Weak-noise limit of Fokker-Planck models and nondifferentiable potentials for dissipative dynamical systems

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The weak-noise limit of Fokker-Planck models is studied for the case where the steady-state probability density in that limit cannot be represented by a continuously differentiable nonequilibrium potential. In a previous paper [J. Stat. Phys. 35, 729 (1984)], we have shown that this corresponds to the general case in systems outside thermodynamic equilibrium. By using an extremum principle, the nondifferentiable potential is constructed, which generalizes the differentiable case. The relation of approximate differentiable potentials to the exact nondifferentiable potential is considered and discussed for two examples with attracting limit cycles, a periodically forced nonlinear oscillator, and two phase-coupled nonlinear oscillators. The relevance of nondifferentiable potentials for nonequilibrium thermodynamics is pointed out.

I. INTRODUCTION

Dissipative dynamical systems under the influence of noise can often be successfully modeled by Fokker-Planck equations. In many physically interesting cases the noise, represented by the diffusion term of the Fokker-Planck equation, can be considered as a small perturbation, and an asymptotic expansion of the solution of the Fokker-Planck equation in this perturbation is desirable.

In a previous paper,[1] henceforth referred to as I, we have examined the conditions under which the time-independent solution $P(q,\eta)$ of the Fokker-Planck equation, assumed to be unique, has the asymptotic form

$$P(q,\eta)\sim \exp(-\phi(q)/\eta) \tag{1.1}$$

with a continuously differentiable potential $\phi(q)$. Here $q=\{q_1,\ldots,q_n\}$ are the macroscopic variables of the dissipative system, the noise intensity η is a small parameter which multiplies the diffusion term of the Fokker-Planck equation. $\phi(q)$ in Eq. (1.1) is defined by the weak-noise limit

$$\phi(q)=\lim_{\eta\to 0}[-\eta \ln P(q,\eta)]\,. \tag{1.2}$$

This function is of considerable interest because, in many respects, it plays the role of a thermodynamic potential for the fluctuating dissipative dynamical system. $\phi(q)$ is nonincreasing along trajectories of the dissipative dynamical system for vanishing noise, and it is stationary in the limit sets of the dynamical system for vanishing noise; it can, therefore, play the role of a Lyapunov function in the vicinity of attractors; in addition it contains information on the asymptotic form of the probability density and other quantities of statistical relevance, such as mean first-passage times.

It was shown in I (see also Ref. 2) that a continuously differentiable $\phi(q)$ in Eq. (1.1) does not exist, unless a certain Hamiltonian dynamical system, which is obtained from the Fokker-Planck equation in the weak-noise limit, has a smooth separatrix $p_\nu=\partial\phi/\partial q^\nu$, a case which is structurally unstable against small changes of the dissipative dynamical system. This result naturally leads to the following question: What are the properties of the potential $\phi(q)$, defined by Eq. (1.2), in the general structurally stable case, where the Hamiltonian system, related to the Fokker-Planck model in the weak-noise limit, has a "wild separatrix"?

In the present paper we answer this question by invoking an extremum principle satisfied by the potential $\phi(q)$. We show that $\phi(q)$ remains a continuous function but loses its continuous differentiability on certain complicated hypersurfaces in q space, owing to the infinitely rapid oscillations of the wild separatrix in the nonintegrable Hamiltonian system. As shown in Ref. 2, thermodynamic equilibrium ensures the smoothness of the separatrix. Therefore, the loss of differentiability of the potential is a nonequilibrium effect.

In Sec. II we present briefly two illustrative physical examples: the periodically forced nonlinear oscillator which was already analyzed in I in some detail and which serves as our main working example, and two phase-coupled nonlinear oscillators. In Sec. III we formulate the extremum principle satisfied by the potential. The extremum principle is used in Sec. IV to provide a construction of the nondifferentiable potential. We consider this construction for the periodically forced nonlinear oscillator in Sec. V and make a comparison with continuously differentiable approximations. In Sec. VI we apply our results and present a continuously differentiable approximation for the example of two phase-coupled nonlinear oscillators. Mathematical details concerning this example are given in Appendixes A–C. In Sec. VII and the related Appendix D we consider the probability distribution in the steady state for the case of a nondifferentiable potential. Finally, we present our conclusions in Sec. VIII.

II. EXAMPLES

A. Periodically forced nonlinear oscillator

The model is defined by the stochastic differential equation

$$\dot{x}=x-x^3+ax\cos y+\sqrt{\eta}\xi(t),\quad \dot{y}=\omega \tag{2.1}$$

with the parameters a,ω and the Gaussian white noise $\xi(t)$ characterized by

$$\langle\xi(t)\rangle=0,\quad \langle\xi(t)\xi(0)\rangle=\delta(t)\ . \tag{2.2}$$

The weak-noise limit of this model has been analyzed in some detail in I to which we refer for details. Inserting the ansatz $P(x,y)\sim\exp[-\phi(x,y)/\eta]$ for the steady-state distribution in the Fokker-Planck equation of (2.1), (2.2) leads to the Hamilton-Jacobi equation

$$\frac{1}{2}\left[\frac{\partial\phi}{\partial x}\right]^2+\frac{\partial\phi}{\partial x}(x-x^3+ax\cos y)+\omega\frac{\partial\phi}{\partial y}=0\ , \tag{2.3}$$

which defines the Hamiltonian dynamical system associated with the Fokker-Planck model in the weak-noise limit. According to I, if the potential $\phi(x,y)$ exists as a continuously differentiable function, it is related to the smooth separatrix of this Hamiltonian system

$$p_x=\tilde{p}_x(x,y),\quad p_y=\tilde{p}_y(x,y) \tag{2.4}$$

by $\tilde{p}_x=\partial\phi/\partial x$, $\tilde{p}_y=\partial\phi/\partial y$. This smooth separatrix connects the limit sets $\{x=0,\ 0\le y<2\pi\}$ and $\{x=\pm f(y),\ 0\le y<2\pi\}$ of the deterministic system Eq. (2.1) with $\eta=0$. Here $f(y)$ is a function which must be determined from Eq. (2.1) with $\eta=0$. However, this smooth separatrix exists only for $a=0$ where the Hamiltonian system (2.3) is integrable.

The actual nonintegrability of the Hamiltonian system has been exhibited in I by plotting the rapid oscillations of the wild separatrix emanating from the limit set $\{x=\pm f(y),\ 0\le y<2\pi\}$. These oscillations were found numerically and also in approximate analytical solutions of (2.3) obtained by an expansion to first order in a. The latter expansion, therefore, serves as a useful practical test for a smooth separatrix. To first order in a and requiring the absence of oscillations near the limit set $\{x=f(y),\ 0\le y<2\pi\}$, the result of I reads

$$\phi(x,y)=-x^2+\tfrac{1}{2}x^4-2ax^2(1-x^2)\times\mathrm{Re}\left[e^{-iy}\frac{F(1;2;2-i\omega/2,1-x^2)}{i\omega-2}\right], \tag{2.5}$$

where F is the hypergeometric function. The infinitely rapid oscillations of

$$p_x^s=\frac{\partial\phi}{\partial x_y}\sim\frac{1}{x}\sin(\omega\ln x-\delta+y) \tag{2.6}$$

near $\{x=0,\ 0\le y<2\pi\}$ obtained from the approximation (2.5) have been discussed in detail in I [cf. Eq. (4.15) of I]. As a result of this numerical and analytical work we conclude that ϕ does not exist as a continuously differentiable function for this simple model.

B. Two phase-coupled nonlinear oscillators

As a second, somewhat more complicated but also more realistic example, we consider two phase-coupled nonlinear oscillators. This problem is, e.g., of relevance for the nonlinear interaction of two laser modes of sufficiently close frequencies whose phases may interact. The weak-noise limit of this problem has been studied previously by Hellwig[3] using the ideas of Refs. 4 and 5. Oscillations of $\phi(q)$, which were found in Ref. 3 but could not be interpreted there, are easily understood in the present context as arising from the nonintegrability of the underlying Hamilton-Jacobi equation. Therefore, the results of I and of the following sections also shed new light on this model.

The model is defined by the following stochastic differential equations for the complex amplitudes x_1,x_2:

$$\begin{aligned}\dot{x}_1&=(a_1-i\omega_1)x_1\\&\quad-(|x_1|^2+k_1|x_2|^2)x_1-k_2x_1^*x_2^2+2\sqrt{\eta}\xi_1\ ,\\ \dot{x}_2&=(a_2-i\omega_2)x_2\\&\quad-(|x_2|^2+k_1|x_1|^2)x_2-k_2x_1^2x_2^*+2\sqrt{\eta}\xi_2\ ,\end{aligned} \tag{2.7}$$

where the Gaussian white-noise terms ξ_1,ξ_2 are characterized by

$$\langle\xi_l(t)\rangle=0,\quad \langle\xi_l(t)\xi_m(0)\rangle=0\ ,$$
$$\langle\xi_l^*(t)\xi_m(0)\rangle=\delta_{lm}\delta(t),\quad l,m=1,2\ .$$

It is convenient to introduce the modulus and phase of the complex amplitudes by $x_l=r_l\exp(i\varphi_l)$ and to derive the Fokker-Planck equation for the new variables. This can be most easily done by using Ito's formula.[6] Since the stationary probability distribution will not depend on the phase $(\varphi_1+\varphi_2)$, we average over φ_1,φ_2 for fixed relative phase $\varphi=\varphi_1-\varphi_2$ and obtain for the probability density $P(r_1,r_2,\varphi)$,

$$\begin{aligned}\frac{\partial P}{\partial t}&=-\frac{\partial}{\partial r_1}\left[[a_1-r_1^2-k_1r_2^2-k_2r_2^2\cos(2\varphi)]r_1+\frac{\eta}{r_1}\right]P\\&\quad-\frac{\partial}{\partial r_2}\left[[a_2-r_2^2-k_1r_1^2-k_2r_1^2\cos(2\varphi)]r_2+\frac{\eta}{r_2}\right]P\\&\quad-\frac{\partial}{\partial\varphi}[\omega+k_2(r_1^2+r_2^2)\sin(2\varphi)]P\\&\quad+\eta\left[\frac{\partial^2}{\partial r_1^2}+\frac{\partial^2}{\partial r_2^2}+\left[\frac{1}{r_1^2}+\frac{1}{r_2^2}\right]\frac{\partial^2}{\partial\varphi^2}\right]P\ .\end{aligned} \tag{2.8}$$

The model has six real parameters: the positive pump parameters a_1,a_2, the intensity-coupling parameter $k_1(|k_1|<1)$, the phase-coupling parameter $k_2(|k_2|<1)$, the frequency difference of the two oscillators $\omega=\omega_1-\omega_2$, and the noise intensity η.

The Hamilton-Jacobi equation for ϕ obtained from the stochastic model (2.7) in the weak-noise limit reads

$$\left[\frac{\partial\phi}{\partial r_1}\right]^2+\left[\frac{\partial\phi}{\partial r_2}\right]^2+\frac{r_1^2+r_2^2}{r_1^2r_2^2}\left[\frac{\partial\phi}{\partial\varphi}\right]^2+[a_1r_1-(r_1^2+k_1r_2^2)r_1-k_2r_1r_2^2\cos(2\varphi)]\frac{\partial\phi}{\partial r_1}$$
$$+[a_2r_2-(r_2^2+k_1r_1^2)r_2-k_2r_1^2r_2\cos(2\varphi)]\frac{\partial\phi}{\partial r_2}+[\omega+k_2(r_1^2+r_2^2)\sin(2\varphi)]\frac{\partial\phi}{\partial\varphi}=0\ . \quad (2.9)$$

The nonintegrability of the Hamiltonian system (2.9) may be put into evidence (but not proven) by calculating ϕ to first order in k_1 and k_2 and exhibiting the infinitely rapid oscillations of its derivatives. In Appendix A we calculate the solution of Eq. (2.9) to first order in k_1 and k_2 in the form

$$\phi=\tfrac{1}{2}[-a_1r_1^2-a_2r_2^2+\tfrac{1}{2}(r_1^4+r_2^4)+k_1r_1^2r_2^2+k_2r_1^2r_2^2\cos(2\varphi)]$$
$$+k_2\left[I(r_1,r_2,\varphi)+G\left[2\varphi+\frac{\omega}{a_1}\ln\frac{r_1^2}{|r_1^2-a_1|},2\varphi+\frac{\omega}{a_2}\ln\frac{r_2^2}{|r_2^2-a_2|}\right]\right], \quad (2.10)$$

where

$$I(r_1,r_2,\varphi)=-\frac{\omega}{2}a_2\int_1^\infty \mathrm{Im}\frac{e^{2i\varphi}z^{i\omega/a_1-1}}{\left[1+\left[\frac{a_1}{r_1^2}-1\right]z\right]\left[1+\left[\frac{a_2}{r_2^2}-1\right]z^{a_2/a_1}\right]}dz\ , \quad (2.11)$$

and $G(x,y)$ is 2π-periodic in x and otherwise arbitrary. The function $I(r_1,r_2,\varphi)$ is analytic in its three variables except for an essential singularity for $r_1^2\to a_1$, $r_2^2\to a_2$. The asymptotic behavior of I for $r_2^2=a_2$ and $r_1^2\to a_1$ is investigated as described in Appendix B and we find that

$$I\sim\frac{a_2\pi\omega}{2\sinh(\pi\omega/a_1)}\cos\left[2\varphi+\frac{\omega}{a_1}\ln\left|\frac{a_1}{a_1-r_1^2}\right|-\frac{\omega}{a_1}[1-\mathrm{sgn}(a_1-r_1^2)]\frac{\pi}{4}\right].$$

This oscillatory behavior of I can be compensated by including a corresponding term with the opposite sign in the function G in Eq. (2.10). However, owing to the particular form of G, this could only be achieved at the cost of introducing an infinitely rapid oscillation of the solution for $r_1^2\to 0$. We conclude by analogy with the first example that the Hamiltonian system described by Eq. (2.9) is nonintegrable, and a continuously differentiable potential does not exist.

III. EXTREMUM PRINCIPLE FOR THE POTENTIAL

In this section we formulate the extremum principle which is satisfied by the potential defined by Eq. (1.2). We only give a heuristic and formal presentation. A rigorous mathematical foundation is given in the book by Freidlin and Wentzell.[7] The role of the extremum principle for solutions of the initial value problem of the Fokker-Planck equation has been discussed by Kitahara.[8] Here we are interested only in the very different problem of the solution of the time-independent Fokker-Planck equation. The basic Fokker-Planck equation on a space M with coordinates q is

$$\frac{\partial P(q,\eta,t)}{\partial t}=\left[-\frac{\partial}{\partial q^\nu}K^\nu(q)+\frac{\eta}{2}\frac{\partial^2}{\partial q^\nu\partial q^\mu}Q^{\nu\mu}\right]P(q,\eta,t)\ . \quad (3.1)$$

For simplicity, we first assume that $Q^{\nu\mu}$ is a nonsingular matrix and generalize our results later. The conditional probability density $P(q\mid q_0,t)$, which is the solution of Eq. (3.1) for the initial condition $P(q\mid q_0,0)=\delta^{(n)}(q-q_0)$, may be written as a functional integral[9–12] which, in the case where $Q^{\nu\mu}(q)$ is nonsingular, takes the general form

$$P(q\mid q_0,t)=\int D\mu[q]\exp\left[-\int_{q(-t)=q_0}^{q(0)=q}d\tau L(q(\tau),\dot q(\tau))\right]. \quad (3.2)$$

An appropriate definition[11,12] of the measure $D\mu[q]$ ensures that the function $L(q,\dot q)$ acts as a Lagrangian for the most probable path[11,13] between two given points $q(-t)=q_0$, $q(0)=q$. L then takes the form[11]

$$L=\frac{1}{2\eta}Q^{-1}{}_{\nu\mu}[\dot q^\nu-h^\nu(q)][\dot q^\mu-h^\mu(q)]+\tfrac{1}{2}\sqrt{Q}\frac{\partial}{\partial q^\nu}\frac{h^\nu}{\sqrt{Q}}+\frac{\eta}{12}R\ , \quad (3.3)$$

where

$$Q=\det Q^{\nu\mu},\quad h^\nu=K^\nu-\frac{\eta}{2}\sqrt{Q}\frac{\partial}{\partial q^\mu}\frac{Q^{\nu\mu}}{\sqrt{Q}}\ ,$$

and R is the Riemannian curvature scalar constructed from the contravariant metric tensor $Q^{\nu\mu}(q)$.

In the following we need $L=L_0/\eta+L_1+\eta L_2$ only to order $1/\eta$. Therefore,

$$L_0=Q^{-1}{}_{\nu\mu}[\dot q^\nu-K^\nu(q)][\dot q^\mu-K^\mu(q)]/2\ . \quad (3.4)$$

We assume that there is a unique steady-state probability

density $P(q,\eta)$ to which the conditional probability density tends for $t\to\infty$ for any given initial distribution. Then, we define

$$\phi(q)=\lim_{\eta\to 0}[-\eta\ln P(q,\eta)]$$
$$=\lim_{\eta\to 0}\lim_{t\to\infty}[-\eta\ln P(q\mid q_0,t)]\,. \tag{3.5}$$

The conditional probability density for small η is evaluated from Eq. (3.2) in saddle-point approximation as

$$P(q\mid q_0,t)\sim\exp\frac{1}{\eta}\left[-\min\int_{q(-t)=q_0}^{q(0)=q}L_0(q(\tau),\dot q(\tau))d\tau\right]. \tag{3.6}$$

Here and in the following, min indicates the absolute minimum. According to our assumption of a unique steady-state probability density, Eq. (3.6) must become independent of q_0 for $t\to\infty$.

If the dynamical system $\dot q^{\nu}=K^{\nu}(q)$ has only a single finite attractor $\mathscr{A}$, and the initial points $q(-\infty)=q_0$ for arbitrary q_0 all lie in its basin of attraction, all trajectories which end at $q(0)=q$ after an infinitely long time interval starting at $q(-\infty)=q_0$ must first visit $\mathscr{A}$, where they spend an infinite amount of time, before going on to q in an infinitely long time interval. In order to achieve the minimum in Eq. (3.6), the initial decay from q_0 to $\mathscr{A}$ must occur by satisfying the equations $\dot q^{\nu}=K^{\nu}(q)$, i.e., $L_0\equiv 0$, and, therefore, this initial decay does not contribute in the time integral in Eq. (3.6). As a result we obtain from Eqs. (3.5) and (3.6)

$$\phi(q)=\min\int_{q(-\infty)\in\mathscr{A}}^{q(0)=q}L_0(q,\dot q)d\tau+\text{const}\,. \tag{3.7}$$

For later reference it is useful to note that the potential (3.7) not only governs the asymptotics of the probability density in the steady state but also the asymptotics of the exit rate of the stochastic process (3.1) from the basin of attraction of the attractor $\mathscr{A}$ in the steady state, if such an exit is possible. If $B(\mathscr{A})$ is the boundary of the domain of attraction of $\mathscr{A}$, we may express the exit rate $R(\mathscr{A},\eta)$ as an integral over $B(\mathscr{A})$ of the probability current density obtained from (3.1) with $P(q,\eta,t)\sim\exp[-\phi(q)/\eta]$. We find for small η

$$R(\mathscr{A},\eta)\sim\int_{B(\mathscr{A})}df_{\nu}\left[K^{\nu}+\tfrac{1}{2}Q^{\nu\mu}\frac{\partial\phi}{\partial q^{\mu}}\right]\exp(-\phi/\eta)\,.$$

The surface integral can be evaluated for $\eta\to 0$ by using the saddle-point approximation and we find

$$\min_{q\in B(\mathscr{A})}\phi(q)=-\lim_{\eta\to 0}[\eta\ln R(A,\eta)]\,.$$

It is clear that in order to have a steady state either $R(\mathscr{A},\eta)$ must vanish (i.e., no exit from the domain of attraction of $\mathscr{A}$ is possible) or there must be an entrance rate into the basin of $\mathscr{A}$ which balances $R(\mathscr{A},\eta)$.

In the general case, where the dynamical system $\dot q^{\nu}=K^{\nu}(q)$ has several attractors $\mathscr{A}_i$, the global potential $\phi(q)$ consists of several local pieces, which are joined together by the condition that $\phi(q)$ is minimal. Locally, in the limit of small fluctuations, the system behaves as if there were only a single attractor. Therefore, the local piece of $\phi(q)$, for q in a sufficiently small vicinity of the attractor $\mathscr{A}_i$, is of the form (3.7)

$$\phi_i(q)=\min\int_{q(-\infty)\in\mathscr{A}_i}^{q(0)=q}L_0(q,\dot q)d\tau+C(\mathscr{A}_i)\,. \tag{3.8}$$

The constants $C(\mathscr{A}_i)$ are still arbitrary. In order to join the local pieces $\phi_i(q)$ into a globally defined function $\phi(q)$, the constants $C(\mathscr{A}_i)$ must be fixed relative to each other, apart from a single additive constant C, which is still contained in $\phi(q)$ [and which may be fixed at the end by requiring that the absolute minimum of $\phi(q)$ vanishes, thereby satisfying the normalization condition for $P(q,\eta)$ in Eq. (1.1)]. The principle by which the $C(\mathscr{A}_i)$ are related is the balance of the probability current between the various attractors. For the purpose of the present paper a detailed discussion of this point is not necessary. Therefore, we defer this discussion to a later work where an explicit example will also be considered. An abstract algorithm for determining the constants $C(\mathscr{A}_i)$ is derived in Ref. 7. After the constants $C(\mathscr{A}_i)$ have been determined, the global potential is obtained from the local pieces by

$$\phi(q)=\lim_{i}\phi_i(q)\,. \tag{3.9}$$

Finally, we turn to the case where the matrix $Q^{\nu\mu}(q)$ in the Fokker-Planck equation is singular. In this case, the Lagrangian functional integral (3.2) does not exist and has to be replaced by its Hamiltonian version.[14] Asymptotically for $\eta\to 0$ it takes the form

$$P(q\mid q_0,t)$$
$$=\int D\left[\frac{iP}{2\pi\eta}\right]D[q]$$
$$\times\exp\left[-\frac{1}{\eta}\int_{q(-t)=q_0}^{q(0)=q}d\tau[p_{\nu}(\tau)\dot q^{\nu}(\tau)-H(p(\tau),q(\tau))]\right] \tag{3.10}$$

with

$$H(p,q)=\tfrac{1}{2}Q^{\nu\mu}(q)p_{\nu}p_{\mu}+K^{\nu}(q)p_{\nu}\,. \tag{3.11}$$

Equation (3.10) is again evaluated in saddle-point approximation. When looking for the stationary points of the exponent in (3.10) we consider variations with respect to only real valued functions $q^{\nu}(\tau)$, similarly as in the Lagrangian version, but allow for variations with respect to complex valued functions $p_{\nu}(\tau)\equiv u_{\nu}(\tau)+iv_{\nu}(\tau)$ as the measure of integration in p is no longer real. It follows from the canonical equations that for stationarity the imaginary part v_{ν} of p_{ν} must satisfy

$$Q^{\nu\mu}v_{\mu}=0;\quad\left[Q^{\nu\kappa}\frac{\partial k^{\mu}}{\partial q^{\kappa}}\right]v_{\mu}=0\,,$$

which, apart from exceptional cases, requires $v_{\mu}=0$. Then, the exponent in Eq. (3.10) is real in its stationary points, and the dominant contribution comes from the absolute maximum of the exponent. We obtain

$$P(q \mid q_0,t) \sim \exp\left[-\frac{1}{\eta}\min \int_{q(-t)=q_0}^{q(0)=q} d\tau[p_\nu \dot{q}^\nu - H(p,q)]\right], \tag{3.12}$$

and hence

$$\phi_i(q)=\min \int_{q(-\infty)\in\mathscr{A}_i}^{q(0)=q} d\tau[p_\nu \dot{q}^\nu - H(p,q)] + C(\mathscr{A}_i) . \tag{3.13}$$

The method of fixing the constants $C(\mathscr{A}_i)$ remains unchanged, and the global potential $\phi(q)$ is still given by Eq. (3.9).

IV. NONDIFFERENTIABILITY OF POTENTIALS

Nondifferentiability of the potential $\phi(q)$ given by Eq. (3.9) may occur for two different reasons:

(i) The different local pieces $\phi_i(q)$ are joined in (3.9) in an obviously continuous but not necessarily differentiable way. Therefore, nondifferentiability may occur when the minimum in (3.9) is transferred from one piece to another piece. This problem is not analyzed here further, but will be taken up in a subsequent paper,[15] where an example of physical interest will also be considered.

(ii) Even within a single local piece $\phi_i(q)$, given by Eq. (3.8), nondifferentiability may occur as a result of taking the minimum on the right-hand side of Eq. (3.8). Here, we analyze this phenomenon and we apply our results in Secs. V and VI to the two examples introduced in Sec. II.

In the following we consider, therefore, the stochastic process (3.1) in the domain of attraction of a single attractor $\mathscr{A}_i$. The index i will be dropped for convenience.

Equation (3.8) is a solution of the Hamilton-Jacobi equation

$$H\left[\frac{\partial\phi}{\partial q}, q\right]=0 , \tag{4.1}$$

where

$$H(p,q)=\tfrac{1}{2}Q^{\nu\mu}(q)p_\nu p_\mu + K^\nu(q)p_\nu$$

with the boundary conditions

$$\phi(q)=C, \quad \frac{\partial\phi}{\partial q^\nu}=0 \quad \text{for } q\in\mathscr{A} , \tag{4.2}$$

and the condition

$$\phi(q)>C \quad \text{for } q\notin\mathscr{A} . \tag{4.3}$$

Equations (4.1) and (4.2) have always the trivial solution $\phi\equiv C$. If there is more than one solution also satisfying Eq. (4.3), Eq. (3.8) singles out the minimum among them. The possibility of more than one solution exists, since the boundary conditions (4.2) are prescribed in a singular set of the Hamiltonian system defined by (4.1). We assume now that at least one differentiable solution of (4.1)–(4.3) exists, at least locally in a sufficiently small neighborhood of $\mathscr{A}$. This local solution $\phi(q)$ then defines via $p_\nu=\partial\phi/\partial q^\nu$, $\nu=1,\ldots,n$ a local piece of a separatrix which forms locally an n-dimensional manifold Λ_n in the $2n$-dimensional phase space, spanned by p,q. On the manifold Λ_n the integral $\int p_\nu dq^\nu$ is locally path independent, and Λ_n is, therefore, called Lagrangian.[16] The theory of Lagrangian manifolds, cf. Ref. 16, guarantees, that Λ_n exists even globally and is uniquely generated from the local piece by transporting the local piece through phase space with the Hamiltonian flow

$$\dot{q}^\nu=\frac{\partial H}{\partial p_\nu}, \quad \dot{p}_\nu=-\frac{\partial H}{\partial q^\nu} .$$

We assume for simplicity that Λ_n is simply connected. Then a global function $\tilde{\phi}$ is defined on Λ_n by the path-independent integral on Λ_n,

$$\tilde{\phi}(q,p)=C(\mathscr{A})+\int_{(q\in\mathscr{A},\, p=0)}^{(q,\,p)} p_\nu dq^\nu . \tag{4.4}$$

In all regions of phase space where the projection of Λ_n on q space is one-to-one, $p_\nu=p_\nu(q)$, the function of q defined by this projection via

$$\phi(q)=\tilde{\phi}(q,p(q)) \tag{4.5}$$

is a smooth, differentiable solution of the Hamilton-Jacobi equation with the required boundary condition.

We now examine the regions of phase space where a one-to-one projection of Λ_n on q space does not exist. In such regions the projection of Λ_n on q space gives rise to $l>1$ different branches $p_\nu=p_\nu^{(k)}(q)$, $k=1,\ldots,l$,

$$\phi=\phi^{(k)}(q)=\tilde{\phi}(q,p^{(k)}(q)) , \tag{4.6}$$

and Eq. (3.8) singles out the one with the smallest value of $\tilde{\phi}$,

$$\phi(q)=\min_k \tilde{\phi}(q,p^{(k)}(q)) . \tag{4.7}$$

$\phi(q)$ may become nondifferentiable for those values of q where the minimum on the right-hand side of Eq. (4.7) is transferred between two different branches, say k_1 and k_2. These values of q are, therefore, determined by the condition

$$\tilde{\phi}(q,p^{(k_1)}(q))=\tilde{\phi}(q,p^{(k_2)}(q)) \tag{4.8}$$

which, generally, form $(n-1)$-dimensional hypersurfaces in q space. Since $(q,\ p=p^{(k_1)}(q))$ and $(q,\ p=p^{(k_2)}(q))$ describe points on two different parts of the same Lagrangian manifold Λ_n, they may be connected by an otherwise arbitrary contour of integration on Λ_n, for which Eqs. (4.4) and (4.8) imply

$$\int_{(q,\,p^{(k_1)}(q))}^{(q,\,p^{(k_2)}(q))} p_\nu dq^\nu=0 . \tag{4.9}$$

Equation (4.9) is well known in equilibrium thermodynamics under the name of Maxwell's rule. Equation (4.9) thus seems to generalize that rule for the nonequilibrium potential $\phi(q)$. However, while true superficially, this statement is a little too simple and on a deeper level there appear basic differences, which may be worth mentioning here. In equilibrium thermodynamics Maxwell's rule appears as a rule for thermodynamic averages $\langle q^\nu\rangle$ and their associated thermodynamic potentials, which are again defined by averages over $P(q,\eta)\sim\exp(-\phi(q)/\eta)$.

Maxwell's rule is not yet needed for the fluctuating variables q and $\phi(q)$ itself, the latter being, in thermodynamic terms, still a coarse-grained potential. Therefore, the appearance of Maxwell's rule (4.9) for the coarse-grained potential $\phi(q)$ is more than a generalization from equilibrium thermodynamics and must really be considered as a typical feature of a nonequilibrium system. Indeed, in thermodynamic equilibrium the property of detailed balancing against microscopically defined time reversal ensures that a differentiable potential $\phi(q)$ exists (cf. I), and Maxwell's rule can appear only for the true thermodynamic potential, defined via the partition function obtained by averaging over $P(q,\eta)$.

Now, we can combine Eq. (4.9) with the results of I. In I it was shown that the Lagrangian manifold Λ_n, in general, undergoes wild oscillations as q approaches the vicinity of the deterministic separatrix, which forms the boundary of the basin of attraction of $\mathscr{A}$. These wild oscillations of Λ_n imply that a one-to-one projection of Λ_n on q space is not possible everywhere in the domain of attraction of $\mathscr{A}$ and a smooth differentiable potential $\phi(q)$ does not exist everywhere in that domain. On the other hand, a nondifferentiable but continuous potential $\phi(q)$ still exists and is given by Eq. (4.7). According to Eq. (1.2) this nondifferentiable potential determines the logarithmic asymptotics of the steady-state probability density in the limit of weak noise.

V. PERIODICALLY FORCED NONLINEAR OSCILLATOR

The results of Sec. IV are now applied to the example introduced in Sec. II A. The phase space of that example is four dimensional with coordinates x,y,p_x,p_y. The phase space is compact in the phase variable y, $0 \le y < 2\pi$ and the potential $\phi(x,y)$ must be 2π-periodic in y. The attractors $\mathscr{A}_i$ of the deterministic system are two limit cycles, which for small a are located close to $x=\pm 1$, $y=\omega t$. The deterministic separatrix between $\mathscr{A}_1,\mathscr{A}_2$ is the curve $x=0$, $y=\omega t$. Owing to the symmetry of the model it is sufficient to consider the attractor $\mathscr{A}_1$ near $x=+1$. The two-dimensional Lagrangian manifold Λ_2 emanating from $\mathscr{A}_1$ (a wild separatrix of the Hamiltonian system) has been computed numerically in I and its one-dimensional intersection with the two-dimensional Poincaré surface $H(x,y,p_x,p_y)=0$, $y=0$ with coordinates x,p_x has been plotted there. This intersection exhibits the wild oscillations of Λ_n as the deterministic separatrix at $x=0$ is approached. Owing to these oscillations a differentiable potential does not exist. However, the results of Sec. IV may now be used to construct the nondifferentiable potential, which governs the logarithmic asymptotics of the steady-state probability density. This is easily done by using Maxwell's rule (4.9) to cut through the oscillating parts of Λ_2. A part of the one-dimensional intersection of the resulting discontinuous manifold near $x=0$ is shown in Fig. 1. The following point is noteworthy: On the way from the attractor $\mathscr{A}_1$ near $x=1$ to the deterministic separatrix at $x=0$ the function $p_x(x)$ first has positive slope, i.e., $\phi(x)$ is upwards concave in x; then $p_x(x)$ passes through a point with horizontal slope where $\phi(x)$ has an inflection point, and then $p_x(x)$ has negative slope, i.e., $\phi(x)$ is downwards concave in x. Geometrically, it is clear, in this special case, that the oscillations of $p_x=p_x(x)$ can only appear after the slope of $p_x(x)$ has become negative, i.e., where $\phi(x)$ is downwards concave in x. The reason is that we have chosen the separatrix which is smooth near the attractor near $x=1$. Hence, $\phi(x)$ is differentiable in regions where it is upwards concave and may lose its differentiability only in regions where it is downwards concave.

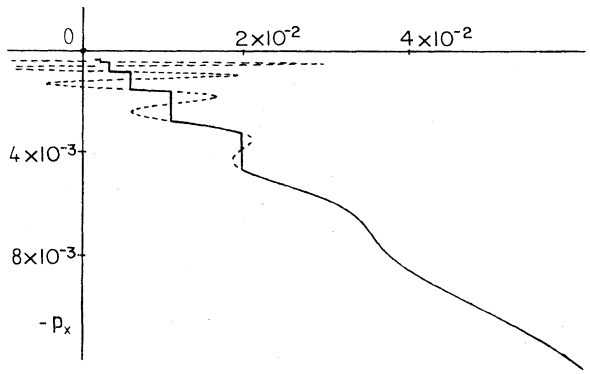

FIG. 1. Reduced unstable manifold of the attractor at $x=1$ in the example of Sec. II A after the oscillating parts (dashed lines) have been cut off by applying Maxwell's rule. (The Poincaré cross section is taken at $y=0$, the parameters are $a=1$, $\omega=10$.)

Now we investigate the behavior of $\phi(x,y)$ in the vicinity of the deterministic separatrix $x=0$. For simplicity, we only consider the intersection of the two-dimensional Lagrangian manifold Λ_2 with the two-dimensional Poincaré surface $y=0$, $H=0$ with coordinates x,p_x. The two-dimensional Lagrangian manifold emanating from $x=0$ intersects the Poincaré surface in a line which is smooth near $x=0$ (but oscillates wildly near $x \in \mathscr{A}_1$) and satisfies the equation $p_x=-\alpha x$ near $x=0$. The constant $\alpha>0$ can be calculated by linearizing the model near $x=0$. The intersection of Λ_2 with the Poincaré surface oscillates wildly for $x \to 0$ around the curve $p_x=-\alpha x$ and intersects that curve approximately horizontally in an infinite number of heteroclinic points. The coordinates of these heteroclinic points close to $x=0$ can be related to each other again by linearizing the model around $x=0$. The linearization allows us to determine the two eigenvalues λ_1,λ_2 of the Poincaré map near the origin, one of which is repelling, $\lambda_1>1$, with its eigenvector along the x direction, while the other is attracting, $\lambda_2=1/\lambda_1<1$, with its eigenvector along the curve $p_x=-\alpha x$. Therefore, if $(x_0,-\alpha x_0)$ is a heteroclinic point near $x=0$, $p_x=0$, then its images $(x_n=\lambda_2^n x_0,-\alpha x_n)$, for $n>0$ integer, are other heteroclinic points in the interval $0<x<x_0$. We note that only next-nearest neighbors of the heteroclinic points are related in this way, being image and preimage of each other.

The areas enclosed by the loops formed by $p_x=-\alpha x$ and the intersection of Λ_2 with the Poincaré surface (cf. Fig. 2) are all equal. For neighboring loops this follows

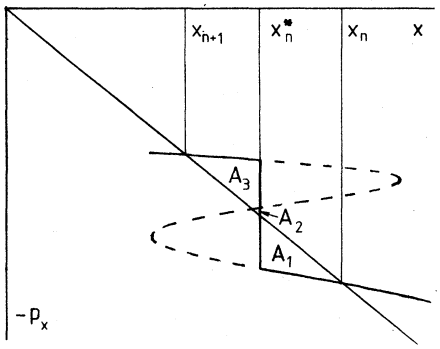

FIG. 2. Magnified part of Fig. 1 illustrating how the coordinate x_n^* of the jump is to be chosen in order to satisfy Maxwell's rule. The straight line is the stable manifold of the hyperbolic point at the origin. x_n and x_{n+1} specify two next-nearest neighboring heteroclinic points.

from the fact that the two manifolds enclosing the loops are both Lagrangian and have the line of intersection in common. To see this, consider the integral $\int (p_x\,dx + p_y\,dy)$ along that line of intersection around a complete round trip $0 \le y < 2\pi$ between two next-nearest neighboring heteroclinic points on the Poincaré surface. This integral may be considered to be carried out on either of the two Lagrangian manifolds. As a result, the contour of integration (with fixed endpoints) may be deformed arbitrarily on either of the two Lagrangian manifolds. This freedom is used to deform the contour of integration in two different ways such that it lies in its first part in the Poincaré surface (see Fig. 2) connecting next-nearest neighboring heteroclinic points along either of the two Lagrangian manifolds, and that in its second part, it makes a complete round trip $0 \le y < 2\pi$ on only one of the Lagrangian manifolds starting and ending in the same x coordinate. The two different ways to carry out the integral along the first part of the contour of integration must give the same result, since the second part of the integration contour is, in both cases, in common and cannot introduce a difference. It then follows in an elementary way that the area enclosed by two neighboring loops must be equal. Next-nearest neighboring loops are related as image and preimage by the Poincaré map, which is area conserving; therefore the area of next-nearest loops must also be equal. Consequently, we may find a general condition for the location of the values $x = x_n^*$ (cf. Fig. 2) where the Maxwell rule is satisfied between two next-nearest neighboring heteroclinic points given by x_n and x_{n+1}. The condition to be fulfilled by x_n^* is the relation $A_1 = A_2 + A_3$ between the areas A_1, A_2, A_3 shown on Fig. 2.

Now we are in a position to determine asymptotically for $x \to 0$ the potential. The near parallelity of the branches intersecting $p_x = -\alpha x$ then implies

$$x_n^* = \frac{x_n + x_{n+1}}{2} = \frac{1+\lambda_2}{2} x_n \ . \tag{5.1}$$

In the interval $x_n^* < x < x_{n-1}^*$ around x_n, the derivative of the potential $\phi(x,0)$ is given by

$$\frac{\partial \phi(x,0)}{\partial x} = -\alpha x_n \ . \tag{5.2}$$

In the endpoints of this interval, $\phi(x,0)$ is not differentiable with respect to x. Equation (5.2) can be integrated in the interval $x_n^* < x < x_{n-1}^*$ and we obtain

$$\phi(x,0) = \phi(x_n^*,0) - \alpha x_n (x - x_n^*), \quad x_n^* < x < x_{n-1}^* \ . \tag{5.3}$$

Using Eqs. (5.1)–(5.3) we can relate $\phi(x_{n+1}^*,0)$ to $\phi(x_n^*,0)$ by

$$\phi(x_{n-1}^*,0) = \phi(x_n^*,0) - \frac{\alpha(1-\lambda_2^2)}{2\lambda_2} x_n^2 \ , \tag{5.4}$$

and hence

$$\phi(x_n^*,0) - \phi(0,0) = \sum_{l=n}^{\infty} [\phi(x_l^*,0) - \phi(x_{l+1}^*,0)]$$

$$= -\frac{\alpha}{2} \lambda_2 x_n^2 \ . \tag{5.5}$$

As a result, the potential $\phi(x,0)$ in the interval around x_n is given by

$$\phi(x,0) = \phi(0,0) - \frac{\alpha}{2} x_n^2 - \alpha x_n (x - x_n) \ ,$$

$$\frac{1+\lambda_2}{2} x_n < x < \frac{1+\lambda_2}{2\lambda_2} x_n \ . \tag{5.6}$$

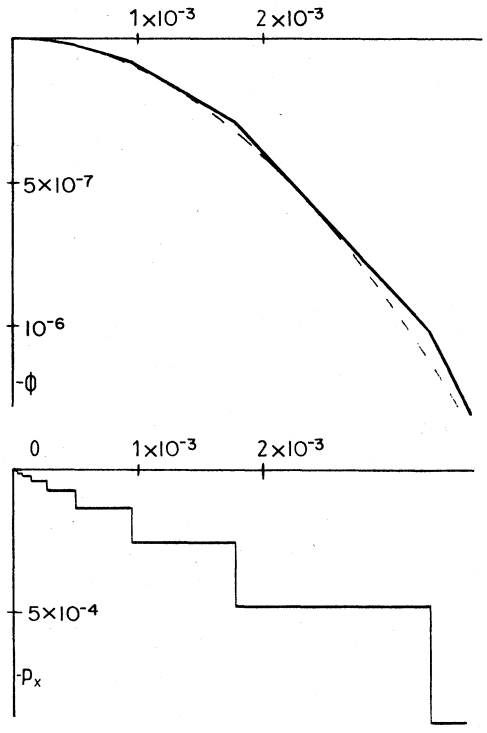

FIG. 3. Continuation of the reduced manifold of Fig. 1 to a close vicinity of the origin (bottom) and the corresponding potential (top). The dashed curve represents the smooth approximate potential $\bar{\phi}$.

Thus $\phi(x)$ for $x\to 0$ is piecewise linear; its linear pieces touch from above the smooth curve $\bar{\phi}=-\alpha x^2/2$. The approximation of the piecewise linear potential $\phi(x,0)$ by the smooth function $\bar{\phi}=-\alpha x^2/2$ becomes exact as $x\to 0$. Figure 3 shows the derivative p_x of the potential obtained after applying Maxwell's rule in a small neighborhood of $x=0$. It also shows the nondifferentiable potential (5.6) itself along with the smooth lower covering curve $\bar{\phi}$.

In I we have constructed a smooth approximate potential for the present model by expanding in the parameter $\epsilon=1/\omega$ [cf. Eq. (2.1)]. We note that these smooth approximations are most accurate near the attractor, where the exact potential is upwards concave and, hence, smooth, and near the separatrix at $x=0$, where the exact nondifferentiable potential closely follows a differentiable curve. The approximation is worst in the regime, somewhere between $x=0$ and the attractor, where the first multivaluedness of the Lagrangian manifold Λ_2 as a function of x occurs.

VI. PHASE-COUPLED NONLINEAR OSCILLATORS

As our next example we take up the phase-coupled nonlinear oscillators defined by Eqs. (2.7). The only deterministic attractor in this model is a limit cycle, which, for small coupling, is close to $r_1^2=a_1$, $r_2^2=a_2$, $\varphi=\omega t$. In addition there is an unstable limit point at $r_1=r_2=0$. We now apply the results of I and Secs. III–V in order to determine $\phi(r_1,r_2,\varphi)$ for this model at least to a reasonable approximation. The details of the calculations are relegated to Appendixes A–C. Here we only summarize the basic procedure and results.

The first step consists in finding the potential in the vicinity of the attractor. This can be done by solving the Hamilton-Jacobi equation (2.9) with the conditions (4.2) and (4.3). In Appendix A the general solution of the Hamilton-Jacobi equation (2.9) to first order in the coupling parameters k_1,k_2 is obtained. The conditions (4.2) and (4.3), following from the extremum principle, now single out a unique particular solution, which is found in Appendix B and given by Eq. (A2) with (B4) or (B5).

The next step is to consider the analytical behavior of this solution near the unstable limit point $r_1=r_2=0$, which is also done in Appendix B. The solution oscillates infinitely rapidly in the vicinity of $r_1=r_2=0$ according to Eq. (B6), indicating that the potential becomes nondifferentiable there.

The nondifferentiable potential near $r_1=r_2=0$ may then be constructed in a similar way as described in Sec. V. As shown in I, the first few heteroclinic intersections of the Lagrangian manifolds emanating from the limit set points, calculated in first-order perturbation theory, already offer a quite accurate estimate of the location of the exact intersection points (which can be found numerically). Thus, a rough estimate of the nondifferentiable potential in a region not too close to $r_1=r_2=0$ could be based on these approximately calculated intersections. Close to $r_1=r_2=0$ the potential becomes piecewise linear and is well approximated from below by the special solution of the Hamilton-Jacobi equation which is smooth near $r_1=r_2=0$. This special solution is given by Eq. (A2) with (A7), taking the arbitrary function G in (A7) equal to 0.

Finally, we look for an approximate potential which is everywhere differentiable. This is possible in the present example, as well as in the preceding one, because a parameter can be identified in which the wild oscillations become exponentially small. This parameter is $1/\omega$ in the present example. An expansion in $1/\omega$, therefore, does not include the oscillations in any order. This is an advantage: It is very reasonable to leave out the oscillations, because the exact nondifferentiable potential does also not include them. The first steps of the $1/\omega$ expansion are carried out in Appendix C. As in the preceding example, the approximation provided by the $1/\omega$ expansion for large ω is expected to be best near the limit sets and to be worst in a region in between, where the first heteroclinic intersections appear.

VII. PROBABILITY DISTRIBUTION FOR FINITE VALUES OF η

It may seem peculiar that the potential describing the logarithmic asymptotics of the probability distribution [see (1.1)] can be nondifferentiable because the probability density itself should satisfy the Fokker-Planck equation and, therefore, is expected to be differentiable. This peculiarity arises from the fact that P is a singular function of η and that according to (1.2) the limit $\eta\to 0$ must be taken. Simple one-dimensional examples are known[7] illustrating that in this limit nondifferentiable potentials may arise. It is interesting, however, to consider in more detail the case of infinitesimally small but finite values of η. We shall see that under this condition the probability distribution is always differentiable although it can be arbitrarily close to a nondifferentiable function.

First, we discuss how the η-independent part of the probability density can be specified. In Appendix D we show that up to the next-to-leading terms in η the distribution is of the form

$$P(q,\eta)\sim z(q)\exp(-\phi(q)/\eta)\ . \tag{7.1}$$

The equation for the prefactor z can then immediately be derived by inserting (7.1) into the Fokker-Planck equation (3.1). Assuming for a moment that $\phi(q)$ is smooth, this reads

$$\left[K^\nu+Q^{\nu\mu}\frac{\partial\phi}{\partial q^\mu}\right]\frac{\partial z}{\partial q^\nu}+\left[\frac{\partial K^\nu}{\partial q^\nu}+\frac{\partial Q^{\nu\mu}}{\partial q^\mu}\frac{\partial\phi}{\partial q^\nu}+\tfrac{1}{2}Q^{\nu\mu}\frac{\partial^2\phi}{\partial q^\nu\partial q^\mu}\right]z=0\ . \tag{7.2}$$

It is convenient to start by solving (7.2) on the attractors of the deterministic system. As the first derivatives of ϕ vanish there, we obtain

$$K^\nu(q)\frac{\partial z_0(q)}{\partial q^\nu}+\left[\frac{\partial K^\nu}{\partial q^\nu}+\tfrac{1}{2}\phi^{\nu\mu}\frac{\partial^2\phi}{\partial q^\nu\partial q^\mu}\right]z_0(q)=0\ , \tag{7.3}$$

where $z_0(q)$ is the prefactor on the attractor $\mathcal{A}_i$. Since the attractors are invariant under the drift K^ν, Eq. (7.3)

can be solved separately there. In solution of (7.3) contains an unspecified multiplicative constant, whose value is specified by normalizing the complete distribution. In the following we assume, for simplicity, that the system under consideration possesses only a single attractor which is nonchaotic, e.g., a limit cycle.

One can now proceed to solve the partial differential equation (7.2). For our purposes it is convenient to give its solution in a parametric representation (cf. Ref. 17). We recall that the canonical equation of q in the Hamiltonian dynamics restricted to the Lagrangian manifold through the attractor reads

$$\dot{q}^{\nu}=K^{\nu}+Q^{\nu\mu}p_{\nu}=K^{\nu}+Q^{\nu\mu}\partial\phi/\partial q^{\nu}\ . \tag{7.4}$$

By means of this relation we may write the first term of (7.2) as a total time derivative of $z(\bar{q}(t))$ where $\bar{q}(t)$ denotes a trajectory of (7.4). This trajectory is just the most probable path starting in the attractor and ending in point q. As we have shown in Sec. III the time interval along this path is infinitely long. Thus the parametric solution of (7.2) reads

$$z(q)=z_0(\bar{q}_0)\exp\int_{-\infty}^{0}A^{\nu}{}_{\nu}(t)dt \tag{7.5}$$

with

$$A^{\nu}{}_{\nu}(t)=\frac{\partial K^{\nu}}{\partial q^{\nu}}+\frac{\partial Q^{\nu\mu}}{\partial q^{\mu}}\frac{\partial\phi}{\partial q^{\nu}}+\tfrac{1}{2}Q^{\nu\mu}\frac{\partial^2\phi}{\partial q^{\nu}\partial q^{\mu}}\bigg|_{q=\bar{q}(t)}, \tag{7.6}$$

where the trajectory $\bar{q}(t)$ has been chosen in such a way that it starts at $q(-\infty)=\bar{q}_0$ on the attractor and ends at $q(0)=q$. One can show that the appearance of $\bar{q}_0$ in (7.5) is only formal, i.e., $z(q)$ does not change if $\bar{q}_0$ is shifted along the attractor. In Appendix D we derive (7.5) and (7.6) from the path-integral solution (3.2) of the Fokker-Planck equation.

Let us now concentrate on a small neighborhood of a point q^* where the global potential (3.9) is not differentiable. We recall that in a nondifferentiability point q^* there are two different trajectories which simultaneously minimalize the action globally. In the vicinity of such a point there must then be two different trajectories yielding approximately the same local minimum value of the action. If η is finite, in a certain region around q^*, both contributions must be kept. Let us denote the actions belonging to the two trajectories by ϕ_1 and ϕ_2. The corresponding solutions of (7.5) and (7.6) are denoted by z_1 and z_2. Note that the function z_0 appearing in (7.5) is the same for both trajectories as they start from the same attractor. According to the path-integral solution, the probability distribution in the vicinity of the point q^* is the sum of both solutions and reads

$$P(q,\eta)\sim z_1(q)\exp[-\phi_1(q)/\eta]+z_2(q)\exp[-\phi_2(q)/\eta]\ . \tag{7.7}$$

For $q=q^*$, ϕ_1 and ϕ_2 are, of course, equal. For any fixed q in the neighborhood of q^* one finds sufficiently small values of η such that one of the two terms is negligible. Therefore, $\phi(q)$ as defined by (1.2) is not differentiable for $q=q^*$. If, however, η is a fixed small number, one can find values of q sufficiently close to q^* such that both terms must be kept. Thus, $P(q,\eta)$ is continuous and differentiable at q^*.

It is easy to find an estimate for the size Δq^{ν}, $\nu=1,2,\ldots,n$ of the region around q^* where both terms of (7.7) are of comparable importance. By expanding the exponents and introducing the jump in the momentum through

$$\Delta p_{\nu}=\frac{\partial\phi_2}{\partial q^{\nu}}\bigg|_{q^*}-\frac{\partial\phi_1}{\partial q^{\nu}}\bigg|_{q^*}, \tag{7.8}$$

one finds

$$|\Delta\rho_{\nu}\Delta q^{\nu}|<c\eta\ , \tag{7.9}$$

where c is of order unity. It may happen that within the region of size Δq^{ν} another nondifferentiability point appears which then means that in a certain region, $P(q,\eta)$ is the sum of several terms of the form $z_i(q)\exp[-\phi_i(q)/\eta]$. Outside the regions of size Δq^{ν} around nondifferentiability points, the probability density $P(q,\eta)$ is accurately approximated by (7.1) for small η.

VIII. CONCLUSIONS

In this paper we have considered the possibility of describing autonomous dissipative dynamical systems in their steady state by a macroscopic potential which is an analog or a generalization of a thermodynamic potential. It may be worth stressing here that the analysis presented applies regardless of whether the systems are in thermodynamic equilibrium or far from it.

In order to put our results into perspective let us give special attention to thermodynamic equilibrium for a moment. There the potential $\phi(q)$ which we have considered has the meaning of a differentiable coarse-grained thermodynamic potential. {It should be clear that $\phi(q)$ does not reduce to a true thermodynamic potentials, since $\phi(q)$ still governs fluctuations via the formula of Boltzmann and Einstein, $P(q)\sim\exp[-\phi(q)/\eta]$, while the corresponding true thermodynamic potential would be given by the average

$$S(u)=-\eta\ln\int dq\exp\{-[\phi(q)+\lambda_{\nu}q^{\nu}]/\eta\}-\lambda_{\nu}u^{\nu}\ ,$$

where λ_{ν} and u^{ν} are related by

$$u^{\nu}=-\eta\frac{\partial}{\partial\lambda_{\nu}}\ln\int dq\exp\left[-\frac{\phi(q)+\lambda_{\nu}q^{\nu}}{\eta}\right].$$

Thus, $S(u)$ is the Legendre transform of the characteristic function of $P(q)$, with the property $\min S(u)=S(\langle q\rangle)$, where $\langle q\rangle$ is the mean value of q. The true thermodynamic potential S is strictly upwards concave; the coarse-grained thermodynamic potential is not, because it is not yet averaged over the fluctuations of q.} In thermodynamic equilibrium the general properties of $\phi(q)$ are well known, such as the Boltzmann-Einstein fluctuation formula, the Arrhenius factor for the average jump rate over a potential barrier, the minimum property of $\phi(q)$ for a "coarse-grained" or "most-probable" or "mean-field" equilibrium state, the H theorem, according to which $\phi(q)$

only decreases deterministically and requires a fluctuation in order to increase, or the definition of conjugate thermodynamic variables p_ν, q^ν by $d\phi = \sum_\nu p_\nu dq^\nu$ and their relation by the "equations of state" $p_\nu(q) = \partial\phi(q)/\partial q^\nu$. In thermodynamic equilibrium it is also well known how to determine $\phi(q)$, either from a basic microscopic description or from a phenomenological set of equations of state.

Outside thermodynamic equilibrium the description of steady states by generalized thermodynamic potentials is far less popular, and the existence of such a description, which carries over all of the above-mentioned properties to nonequilibrium steady states, seems not to be generally appreciated. From a physical point of view such a description has been advocated, e.g., in Refs. 9, 10, 18–20. Important mathematical work on this problem has been done in the last decade by Freidlin and Wentzell (Ref. 7 and references given therein). It seems clear that these potentials really should be considered as the central objects of nonequilibrium thermodynamics, and therefore the general properties and methods to construct them are of high interest.

In the present paper we have focused our attention on the particular property of nondifferentiability of the potential $\phi(q)$ which may appear in nonequilibrium steady states. We have also discussed how the nondifferentiability of the potential is reconciled with the differentiability of the probability density in the steady state for small but fixed noise intensity. Nondifferentiability of $\phi(q)$ has been shown here to arise from two different causes: (i) the crossing of different branches of $\phi(q)$ arising from two different attractors; this cause was not analyzed here further, but will be considered in detail in a subsequent paper,[15] where its origin in the nonequilibrium situation will also become clear; (ii) the crossing of different branches of $\phi(q)$ emerging from the same attractor; here and in I, this case was shown to be typical for nonequilibrium steady states.

We have presented two examples of driven systems with attracting limit cycles which show this phenomenon. We have also elucidated the central role of the minimum principle satisfied by $\phi(q)$ in defining a unique global potential, by joining the various differentiable pieces together via Maxwell's rule. The minimum principle (3.7) is again well known for the case of equilibrium thermodynamics (including relaxation towards thermodynamic equilibrium) as the principle of minimum entropy production or maximum energy dissipation (cf. Ref. 20). The emergence of Maxwell's rule from the minimum principle leads to the result that nondifferentiability in a certain variable typically occurs near maxima of $\phi(q)$ in that variable. In thermodynamic equilibrium, as was shown in Ref. 2, detailed balancing ensures that the Hamiltonian has a smooth separatrix, the coarse-grained potential $\phi(q)$ is, therefore, differentiable, and Maxwell's rule is not needed for $\phi(q)$.

We have also presented explicit results for the potential $\phi(q)$ in the two examples of nonequilibrium systems with attracting limit cycles. The application of Maxwell's rule was considered in detail for the simpler of the two examples, but the results can easily be generalized to the more complicated one. In particular, the approximation of the exact nondifferentiable potential by analytical differentiable functions obtained by expanding in a suitably determined parameter has been considered for both examples. The most important problem left open here is the nondifferentiability of the potential $\phi(q)$ due to the coexistence of several attractors. This problem will be analyzed in a subsequent paper.[15]

APPENDIX A

In this appendix we derive the solution of the Hamilton-Jacobi equation (2.9) which after taking into account the boundary conditions yields the potential for the system of coupled oscillators to first order in the parameters k_1, k_2. The method is essentially the same as that used by Hellwig.[31]

In the case of uncoupled oscillators the potential is given by

$$\bar{\phi}_0 = -\tfrac{1}{2}(a_1 r_1^2 + a_2 r_2^2) + \tfrac{1}{4}(r_1^4 + r_2^4) \tag{A1}$$

as this is a solution of (2.9) whose first derivatives vanish in all limit sets of the deterministic system (i.e., for $\{r_1^2 = 0 \text{ or } a_1,\ r_2^2 = 0 \text{ or } a_2,\ 0 \le \varphi < 2\pi\}$). When looking for how the potential is modified in the presence of couplings, we write to first order in k_1, k_2

$$\phi = \bar{\phi}_0 + [k_1 + k_2 \cos(2\varphi)] r_1^2 r_2^2/2 + k_2 W(r_1, r_2, \varphi) . \tag{A2}$$

The linear equation obtained for W is then particularly simple:

$$r_1(r_1^2 - a_1)\frac{\partial W}{\partial r_1} + r_2(r_2^2 - a_2)\frac{\partial W}{\partial r_2} + \omega\frac{\partial W}{\partial \varphi} = \omega r_1^2 r_2^2 \sin(2\varphi) . \tag{A3}$$

The characteristics of (A3) are given by

$$2\varphi + \frac{\omega}{a_i}\ln\left|\frac{r_i^2}{a_i - r_i^2}\right| = c_i, \quad i = 1,2 . \tag{A4}$$

Therefore, the solution of the homogeneous equation is an arbitrary function of the combinations c_1 and c_2, and the particular solution of the inhomogeneous equation obtained by integrating (A3) along the characteristics, is of the form

$$I(r_1, r_2, \varphi) = \int_{\varphi_0}^{\varphi} R_1(x) R_2(x) \sin(2x) dx , \tag{A5}$$

where

$$R_i(x) = a_i\left[1 + \left[\frac{a_i}{r_i^2} - 1\right]\exp\frac{2a_i}{\omega}(x - \varphi)\right]^{-1}, \quad i = 1,2 . \tag{A6}$$

It is convenient to choose $\varphi_0 = \infty$ in (A5) since then the integral is well defined and manifestly 2π-periodic in φ. Thus, after introducing new variables, we may write the general solution of (A3) as

$$W=I+G=-\frac{\omega}{2}a_2\int_1^\infty\left[1+\left[\frac{a_1}{r_1^2}-1\right]z\right]^{-1}\left[1+\left[\frac{a_2}{r_2^2}-1\right]z^{a_2/a_1}\right]^{-1}\mathrm{Im}(e^{2i\varphi}z^{i\omega/a_1-1})dz$$
$$+G\left[2\varphi+\frac{\omega}{a_1}\ln\left|\frac{r_1^2}{a_1-r_1^2}\right|,2\varphi+\frac{\omega}{a_2}\ln\left|\frac{r_2^2}{a_2-r_2^2}\right|\right], \tag{A7}$$

where G must be an arbitrary function 2π-periodic in φ as it follows from the requirement of periodicity of the potential. (A1), (A2), and (A5) taken together yield the solution given in (2.10).

In order to find an explicit expression for the integral term I, we consider the case $a_1=a_2\equiv a$. By decomposing the integrand into partial fractions, one obtains

$$I=\frac{\omega}{2}\frac{r_1^2r_2^2}{r_1^2-r_2^2}\mathrm{Im}\left[e^{2i\varphi}\left[\frac{a}{a-i\omega}[F(1,1-i\omega/a;2-i\omega/a;r_1^2/(r_1^2-a))-F(1,1-i\omega/a;2-i\omega/a;r_2^2/(r_2^2-a))]\right]\right], \tag{A8}$$

where F stands for hypergeometric functions.

APPENDIX B

We now investigate the influence of the boundary conditions on the behavior of the solution. The existence of a differentiable potential requires a smooth solution of (2.9) with vanishing first derivatives in the limit sets of the deterministic system.[1] In other words, there must exist a single smooth separatrix in the associated Hamiltonian system, given by $p_i=\partial\phi/\partial r_i$, $i=1,2$, $p_\varphi=\partial\phi/\partial\varphi$ which joins the limit sets.

It follows immediately from (A5)–(A7) that such a smooth potential does not exist in the presence of phase coupling, $k_2\neq0$. The reason is that the function $I(r_1,r_2,\varphi)$ exhibits an essential singularity for $r_1^2\to a_1$, $r_2^2\to a_2$ and although it may be compensated by an appropriate choice of G, the same choice generates nonanalytic behavior at the origin ($r_1=r_2=0$) owing to the special form of G.

In order to be more explicit, we turn to the case $a_1=a_2\equiv a$. I as given by (A8) is analytic for $r_1^2,r_2^2<a$. By applying the transformation formulas of the hypergeometric function,[21] (A8) may be rewritten as

$$I=-\frac{\omega}{2}\frac{r_1^2r_2^2}{r_1^2-r_2^2}\mathrm{Re}\left[e^{2i\varphi}\left\{\frac{\pi}{\sinh(\pi\omega/a)}\left[\left[\frac{a-r_1^2}{r_1^2}\right]^{1-i\omega/a}-\left[\frac{a-r_2^2}{r_2^2}\right]^{1-i\omega/a}\right]\right.\right.$$
$$\left.\left.-\frac{ia}{a-i\omega}[F(1,i\omega/a-1;i\omega/a;(r_1^2-a)/r_1^2)-F(1,i\omega/a-1;i\omega/a;(r_2^2-a)/r_2^2)]\right\}\right]. \tag{B1}$$

The term proportional to $\pi/\sinh(\pi\omega/a)$ exhibit here nonanalytic oscillations for $r_1^2,r_2^2\to a$. At $r_2^2=a$, for example, the asymptotic behavior of I is given for $r_1^2\to a\,(r_1^2<a)$ by

$$I\sim\frac{1}{2}\frac{\pi\omega a}{\sinh(\pi\omega/a)}\cos\left[2\varphi-\frac{\omega}{a}\ln\frac{a-r_1^2}{a}\right]. \tag{B2}$$

The investigation above shows that with an identically vanishing G in (A7) one would find a potential which is smooth at $r_1=r_2=0$ but no longer analytic at the other limit sets. One might hope that by an appropriately chosen G it is possible to eliminate the oscillations. In fact, by taking, for example,

$$G(u_1,u_2)=\frac{\omega}{2}\frac{\pi a}{\sinh(\pi\omega/a)}\times\frac{\cos u_1-\exp[a(u_1-u_2)/\omega]\cos u_2}{-1+\exp[a(u_1-u_2)/\omega]} \tag{B3}$$

for $r_1^2,r_2^2<a$ in (A7), the oscillating part of W at $r_1^2=r_2^2=a$ disappears; at the origin, however, a new one will be created. Thus, no solution can be found which is smooth everywhere.

Therefore, we may demand the smoothness of the potential in one of the limit sets only. According to the extremum principle formulated in Sec. III this particular limit set is naturally chosen to be the only attractor of our system, which is at $r_1^2=r_2^2=a$ in leading order in k_2, where the probability distribution has its absolute maximum. So we find that the potential of the coupled oscillator system is given by (A2) with

$$W=-\frac{\omega}{2}\frac{r_1^2r_2^2}{r_1^2-r_2^2}\mathrm{Im}\left[e^{2i\varphi}\frac{a}{a-i\omega}[F(1,i\omega/a-1;i\omega/a;(r_1^2-a)/r_1^2)-F(1,i\omega/a-1;i\omega/a;(r_2^2-a)/r_2^2)]\right]. \tag{B4}$$

By means of identities among hypergeometric functions we rewrite W as

$$W=\frac{\omega}{2}\frac{r_1^2r_2^2}{r_1^2-r_2^2}\,\mathrm{Im}\left[e^{2i\varphi}\left\{\frac{i\pi}{\sinh(\pi\omega/a)}\left[\left[\frac{a-r_1^2}{r_1^2}\right]^{1-i\omega/a}-\left[\frac{a-r_2^2}{r_2^2}\right]^{1-i\omega/a}\right]\right.\right.$$
$$+\frac{a}{a-i\omega}[F(1,1-i\omega/a;2-i\omega/a;r_1^2/(r_1^2-a))$$
$$\left.\left.-F(1,1-i\omega/a;2-i\omega/a;r_2^2/(r_2^2-a))]\right\}\right]. \tag{B5}$$

One sees from this form that the potential at any fixed value of r_2 (r_1) oscillates for $r_1\to 0$ ($r_2\to 0$) as

$$W\sim-\frac{\omega}{2}\frac{\pi a}{\sinh(\pi\omega/a)}\cos\left[2\varphi+\frac{\omega}{a}\ln\frac{r_1^2}{a}\right]. \tag{B6}$$

APPENDIX C

Finally, we turn to a short discussion of how an *approximate* smooth potential can be constructed. The method is based on the observation that the oscillating part of the potential obtained in a perturbative calculation in a certain parameter of the system is generally a nonanalytic function of another parameter (cf. I). In the present example this parameter is $1/\omega$ [see (B1), (B2), and (B5), (B6)]. Therefore, in a $1/\omega$ expansion the oscillations are entirely suppressed and one obtains just the nonoscillating part of the separatrices and the corresponding smooth approximate potential. Especially for large values of ω, where the prefactor of the oscillating part is extremely small [see (B2) and (B6)], the method provides a very accurate approximation.

We look for a potential in the form

$$\bar{\phi}(r_1,r_2,\varphi)=\sum_{n=0}^{\infty}\omega^{-n}\phi_n(r_1,r_2,\varphi), \tag{C1}$$

where $\bar{\phi}$ is assumed to differ from a solution of (2.9) only by terms exponentially small in $1/\omega$. By substituting (C1) into (2.9) one finds only one term proportional to ω, consequently $\partial\phi_0/\partial\varphi=0$. From the ω-independent part,

$$\left[\frac{\partial\phi_0}{\partial r_1}\right]^2+\left[\frac{\partial\phi_0}{\partial r_2}\right]^2$$
$$+\frac{\partial\phi_0}{\partial r_1}\{a_1-r_1^2-r_2^2[k_1+k_2\cos(2\varphi)]\}r_1$$
$$+\frac{\partial\phi_0}{\partial r_2}\{a_2-r_2^2-r_1^2[k_1+k_2\cos(2\varphi)]\}r_2=-\frac{\partial\phi_1}{\partial\varphi} \tag{C2}$$

follows. The sum of φ-independent terms must vanish, otherwise ϕ_1 could not be periodic in φ. This determines ϕ_0 to be

$$\phi_0=-\tfrac{1}{2}(a_1r_1^2+a_2r_2^2)+\tfrac{1}{4}(r_1^4+r_2^4)+\tfrac{1}{2}k_1r_1^2r_2^2. \tag{C3}$$

The integration of (C2) then yields

$$\phi_1=\tfrac{1}{2}k_2r_1^2r_2^2[(r_1^2+r_2^2)(1+k_1)-a_1-a_2]\sin(2\varphi)$$
$$+g_1(r_1,r_2), \tag{C4}$$

where g_1 is an unspecified function at this stage. In first order of $1/\omega$ one obtains an equation for g_1 from the requirement of periodicity. This equation allows for the trivial solution besides a nontrivial one. However, as the nontrivial solution of this equation is not everywhere bounded, $g_1\equiv 0$ is chosen.

The approximate potential provides a smooth function whose derivatives vanish in *all* limit sets. One may easily see that the expression (C1) with (C3) and (C4) is consistent with that found in Appendix B because up to terms proportional to $1/\omega$ the expansion of (B4) agrees with the present result to leading order in k_1,k_2 after $a_1=a_2\equiv a$ has been set.

APPENDIX D

Our aim here is to calculate the stationary probability distribution of a Fokker-Planck model up to next-to-leading terms in the noise intensity, starting from its representation as a functional integral. This representation is provided by the path integral (3.2) for $t=\infty$ with the Lagrangian $L=L_0/\eta+L_1$, where L_0 is given by (3.4) and

$$L_1=\frac{1}{2}\frac{\partial K^\nu}{\partial q^\nu}+\tfrac{1}{2}Q^{-1}{}_{\nu\mu}\frac{\partial Q^{\mu\lambda}}{\partial q^\lambda}(\dot{q}^\nu-K^\nu)-\frac{1}{4}\frac{d}{d\tau}\ln Q \tag{D1}$$

follows from (3.3). When performing the functional integral it is sufficient, for the desired accuracy, to use the most probable path (7.4) in $\int L_1\,d\tau$, while in L_0 one has to go beyond the saddle-point approximation by considering also Gaussian fluctuations around the most probable path. Therefore, it is convenient to rewrite L_0 as

$$L_0=\frac{d\phi}{d\tau}+\tilde{L}_0$$
$$=\frac{d\phi}{d\tau}+\tfrac{1}{2}Q^{-1}{}_{\nu\mu}\left[\dot{q}^\nu-K^\nu-Q^{\nu\lambda}\frac{\partial\phi}{\partial q^\lambda}\right]$$
$$\times\left[\dot{q}^\mu-K^\mu-Q^{\mu\lambda}\frac{\partial\phi}{\partial q^\lambda}\right], \tag{D2}$$

which is an identity by virtue of the Hamilton-Jacobi equation

$$\tfrac{1}{2}Q^{\nu\mu}\frac{\partial\phi}{\partial q^\nu}\frac{\partial\phi}{\partial q^\mu}+K^\nu\frac{\partial\phi}{\partial q^\nu}=0 \tag{D3}$$

satisfied by $\phi(q)$. In the path integral we choose $y^\nu(\tau)$, the derivation between the actual and the most probable trajectory, as the variable of integration. Since $\tilde{L}_0$ vanishes along the most probable path, which we denote by

$q=\bar{q}(\tau)$, $\tilde{L}_0$ up to quadratic terms in y is given by

$$\tilde{L}_0(y,\dot{y})=\tfrac{1}{2}Q^{-1}\nu\mu(\tau)[\dot{y}^{\nu}-A^{\nu}_{\lambda}(\tau)y^{\lambda}][\dot{y}^{\mu}-A^{\mu}_{\lambda}(\tau)y^{\lambda}] \tag{D4}$$

with

$$A^{\nu}_{\mu}(\tau)=\frac{\partial K^{\nu}}{\partial q^{\mu}}+\frac{\partial Q^{\nu\lambda}}{\partial q^{\mu}}\frac{\partial\phi}{\partial q^{\lambda}}+Q^{\nu\mu}\frac{\partial^2\phi}{\partial q^{\nu}\partial q^{\mu}}\bigg|_{q=\bar{q}(\tau)},$$
$$Q^{\nu\mu}(\tau)=Q^{\nu\mu}(q)\,|_{q=\bar{q}(\tau)}\,. \tag{D5}$$

In the following considerations we restrict ourselves to the case of a single attractor of the dynamical system $\dot{q}^{\nu}=K^{\nu}(q)$ and consider only points in its domain of attraction. In Sec. III we have seen that the most probable trajectory which starts at an arbitrary point $q^{(0)}$ and ends at a given point $q(0)=q$ after an infinitely long time must visit the attractor where it spends an infinite amount of time. Therefore, we split the most probable trajectory into an initial part between $q^{(0)}$ and $\bar{q}_0$, where $\bar{q}_0$ belongs to the attractor, and into a final part between $\bar{q}_0$ and q. The time interval on both parts of the path is infinitely long. The path integral yielding the stationary probability distribution $P(q,\eta)$ up to the desired accuracy can then be written as

$$P(q,\eta)=\exp\{-[\phi(q)-\phi(\bar{q}_0)]/\eta\}\exp\left[-\int_{-\infty}^{0}d\tau L_1(\tau)\right]$$
$$\times\int D\mu[y]\,dy(-\infty)\exp\left[-\tfrac{1}{2}\int_{y(-\infty)}^{y(0)=0}d\tau\tilde{L}_0(y,\dot{y})\right]P(\bar{q}_0+y(-\infty),\eta)\,. \tag{D6}$$

Here $P(\bar{q}_0+y(-\infty),\eta)$ is the result of the path integral over the fluctuations from the initial part of the most probable path. The measure appearing in (D6) is given on a time lattice of size ϵ through

$$D\mu[y]=\lim_{N\to\infty}\prod_{j=-N+1}^{-1}\frac{d^n y(\tau_j)}{\sqrt{(2\pi\epsilon\eta)^n Q[\bar{q}(\tau_j)]}}$$
$$\times\frac{1}{\sqrt{(2\pi\epsilon\eta)^n Q(q)}} \tag{D7}$$

(cf. Ref. 22), where n is the dimensionality of the stochastic system.

For the sake of clarity we evaluate here the path integral (D6) and (D7) for a general process of one variable only. The extension to several variables is then straightforward. It is useful to introduce an extra integration over $y(0)$ by introducing $\delta[y(0)]$ in the integrand and to replace $\delta[y(0)]$ by its Fourier integral. Next, we introduce the new variable of integration $u(\tau)$ by

$$\dot{u}(\tau)=\dot{y}(\tau)-A(\tau)y(\tau)\,, \tag{D8}$$

where $A(\tau)$ is given by (D5) specialized for the one-dimensional case. The Jacobian of the transformation (D8), whose integrated form is a Volterra equation, is $J=|\delta y/\delta u|=\exp\int_{-\infty}^{0}d\tau A(\tau)/2$. Equation (D6) now appears in the form

$$P(q,\eta)=\exp\{-[\phi(q)-\phi(\bar{q}_0)]/\eta\}\exp\int_{-\infty}^{0}d\tau[A(\tau)/2-L_1(\tau)]$$
$$\times\int D\mu[u]\,dy(-\infty)\frac{d\alpha}{2\pi}\exp\left[-\frac{1}{2\eta}\int d\tau Q^{-1}(\tau)\dot{u}^2(\tau)+i\alpha y_0[\dot{u}]\right]$$
$$\times P(\bar{q}_0+y(-\infty),\eta)\,, \tag{D9}$$

where, according to (D8)

$$y_0[\dot{u}]=y(-\infty)\exp\int_{-\infty}^{0}d\tau A(\tau)+\int_{-\infty}^{0}d\tau\exp\left[\int_{\tau}^{0}d\tau' A(\tau')\right]\dot{u}(\tau)\,. \tag{D10}$$

After doing the Gaussian path integral over u and properly taking into account the explicit time dependence of the prefactor of $\dot{u}^2(\tau)$ in the discretized form of the path integral (cf. Ref 22) one finds

$$P(q,\eta)=\exp\{-[\phi(q)-\phi(\bar{q}_0)]/\eta\}\exp\int_{-\infty}^{0}d\tau[A(\tau)/2-L_1(\tau)]\left[\frac{Q(\bar{q}_0)}{Q(q)}\right]^{1/4}$$
$$\times\int dy(-\infty)\frac{d\alpha}{2\pi}\exp\left[i\alpha y(-\infty)\exp\int_{-\infty}^{0}d\tau A(\tau)\right.$$
$$\left.-\frac{\eta}{2}\alpha^2\int_{-\infty}^{0}d\tau Q(q)\exp 2\int_{\tau}^{0}d\tau' A(\tau')\right]. \tag{D11}$$

Within the desired accuracy in η we may now neglect the term proportional to η in the exponent of Eq. (D11). The integration over α then yields a δ function which selects the point $y(-\infty)=0$ in the integration over $y(-\infty)$. We obtain

$$P(q,\eta)=\exp[-\phi(q)/\eta]\exp\left[-\int_{-\infty}^{0}d\tau\left[\frac{\partial K}{\partial q}+\frac{\partial Q}{\partial q}\frac{\partial\phi}{\partial q}+\tfrac{1}{2}Q\frac{\partial^2\phi}{\partial q^2}\right]\Bigg|_{q=\bar{q}(\tau)}\right]z_0(\bar{q}_0)\,, \tag{D12}$$

where

$$z_0(\bar{q}_0)=\lim_{\eta\to 0}\,[P(\bar{q}_0,\eta)\exp\phi(\bar{q}_0)/\eta]$$

is a function on the attractor.

In the case of several variables the argument of the τ integral in (D12) is to be replaced by the trace of the corresponding matrix, and $z_0(\bar{q}_0)$ again denotes the prefactor on the attracting set. Thus, the path-integral solution recovers the result (7.5) if there is a single trajectory minimizing the action. But the method also enables us to consider cases where two or more different trajectories starting on the attractor and ending at the same point q approximately yield the same minimum value of the action. Then, $P(q,\eta)$ is given by a sum of terms each like in (D12) with ϕ and $\bar{q}(\tau)$ belonging, respectively, to the different trajectories in question. For the case of two such most-probable trajectories the expression for the probability distribution is given by (7.7).

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