correction due to the inner-vertex graph is

$$\Gamma_{\rm IV}=\frac{m}{2^7\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_3\sum_{\epsilon_1,\epsilon_2,\epsilon_3}\tfrac{1}{3}\sum_{\epsilon_m}\frac{1}{3!}2\,{\rm Re}[(M_{\rm LO})^*M_{\rm IV}]\ , \tag{4}$$

where $\epsilon_1,\epsilon_2,\epsilon_3$ are photon polarization vectors and ϵ_m represents the orthopositronium spin.

Expressions for contributions to M are obtained by evaluating the corresponding Feynman diagrams. The lowest-order invariant matrix element [Fig. 1(a)] is

$$M_{\rm LO}\simeq-i\pi\alpha^3\sum_{\sigma\in S_3}\frac{x_{\sigma(2)}}{x_1x_2x_3}{\rm tr}\left[\gamma\epsilon_{\sigma(3)}(-\gamma R_{\sigma(3)}+1)\gamma\epsilon_{\sigma(2)}(\gamma R_{\sigma(1)}+1)\gamma\epsilon_{\sigma(1)}\begin{bmatrix}0 & \sigma\cdot\hat{\epsilon}_m\\ 0 & 0\end{bmatrix}\right]\ . \tag{5}$$

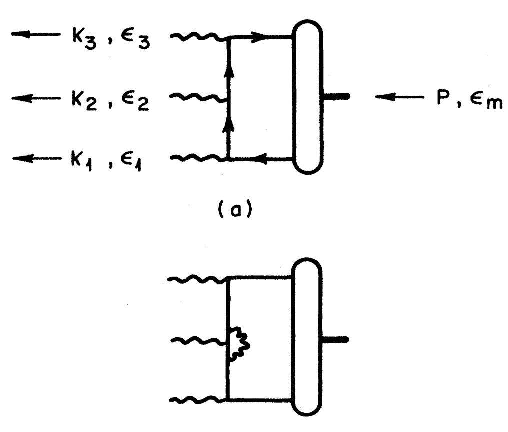

FIG. 1. (a) The lowest-order and (b) inner-vertex orthopositronium decay graphs.

The sum in (5) is over the 3! permutations of the final photons, and the R's are dimensionless momentum vectors: $R_i = N - K_i$ where $N=(1,0)$ and $K_i=(\omega_i/m, \mathbf{k}_i/m)$. The invariant matrix element $M_{\rm IV}$ [Fig. 1(b)] is obtained from (5) by the replacement $\gamma\epsilon_{\sigma(2)} \to \Lambda_\lambda(x_{\sigma(3)}, x_{\sigma(1)})\epsilon^{\lambda}_{\sigma(2)}$, where

$$\Lambda^\lambda(x_3,x_1) = \frac{\alpha}{4\pi}[\gamma^\lambda F_0(x_3,x_1) + \gamma R_3 \gamma^\lambda \gamma R_1 F_1(x_3,x_1) + \gamma S S^\lambda F_2(x_3,x_1) + \gamma K_2 S^\lambda F_3(x_3,x_1) + S^\lambda F_4(x_3,x_1)] \quad (6)$$

with $S^\lambda = (K_3 - K_1)^\lambda$. The F factors in (6) are

$$F_0(x_3,x_1) = D + \left[\frac{2x_1(x_1+2x_3-2)}{(x_3-x_1)(1-2x_1)} \ln(2x_1) + \frac{1}{(x_3-x_1)} \mathrm{Li}_2(1-2x_1) + (1\leftrightarrow 3) \right], \quad (7a)$$

$$F_1(x_3,x_1) = \frac{2x_1}{(x_3-x_1)(1-2x_1)} \ln(2x_1) + (1\leftrightarrow 3), \quad (7b)$$

$$F_2(x_3,x_1) = \frac{x_1}{(x_3-x_1)(1-2x_1)} \left[\frac{1}{2} + \frac{2-3x_1}{1-2x_1} \ln(2x_1) \right] + (1\leftrightarrow 3), \quad (7c)$$

$$F_3(x_3,x_1) = \frac{1}{2(x_3-x_1)} \left[\frac{1}{2} + \frac{x_1(x_1+x_3-1)}{(x_3-x_1)(1-2x_1)} + \frac{2x_1}{(x_3-x_1)(1-2x_1)} \left[1 + \frac{x_1(x_1+x_3-1)}{(1-2x_1)} \right] \ln(2x_1) - \frac{1}{(x_3-x_1)} \mathrm{Li}_2(1-2x_1) + (1\leftrightarrow 3) \right], \quad (7d)$$

$$F_4(x_3,x_1) = -2F_1(x_3,x_1), \quad (7e)$$

where Li_2 is the dilogarithm function discussed by Lewin.[8] Useful tables for integrals involving dilogarithms are given by Barbieri, Mignaco, and Remiddi.[9] The vertex function was evaluated in Feynman gauge using dimensional regularization. The divergent constant is

$$D = \frac{1}{2-\omega} - \gamma_E + \ln(4\pi), \quad (8)$$

where $d=2\omega$ is the number of dimensions and γ_E is Euler's constant. There is actually no need to symmetrize in $M_{\rm IV}$, because both $M_{\rm LO}$ and the phase space are already symmetric under photon interchanges. Therefore the vertex correction can be taken to act on photon 2 only.

Expression (4) is evaluated by performing the polarization sums, the spin sum, and the resulting trace [using REDUCE (Ref. 10)]. One finds that

$$\Gamma_{\rm IV} = \frac{m\alpha^6}{6\pi}\frac{\alpha}{\pi} \int_0^1 dx_1 \int_{1-x_1}^1 dx_3 \frac{x_2}{(x_1x_2x_3)^2} \sum_{i=0}^4 F_i(x_3,x_1) T_i, \quad (9)$$

where $x_2 = 2 - x_1 - x_3$. The traces written in terms of $x^{nm} = (x_3)^n (x_1)^m$ are

$$T_0 = \tfrac{1}{2}(-x^{31} - 4x^{30} - 4x^{22} - 7x^{21} + 16x^{20} - x^{13} - 7x^{12} + 28x^{11} - 20x^{10} - 4x^{03} + 16x^{02} - 20x^{01} + 8), \quad (10a)$$

$$T_1 = \tfrac{1}{2}(-2x^{32} + 7x^{31} - 2x^{23} + 8x^{22} + 3x^{21} - 12x^{20} + 7x^{13} + 3x^{12} - 32x^{11} + 20x^{10} - 12x^{02} + 20x^{01} - 8), \quad (10b)$$

$$T_2 = 4x^{32} + 3x^{31} - 8x^{30} + 4x^{23} + 6x^{22} - 40x^{21} + 30x^{20} + 3x^{13} - 40x^{12} + 72x^{11} - 34x^{10} - 8x^{03} + 30x^{02} - 34x^{01} + 12, \quad (10c)$$

$$T_3 = (x_3 - x_1)(-2x^{22} - 3x^{21} + 6x^{20} - 3x^{12} + 16x^{11} - 14x^{10} + 6x^{02} - 14x^{01} + 8), \quad (10d)$$

$$T_4 = -x^{30} - 7x^{21} + 8x^{20} - 7x^{12} + 20x^{11} - 13x^{10} - x^{03} + 8x^{02} - 13x^{01} + 6. \quad (10e)$$

The integral for $\Gamma_{\rm IV}$ can be divided into three parts. The piece proportional to D is

$$\Gamma_{\rm LO} \left[\frac{\alpha}{\pi} \right] (\tfrac{1}{2}D). \quad (11)$$

The piece that involves no dilogarithms in the F's is

$$\frac{m\alpha^6}{6\pi}\frac{\alpha}{\pi}[-4 - \tfrac{34}{3}\ln 2 - \tfrac{1057}{36}\zeta(2) + \tfrac{2405}{108}\zeta(2)\ln 2 + \tfrac{2971}{108}\zeta(3)], \quad (12)$$

where $\zeta(2) = \pi^2/6$ and $\zeta(3) = 1.202\,056\,903\,2$. The piece that does involve dilogarithms is

$$\frac{m\alpha^6}{6\pi}\frac{\alpha}{\pi}[6\zeta(2)-\tfrac{32}{3}\zeta(2)\ln 2+\tfrac{23}{12}\zeta(3)+\tfrac{17}{40}\zeta^2(2)-a_4-\tfrac{7}{8}\zeta(3)\ln 2+\tfrac{5}{2}\zeta(2)\ln^2 2-\tfrac{1}{24}\ln^4 2]\,, \tag{13}$$

where

$$a_4=\mathrm{Li}_4(\tfrac{1}{2})=\sum_{n=1}^{\infty}\frac{1}{n^4 2^n}=0.517\,479\,061\,674\;. \tag{14}$$

In all we have

$$\Gamma_{\rm IV}=\Gamma_{\rm LO}\left[\frac{\alpha}{\pi}\right]\left[\tfrac{1}{2}D+\frac{3}{4(\pi^2-9)}[-4-\tfrac{34}{3}\ln 2-\tfrac{841}{36}\zeta(2)+\tfrac{1253}{108}\zeta(2)\ln 2+\tfrac{1589}{54}\zeta(3)+\tfrac{17}{40}\zeta^2(2)-a_4-\tfrac{7}{8}\zeta(3)\ln 2+\tfrac{5}{2}\zeta(2)\ln^2 2-\tfrac{1}{24}\ln^4 2]\right]=\Gamma_{\rm LO}\left[\frac{\alpha}{\pi}\right](\tfrac{1}{2}D+0.160\,677)\;. \tag{15}$$

The number in (15) compares favorably with the numerical result 0.1605(5) obtained in Ref. 5. This agreement provides an important check of the result.

The sum of all vertex contributions is

$$\Gamma_{\rm IV}+\Gamma_{\rm OV}=\Gamma_{\rm LO}\left[\frac{\alpha}{\pi}\right](\tfrac{3}{2}D+3.131\,816)\;, \tag{16}$$

where $\Gamma_{\rm OV}$ is given in Ref. 4. This can be compared to previous numerical results 3.10(15) of Stroscio and Holt,[11] 3.1284(36) of Stroscio,[12] 3.132(3) of Caswell, Lepage, and Sapirstein,[13] and 3.1312(9) of the present author.[5] The numerical results all agree with (16) within their quoted errors.

The contributions of the various graphs can be ranked according to their maximum degree, where the notion of degree is discussed in Appendix I of Ref. 9. In brief, the degree of a product is the sum of the degrees of the factors, and $\ln^n(x)$, $\mathrm{Li}_n(x)$, and $\zeta(n)$ all have degree n. The self-energy result of Ref. 3 has maximum degree three, which comes from the double integration of the logarithm in the self-energy function. The outer-vertex result of Ref. 4 and the inner-vertex result here both have maximum degree four, because the vertex function contains a dilogarithm. One would expect that the double-vertex graph and the annihilation graph will have maximum degree five, and the binding graph will have maximum degree six. Analytic evaluations of these graphs will be correspondingly difficult.

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