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Photon antibunching in a free-electron laser

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A perturbative solution to the difference-differential equation describing the time evolution of a free-electron laser based on the Bambini-Renieri Hamiltonian is obtained. This solution indicates that the photon number distribution is always sub-Poissonian as long as the perturbation approximation is valid. This photon antibunching becomes more prominent as more photons are emitted; so it can enhance the coherent properties of the free-electron-laser radiation in a way that is very welcome.

The free-electron laser (FEL) is attractive because it has the potential to become the most efficient way to generate coherent radiation which is also tunable over a wide frequency range. This is made possible by eliminating the "middle man," i.e., the atoms, molecules, or crystals in the traditional laser system. Using quantum-mechanical analysis, Madey[1] first explained the gain mechanism in propagating relativistic electrons through a static magnetic field, the wiggler, of an FEL system. When the first FEL amplifier became operative,[2] it indicated that most essential features of FEL could be understood in terms of classical mechanics.[3] However, the problem of the photon statistics of FEL and, consequently, the very question of whether or not FEL is a laser in the sense that its photon number distribution is Poissonian can only be studied quantum mechanically.

Quantum-mechanical analyses of the FEL often start from the Bambini-Renieri Hamiltonian[4] which describes the system in a frame moving at a speed very close to that of light so that (1) the wiggler field appears almost as a plane-wave radiation (Weizsäcker-William approximation), (2) the frequency of the wiggler field coincides with that of the laser, and (3) the electron dynamics is nonrelativistic.

Using the Bambini-Renieri Hamiltonian, Becker and Zubairy[5] have studied the photon statistics of an FEL in the small-signal regime based on an evaluation of the time-evolution operator to first order in the quantum-mechanical recoil. On the other hand, Bambini and Stenholm[6] and Dattoli and co-workers[7] have solved various simplified versions of the so-called spherical Raman-Nath equation which is a difference-differential equation for the probability amplitudes of different photon numbers as functions of time.

We follow the approach of Dattoli and co-workers[7] to obtain a perturbative solution to the spherical Raman-Nath equation based on small quantum recoils compared with the initial momentum of the electron in the moving frame. Using this solution we can calculate the evolution of the photon statistics starting from vacuum. It is found that the photon number distribution is always narrower than Poissonian, as long as the perturbation approximation is valid. This phenomenon can serve to compensate other factors which tend to broaden the photon number distribution; therefore, it can enhance the coherent properties of the FEL radiation. Furthermore, this photon antibunching gets stronger as the number of photons emitted increases. This is very encouraging news, especially coming after the pessimistic conclusion of Ref. 5 that "the FEL is a laser in the sense that it produces a coherent state only if it is not a laser in the sense that it does not amplify."

The Bambini-Rerieri Hamiltonian is given by Ref. 6 as

$$H=\frac{p^2}{2m}+\hbar\omega(a_L^\dagger a_L+a_W^\dagger a_W)+\hbar\Lambda(a_L^\dagger a_W e^{-2ikz}+a_W^\dagger a_L e^{2ikz}) \ , \tag{1}$$

where p is the momentum of the electron, $a_L^\dagger$ ($a_W^\dagger$) the laser (wiggler) creation operator, and

$$\Lambda=e^2/2m\omega\epsilon_0 V_W \tag{2}$$

is the coupling constant with V_W being the wiggler volume. The last term of Eq. (1) describes the elastic backscattering of a wiggler "photon" into a laser photon and vice versa, with the electron providing for the necessary momentum change.

Starting from a quantum state, wherein the electron has the initial momentum p_0, the wiggler field has n_W "photons" and the laser field has n_L photons, written as

$$|\psi_0\rangle=Ke^{ip_0z/\hbar}|n_W\rangle|n_L\rangle \ , \tag{3}$$

where K is the normalization constant which does not need to be specified. Because of the conservation of momentum and total "photon" number, the wave function at a later time t can be written as

$$|\psi(t)\rangle=\exp[-i(p_0^2/2m\hbar+n_W\omega+n_L\omega)t]\sum_{n=-n_L}^{n_W}C_n(t)|n\rangle \ , \tag{4}$$

where we have chosen the phase factor in a convenient way and have adopted the simple notation

$$|n\rangle\equiv K\exp[i(p_0/\hbar-2nk)z]|n_W-n\rangle|n_L+n\rangle \ . \tag{5}$$

Substituting Eq. (4) into the Schrödinger equation with the Hamiltonian given by Eq. (1), we obtain the following difference-differential equation

$$i\frac{d}{dt}C_n(t)=(-2n\delta+n^2\epsilon)C_n(t)+\Lambda[(n_W-n)(n_L+n+1)]^{1/2}C_{n+1}(t)+\Lambda[(n_W-n+1)(n_L+n)]^{1/2}C_{n-1}(t) \ , \tag{6}$$

where $C_n(t)$ is the probability amplitude that n photons have been emitted at the time t and the two parameters

$$\delta\equiv kp_0/m, \quad \epsilon\equiv 2\hbar k^2/m \ , \tag{7}$$

are related to the initial momentum and the quantum recoil, respectively, of the electron.

To be more specific, we consider the case when the laser field starts from vacuum; so the total number of photons is

$$N \equiv n_W + n_L = n_W = (\omega\epsilon_0/\hbar) V_W A_W^2 \quad , \tag{8}$$

where A_W is the amplitude of the wiggler vector potential. Then Eq. (6) reduces to

$$\begin{aligned} i\frac{d}{dt}C_n(t) = &(-2n\delta + n^2\epsilon)C_n(t) \\ &+\Lambda[(N-n)(n+1)]^{1/2}C_{n+1}(t) \\ &+\Lambda[(N-n+1)n]^{1/2}C_{n-1}(t) \end{aligned} \tag{9}$$

with the initial condition

$$C_n(0) = \delta_{n,0} \quad . \tag{10}$$

The $n^2\epsilon$ term is the main obstacle in solving Eq. (9). So we try a perturbative solution to the first order of ϵ based on the assumption that

$$N\epsilon/\delta < 1 \quad , \tag{11}$$

which simply means that the electron momentum remains positive in the moving frame after the emission of N laser photons. Let

$$C_n(t) = A_n(t) + \epsilon B_n(t) \quad . \tag{12}$$

Substitution of Eq. (12) into Eq. (9) yields two equations:

$$i\frac{d}{dt}A_n(t) = -2n\delta A_n(t) + \Lambda[(N-n)(n+1)]^{1/2}A_{n+1}(t) + \Lambda[(N-n+1)n]^{1/2}A_{n-1}(t) \quad , \tag{13}$$

$$i\frac{d}{dt}B_n(t) = n^2A_n(t) - 2n\delta B_n(t) + \Lambda[(N-n)(n+1)]^{1/2}B_{n+1}(t) + \Lambda[(N-n+1)n]^{1/2}B_{n-1}(t) \quad . \tag{14}$$

The solution to Eq. (13) is the following[8]

$$A_n(t) = (-i)^n e^{iN\delta t}\binom{N}{n}^{1/2} W^{N-n}(t)[(\Lambda/\Omega)\sin\Omega t]^n \quad , \tag{15}$$

where

$$\Omega \equiv (\delta^2 + \Lambda^2)^{1/2} \tag{16}$$

and

$$W(t) \equiv \cos\Omega t - i(\delta/\Omega)\sin\Omega t \quad . \tag{17}$$

Using Eq. (15) we obtain a solution to Eq. (14) of the form

$$B_n(t) = A_n(t)[F(t) + nG(t) + n^2H(t)] \quad , \tag{18}$$

where

$$H(t) \equiv -i[W(t)\sin\Omega t]^{-2}\int W^2(t)\sin^2\Omega t\,dt \quad , \tag{19a}$$

$$G(t) \equiv [2(N-1)W(t)\cos\Omega t - 2N + 1]H(t) - iN[W(t)\sin\Omega t]^{-1}\int W(t)\sin\Omega t\,dt + i\frac{N-1}{2\Omega}W(t)\sin\Omega t \quad , \tag{19b}$$

and

$$\begin{aligned} F(t) \equiv &-iN(N-1)(\Lambda/\Omega)^2\sin^2\Omega t H(t) + iN^2\int W(t)\cos\Omega t\,dt \\ &+ iN^2[\sin\Omega t + i(\delta/\Omega)\cos\Omega t]W^{-1}(t)\int W(t)\sin\Omega t\,dt - iN(N-1)(\Lambda^2/8\Omega^3)\sin 2\Omega t + i[N(N-1)\Lambda^2/4\Omega^2 - N^2]t \quad . \end{aligned} \tag{19c}$$

All the integrals that appear in Eqs. (19) can be carried out explicitly.

The probability that n photons have been emitted at time t can be written as

$$P_n(t) \equiv C_n^*(t)C_n(t) = \binom{N}{n}[1 - V(t)]^{N-n}V^n(t)\{1 + \epsilon[F(t) + nG(t) + n^2H(t)] + \text{c.c.}\} \quad , \tag{20}$$

where we have defined

$$V(t) \equiv (\Lambda/\Omega)^2\sin^2\Omega t \quad . \tag{21}$$

The normalization condition for the probability distribution of Eq. (20) implies that

$$\text{Re}\{F(t) + NVG(t) + [N^2V^2 + NV(1-V)]H(t)\} = 0 \quad , \tag{22}$$

which can also be established by using the explicit expressions given by Eqs. (19). Using Eqs. (20)–(22), we can carry out the photon statistics calculations as follows:

$$\begin{aligned} \langle n \rangle = &NV + \epsilon 2NV(1-V) \\ &\times \text{Re}\{G(t) + (2NV + 1 - 2V)H(t)\} \quad , \end{aligned} \tag{23}$$

$$\begin{aligned} \langle n^2 \rangle = &N^2V^2 + NV(1-V) \\ &+ \epsilon 2NV(1-V)\,\text{Re}\{(2NV + 1 - 2V)G(t) \\ &+ [4N^2V^2 + NV(6 - 10V)]H(t)\} \end{aligned} \tag{24}$$

and

$$Q \equiv \frac{\langle n^2 \rangle - \langle n \rangle^2 - \langle n \rangle}{\langle n \rangle}$$

$$= -V - \epsilon 2V(1-V)\,\mathrm{Re}\{G(t) - 2N(1-2V)H(t)\} \quad . \quad (25)$$

The parameter Q has been used by Mandel[9] as a natural measure of the departure of the variance of the photon number n from that of a Poisson distribution, for which $Q = 0$. A negative value of Q implies photon antibunching. As displayed in Eq. (25), we always have $Q < 0$ as long as the perturbation approximation is valid.

Since V, as defined by Eq. (21), is proportional to $\sin^2 \Omega t$, we see that the antibunching effects become more prominent as time passes; it reaches its peak at the same time when the laser output is at the maximum. It should be interesting to examine the situation at $t = \pi/2\Omega$. We have $V = \Lambda^2/\Omega^2$, $\mathrm{Re}\{F\} = -N\Lambda^2/2\delta\Omega^2$, $\mathrm{Re}\{G\} = -N\Lambda^2/2\delta\Omega^2 + \Lambda^2/2\delta\Omega^2$, and $\mathrm{Re}\{H\} = 1/2\delta$. Substitution of these values in Eq. (25) gives

$$Q(t = \pi/2\Omega) = -(\Lambda/\Omega)^2[1 + \epsilon N\delta(\Lambda^2 - 2\delta^2)/\Omega^4] \quad . \quad (26)$$

In conclusion, we have carried out a perturbative analysis of the fundamental process in a free-electron laser, a single-electron interacting with a single-mode radiation which is in vacuum state initially, described in the Bambini-Renieri frame. The photon statistics of this simple system shows antibunching as long as the perturbation approach is viable.

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