Exactly solvable model of a particle interacting with a field: The origin of a quantum-mechanical divergence

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The dynamically exactly solvable system of a local oscillator coupled harmonically to a finitely extended one-dimensional string (continuum transmission line field) is studied in detail. An earlier recognized quantum-mechanical logarithmic ultraviolet divergence in the oscillator's kinetic fluctuations is shown to originate in the infinitely strong (rigid) coupling limit. For any finite coupling strength the fluctuations are finite. *Inter alia* attention is given to such aspects as mode stability, nonorthogonality of the eigenfunctions, the essential singular nature of the eigenmatrix, completeness relations, the initial-value problem, infinite-system limit, quantum exponential decay, coherent-state evolution, and the thermal string model.

I. INTRODUCTION

Physical observations (measurements) are generally made on subsystems. The system of interest is embedded in a (much) large system, and the interaction between the two parts is usually considered to be weak in some sense. Normally, the precise dynamical features of the "rest of the system" are eliminated from the modeling of the "subsystem under observation" in favor of a statistical description. In this spirit the (infinitely) large environment is typically treated as a heat bath (or thermal reservoir). In consequence of the coupling the subsystem loses part of its coherent (i.e., directed) energy from its few degrees of freedom to the reservoir, where this energy is (often rapidly) distributed randomly among the very many degrees of freedom of the environment. In turn, the reservoir then couples incoherent energy (noise) back into the subsystem. In this way the well-known relation between dissipation and fluctuations is physically realized.

The precise mathematical formulation of the above-described features has been recognized to be highly nontrivial. In view of the encumbering difficulties, and also for practical purposes since the noise is often small, dissipation has been described historically most frequently by means of classical phenomenology within a Newtonian framework. The standard textbook example is the simple linearly damped harmonic oscillator, i.e., $\ddot{\zeta}+2\lambda\dot{\zeta}+\Omega^2\zeta=0$, where λ is often called the friction coefficient (see, e.g., Ref. 1). However, for good reasons modern mathematical physics relies heavily on the Lagrange-Hamilton formulation of mechanics. This is true in particular for quantum theory. Unfortunately, it is highly problematic to cope with time-irreversible phenomena within this framework in any simple consistent manner. An interesting, though still controversial, attempt is known in the so-called time-dependent Caldirola-Kanai Hamiltonian (see, e.g., Refs. 2—5). Another example may be provided by the author's quasi-Hamiltonian theory, invoking complex dynamical variables (see, e.g., Refs. 6—8). Although this approach can be pushed quite far even in its quantum-mechanical consequences (see, e.g., Refs. 9 and 10), it obviously never really gets beyond the descriptional level.

At the more conceptual level one indeed begins with a closed Hamiltonian system and investigates the dynamics of a nonisolated subsystem. It is useful to divide such modeling into two categories. First, there are models considered to be physically realistic, such as for an atomic electron interacting with the electromagnetic field, for instance, in a laser cavity. These models can be solved only perturbationally in terms of the coupling strength, if it is weak enough. Generally it is difficult to decide, for example, on the problem of convergence or analyticity, or completeness of the solution. Second, there are models that are hopefully still relevant for the crucial questions but are so simple that they can be solved exactly. The system to be discussed in the present paper belongs to this second category. It is a generalization of an earlier version, discussed in some detail in Refs. 11 and 12, which is closely related to work of Lamb,[13] Stevens,[14] Yurke and Denker,[15,16] van Kampen,[17] Schwabl and Thirring,[18] Ford, Kac, and Mazur,[19] Ullersma,[20] and Lodder.[21] See also Refs. 22 and 23.

The treatment in Ref. 11 in particular has clearly shown how dissipation may arise. Suppose that the physical object of interest (say, an oscillator, be it purely mechanical or electrical; see Ref. 11 for details) interacts with its environment in such a way that it is capable of emitting and receiving waves (of some kind), and suppose further that it is possible to distinguish clearly between so-called "outgoing" and "incoming" waves *à la* d'Alembert. This will be the case, for instance, for an oscillating "ball" attached rigidly to a long "cord" that is kept under some tension. Then the damping of the oscillator's motion takes place through the generation of outgoing waves (emission). If the velocity of propagation of these waves is now finite, while the environment is of infinite extension, then these outgoing waves will never return to the oscillator in any arbitrarily large finite time interval. However, the environment itself may and in general will also generate waves, for instance, by thermal excitations. Part of those waves are incoming for the oscillator and represent the noise.

Unfortunately, the earlier version of the model gives rise to two difficulties. The least troublesome one is that the exact dynamical solution, obtained by means of an eigenfunction expansion, requires a *finite* renormalization of the initial conditions (see, e.g., Appendix B, and also Refs. 20 and 22). This problem manifests itself in eigenfrequency sums or integrals of the type $\int_0^\infty d\omega\,\omega^{-1}\sin(\omega t)=\pi/2$ independent of t, whereas each separate mode $\sin(\omega t)\rightarrow 0$ as $t\rightarrow 0$. Apparently, this is due to the fact that the model described in Refs. 11 and 12 does not possess any intrinsic cutoff frequency. The more serious difficulty, however, shows up in the model's quantum mechanics although it is intimately connected with the above-mentioned absence of a spectral cutoff mechanism. Namely, it turns out to be impossible to calculate the frequency sum or integral pertinent to the oscillator's kinetic quantum noise because it diverges logarithmically at the high-frequency end. The calculation of Ref. 11 anticipated the limit of infinite extension of the environment (a string, or continuum transmission line) at an early stage, but in Ref. 12 is was definitely shown that this *infinite* result is an exact consequence of that model for any size of the system. Further, this logarithmic infinity could not be removed by means of some renormalization procedure or other mathematical manipulation.[21]

In Ref. 22 a first brief attempt has been made to explain and eliminate the above-noted deficiencies in a quantitative manner within the context of a somewhat different model. It was clearly pointed out that, as conjectured (see also Ref. 23), such infinities vanish only if there are no particles in the system which are "rigidly" (or "stiffly") coupled—in a sense that will be made more precise in the present article—to a continuum transmission line (or string) as the environment.

Therefore we will consider the simplest nontrivial generalization of the model treated in Refs. 11 and 12. On one hand it is still fairly easy to solve exactly, but on the other hand it is structurally rich enough to provide the proper framework for a comprehensive quantitative discussion of all relevant aspects. In macroscopic terms (see Fig. 1) the system can be said to consist of a particle with mass m, which has been connected first of all to a rigid support by means of a spring with Hooke's constant b, in such a way that it is constrained to move frictionlessly up and down only. Next, the particle has been attached to a string of length Λ via another spring with Hooke's constant B, which is generally different from b. The string is under tension $\mathcal{T}$ and has a mass ρ per unit length, so that the velocity of propagation for transverse waves becomes $c=(\mathcal{T}/\rho)^{1/2}$.

In Sec. II we essentially solve the dynamics of the system in terms of its natural modes. In Sec. III we discuss certain interesting features of this solution. Quantum-mechanical aspects of the model are treated in Sec. IV, all considerations being exact for any finite length Λ of the string. Then, in Sec. V the limit $\Lambda\rightarrow\infty$ is considered. Section VI contains a final discussion and some concluding remarks. Further aspects are discussed in the appendixes. In Appendix A the singular character of the matrix of eigenfunctions on the string is proven in detail. The quantum-mechanical law of exponential decay is considered in Appendix B, where we present a calculation based on the work of Schwabl and Thirring. Appendix C has been devoted to some aspects of the system's coherent states. Finally, the case of a thermal string is treated in Appendix D.

II. CLASSICAL MECHANICS

Consider Fig. 1. This model has already been explicated in the Introduction. The transverse displacement from equilibrium ($z\equiv 0$) of the string is indicated by $z(x,t)$; the displacement of the particle is denoted as $\zeta(t)$. The Lagrangian reads

$$L=\tfrac{1}{2}m\dot{\zeta}^2-\tfrac{1}{2}b\zeta^2-\tfrac{1}{2}B(\zeta-z_0)^2+\tfrac{1}{2}\rho\int_0^\Lambda (z_{,t}^2-c^2z_{,x}^2)dx\ , \tag{2.1}$$

where $\rho c^2\equiv\mathcal{T}$, $z_{,t}\equiv\partial z(x,t)/\partial t$, etc., and where we have set $z_0\equiv z(0,t)$ for convenience. It is straightforward to obtain from (2.1) the dynamics as

$$z_{,tt}-c^2z_{,xx}=0 \quad \text{on } x\in(0,\Lambda) \tag{2.2}$$

$$z_{,x}(\Lambda,t)=0\ , \tag{2.3}$$

$$B(\zeta-z_0)+\rho c^2 z_{,x}(0,t)=0\ , \tag{2.4}$$

$$m\ddot{\zeta}+b\zeta+B(\zeta-z_0)=0\ . \tag{2.5}$$

The Neumann condition (2.3) at $x=\Lambda$ is the natural condition following from (2.1), where the far end of the string is free. Pinning the end at $x=\Lambda$ to $z=0$ would have lead, of course, to the Dirichlet condition $z(\Lambda,t)=0$. This results in effect only in an overall phase shift of $\pi/2$ in the eigenmodes (see, e.g, Ref. 22). In the limit $\Lambda\rightarrow\infty$ this difference even becomes totally irrelevant. In view of earlier work we adhere to (2.3). Using (2.4) to eliminate $\zeta(t)$ from (2.5), one gets

$$m\ddot{z}_0+bz_0=(\rho c^2/B)[m\ddot{z}_{,x}+(B+b)z_{,x}]_0\ . \tag{2.6}$$

Once using (2.2) on the right-hand side and dividing by m, one may write (2.6) as

$$\ddot{z}_0+\Omega^2 z_0=2\lambda c(z_{,x}+c^2\tau^2 z_{,xxx})_0\ , \tag{2.7}$$

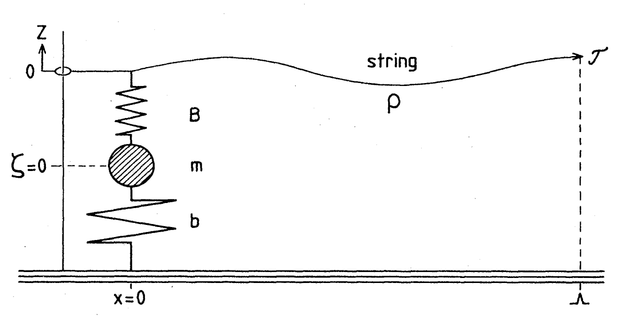

FIG. 1. Model of an oscillating particle coupled flexibly to a mechanical string of finite length. See the Lagrangian (2.1) in the text.

where[24]

$$\Omega^2 \equiv \frac{b}{m}, \quad 2\lambda \equiv \frac{\rho c}{m}\left[1+\frac{b}{B}\right], \quad \tau^2 \equiv \frac{m}{B+b} . \qquad (2.8)$$

Equation (2.7) simply plays the role of the (dynamical) boundary condition on the wave equation (2.2) at $x=0$. Subject to the Neumann boundary condition (2.3) at $x=\Lambda$, the wave equation is solved by

$$z(x,t)=\sum_k \left[\frac{\hbar}{2\omega_k}\right]^{1/2}(a_k e^{-i\omega_k t}+a_k^* e^{i\omega_k t})\Phi_k(x) , \qquad (2.9)$$

$$\Phi_k(x)=\sqrt{2/\Lambda}\cos(\omega_k x/c-\varphi_k) , \qquad (2.10)$$

$$\omega_k\Lambda/c-\varphi_k=k\pi \quad \text{with } k=0,1,2,\ldots , \qquad (2.11)$$

and where the scale of the coefficients in (2.9) has been set by convention, anticipating quantum-mechanical considerations. Inserting now (2.9) and (2.10) into (2.7), one obtains the phase shifts

$$\varphi_k=\text{arccot}[(\Omega^2-\omega_k^2)/2\lambda\omega_k(1-\omega_k^2\tau^2)] . \qquad (2.12)$$

This completes, in fact, the solution of the string's dynamics. The explicit behavior of the particle is now easily obtained from either (2.4) or (2.5). Employing the latter is the more direct route.[25] One finds

$$\zeta(t)=\sum_k \left[\frac{\hbar}{2\omega_k}\right]^{1/2}(a_k e^{-i\omega_k t}+a_k^* e^{i\omega_k t})\Phi_k^0(0) , \qquad (2.13)$$

$$\Phi_k^0(0)\equiv[(1-\Omega^2\tau^2)/(1-\omega_k^2\tau^2)]\Phi_k(0) , \qquad (2.14)$$

where from (2.8) we have inferred that $(1+b/B)^{-1}=1-\Omega^2\tau^2$.

III. PROPERTIES OF THE SOLUTION

It is of interest to investigate a number of important features of the dynamical solution obtained in Sec. II.

A. Mode stability

Self-accelerating or runaway modes might occur if the Hamiltonian belonging to the Lagrangian (2.1) were not a positive definite form. Such a situation arises, for instance, in the theory of the (point) electron, the electromagnetic field playing the role of the transmission line, typically due to the involved (mass-) renormalization procedure. In view of the absence of any such procedure in the treatment of Sec. II, and considering the apparent positive definiteness of the Hamiltonian belonging to (2.1), namely,

$$H=\tfrac{1}{2}m\dot{\zeta}^2+\tfrac{1}{2}b\zeta^2+\tfrac{1}{2}B(\zeta-z_0)^2 + \tfrac{1}{2}\rho\int_0^\Lambda (z_{,t}^2+c^2 z_{,x}^2)dx , \qquad (3.1)$$

the present model should be free of such instabilities. This is confirmed explicitly by the characteristic equation

$$2\lambda\omega_k\tan(\omega_k\Lambda/c)=(\Omega^2-\omega_k^2)/(1-\omega_k^2\tau^2) , \qquad (3.2)$$

which arises from taking the tangent of (2.11) and using (2.12) to eliminate φ_k. Indeed, setting $\omega_k=i\kappa$, one immediately notes the inequality

$$2\lambda\kappa(1+\kappa^2\tau^2)\tanh(\kappa\Lambda/c)+\kappa^2+\Omega^2\geq\Omega^2>0 . \qquad (3.3)$$

Hence, there are no imaginary ω_k, which is what we set out to show.

B. Nonorthogonality

As a necessary consequence of the attachment of the oscillator to the string, the eigenfunctions (2.10) are not orthogonal on $x\in(0,\Lambda)$. This consequence is indeed necessary, for if the $\Phi_k(x)$ were orthogonal on the string, we could readily invert (2.9) at $t=0$ in order to express the a_k and a_k^* in terms of the initial conditions of the string only. But by (2.13) that would imply that the dynamics of the particle could be given in terms of those initial conditions only, i.e., without specifying the particle's own initial conditions. Clearly, this can not be correct. Mathematically this lack of orthogonality arises because the boundary condition (2.7) at $x=0$ is of a dynamical nature, so that per mode it depends itself explicitly on the eigenfrequency (see also, e.g., Ref. 26). Defining

$$\eta_{kl}\equiv\int_0^\Lambda \Phi_k(x)\Phi_l(x)dx , \qquad (3.4)$$

inserting (2.10) for the $\Phi_k(x)$, evaluating the integral, and using (2.11) at $x=\Lambda$, one finds

$$\eta_{kk}=1+\left[\frac{c}{\Lambda}\right]\frac{\sin\varphi_k\cos\varphi_k}{\omega_k} , \qquad (3.5)$$

$$\eta_{kl}=\left[\frac{c}{\Lambda}\right]\left[\frac{\sin(\varphi_k-\varphi_l)}{\omega_k-\omega_l}+\frac{\sin(\varphi_k+\varphi_l)}{\omega_k+\omega_l}\right] , \quad k\neq l . \qquad (3.6)$$

By means of some elementary goniometric manipulations one easily rewrites (3.6) as

$$\eta_{kl}=\left[\frac{2c}{\Lambda}\right]\frac{\cos\varphi_k\cos\varphi_l}{\omega_k^2-\omega_l^2}(\omega_k\tan\varphi_k-\omega_l\tan\varphi_l) , \quad k\neq l . \qquad (3.7)$$

Now employing (2.12) for $\tan\varphi_k$ and $\tan\varphi_1$, and combining terms, one obtains

$$\eta_{kl}=-\left[\frac{2c}{\Lambda}\right]\frac{1-\Omega^2\tau^2}{2\lambda}\frac{\cos\varphi_k}{1-\omega_k^2\tau^2}\frac{\cos\varphi_l}{1-\omega_l^2\tau^2}, \quad k\neq l . \qquad (3.8)$$

Finally, using (2.10) at $x=0$, and (2.8) for 2λ, the result reads

$$\eta_{kl}=-\frac{m}{\rho}\Phi_k^0(0)\Phi_l^0(0), \quad k\neq l . \qquad (3.9)$$

The effective oscillator eigenfunctions $\Phi_k^0(0)$ have been defined in (2.14). Note that, as expected, the off-diagonal elements $\eta_{k\neq l}$ become zero if either (i) $B=0$, so that $\Omega^2\tau^2=1$, i.e., complete decoupling of string and particle, or (ii) $m=0$, i.e., complete absence of the particle. In Appendix A it is shown that the η_{kl} not only are nonzero for $k \neq l$, but that they essentially form a singular matrix, i.e., η_{kl}^{-1} does not exist. This unequivocally rules out the possibility to use η_{kl} to invert (2.9) in order to express the dynamical solution in terms of the initial conditions. Fortunately, from (3.9) it is now evident that the matrix

$$\vartheta_{kl} \equiv \rho \eta_{kl} + m \Phi_k^0(0) \Phi_l^0(0) \tag{3.10}$$

will be exactly diagonal by construction. Hence, it can be inverted trivially. Two comments on (3.10) seem in place. First, it is important to note the structure of the added term as the product of two (effective) eigenfunctions. This property will be crucial in the application of (3.10), already in Sec. III C. Second, the structure of ϑ_{kl} as consisting of the usual form (3.4) along the transmission line (with density ρ) plus an added product term pertaining solely to the attached particle (with weight m) readily generalizes if more particles are attached. See for an example of such a situation Ref. 22. Finally, using (3.5) for η_{kk} and (2.12) for $\tan\varphi_k$, an explicit calculation of (3.10) yields

$$\vartheta_{kl} = \delta_{kl} \left[\rho + \frac{m}{\Lambda} \left[\frac{\Omega^2 + \omega_k^2}{\omega_k^2} (1 + \omega_k^2 \tau^2) - (2\Omega\tau)^2 \right] \times (1 - \Omega^2 \tau^2) \left[\frac{\cos\varphi_k}{1 - \omega_k^2 \tau^2} \right]^2 \right] . \tag{3.11}$$

This formula reduces to the earlier obtained results in Refs. 12 and 27 if $\tau = 0$.[28]

C. Completeness relations

The eigenfunctions (2.10) obey a number of interesting completeness relations. All considerations in this subsection are at the arbitrarily chosen time $t = 0$. First consider $z(x) \equiv z(x,0)$ from (2.9). Define the spectral coordinates

$$q_k \equiv \left[\frac{\hbar}{2\omega_k} \right]^{1/2} (a_k + a_k^*) . \tag{3.12}$$

Then multiply (2.9) at $t = 0$ by $\Phi_l(x)$ and integrate from $x = 0$ to Λ. Using the definition (3.4) of η_{kl}, one thus has

$$\int_0^\Lambda \Phi_k(x) z(x) dx = \sum_l \eta_{kl} q_l . \tag{3.13}$$

As shown in Appendix A, this result cannot be inverted, which is as it should be. Now multiply (2.13) for the oscillator at $t = 0$ by $m \Phi_l^0(0)$ and add this to ρ times (3.13). Noticing the definition (3.10) by ϑ_{kl} and the diagonality property expressed in (3.11), one then readily obtains

$$q_k = \vartheta_{kk}^{-1} \left[m \Phi_k^0(0) \zeta(0) + \rho \int_0^\Lambda \Phi_k(x) z(x) dx \right] . \tag{3.14}$$

Substituting these q_k back into (2.9) and (2.13) at $t = 0$, one arrives at the two identities

$$z(x) = \left[\sum_k m \vartheta_{kk}^{-1} \Phi_k(x) \Phi_k^0(0) \right] \zeta(0) + \int_0^\Lambda \left[\sum_k \rho \vartheta_{kk}^{-1} \Phi_k(x) \Phi_k(x') \right] z(x') dx' , \tag{3.15}$$

$$\zeta(0) = \left[\sum_k m \vartheta_{kk}^{-1} \Phi_k^0(0)^2 \right] \zeta(0) + \int_0^\Lambda \left[\sum_k \rho \vartheta_{kk}^{-1} \Phi_k^0(0) \Phi_k(x) \right] z(x) dx . \tag{3.16}$$

Hence,

$$\sum_k \rho \vartheta_{kk}^{-1} \Phi_k(x) \Phi_k(x') = \delta(x - x') , \tag{3.17}$$

$$\sum_k \vartheta_{kk}^{-1} \Phi_k(x) \Phi_k^0(0) = 0 , \tag{3.18}$$

$$\sum_k m \vartheta_{kk}^{-1} \Phi_k^0(0)^2 = 1 . \tag{3.19}$$

It is an interesting exercise to verify these relations, even if $\tau = 0$, for instance in the limit $\Lambda \to \infty$, so that the sums can be replaced by integrals (see Sec. V) and one can resort to the standard mathematics of complex functions. See also Refs. 11, 12, and 22.

D. Initial conditions

In this subsection the dynamical solutions (2.9) and (2.13) will be expressed explicitly in terms of the initial conditions of both the transmission line and the particle. Using essentially the same completeness relations procedure as discussed in Sec. III C also for $\dot{z}(x) \equiv \dot{z}(x,0) \equiv z_{,t}(x,0)$ and $\dot{\zeta}(0)$, one finds that the spectral momenta defined by

$$p_k \equiv -i \left[\frac{\hbar \omega_k}{2} \right]^{1/2} (a_k - a_k^*) , \tag{3.20}$$

may be given as

$$p_k = \vartheta_{kk}^{-1} \left[m \Phi_k^0(0) \dot{\zeta}(0) + \rho \int_0^\Lambda \Phi_k(x) \dot{z}(x) dx \right] . \tag{3.21}$$

Introducing now both (3.14) for q_k and (3.21) for p_k into (2.9) and (2.13), one obtains

$$z(x,t) = m \dot{A}(x,t) \zeta(0) + m A(x,t) \dot{\zeta}(0) + \int_0^\Lambda [\dot{A}(x,x',t) z(x',0) + A(x,x',t) \dot{z}(x',0)] dx' , \tag{3.22}$$

$$\zeta(t) = \dot{A}(t) \zeta(0) + A(t) \dot{\zeta}(0) + \int_0^\Lambda \rho [\dot{A}(x,t) z(x,0) + A(x,t) \dot{z}(x,0)] dx , \tag{3.23}$$

where

$$A(x,x',t) \equiv \sum_k \rho \vartheta_{kk}^{-1} \Phi_k(x) \Phi_k(x') \omega_k^{-1} \sin(\omega_k t) , \tag{3.24}$$

$$A(x,t) \equiv \sum_k \vartheta_{kk}^{-1} \Phi_k(x) \Phi_k^0(0) \omega_k^{-1} \sin(\omega_k t) , \tag{3.25}$$

$$A(t) \equiv \sum_k m \vartheta_{kk}^{-1} \Phi_k^0(0)^2 \omega_k^{-1} \sin(\omega_k t) . \tag{3.26}$$

Notice that the completeness relations (3.17)—(3.19) imply $\dot{A}(x,x',0)=\delta(x-x')$, $\dot{A}(x,0)=0$, and $\dot{A}(0)=1$. Thus they supply an explicit check on the proper initial conditions in (3.22) and (3.23). Finally, it is of interest to consider a special case in slightly more detail. If we suppose that the string is initially in a state of complete rest, in the sense that $z(x,0)=\dot{z}(x,0)=0$, then the behavior of the particle is obviously simply given by

$$\zeta(t)=\dot{A}(t)\zeta(0)+A(t)\dot{\zeta}(0) \ . \tag{3.27}$$

Using the definition (2.14) of the $\Phi_k^0(0)$, (2.10) for the $\Phi_k(0)$, and (2.12) for the phase shifts φ_k, one obtains *in extenso*

$$A(t)=\frac{2m}{\Lambda}\sum_k \vartheta_{kk}^{-1}\frac{4\lambda^2\omega_k^2(1-\Omega^2\tau^2)^2}{(\Omega^2-\omega_k^2)^2+4\lambda^2\omega_k^2(1-\omega_k^2\tau^2)^2}\frac{\sin(\omega_k t)}{\omega_k} \ . \tag{3.28}$$

It is recalled that ϑ_{kk} has been given explicitly in (3.11), and that the frequencies ω_k follow from (3.2) for arbitrary Λ.

IV. QUANTUM MECHANICS

In view of the conservative nature of the system described by the Lagrangian (2.1), the Hamiltonian (3.1) will be a constant of the motion so that the following calculations can be done at $t=0$ for convenience. But before substituting the explicit solutions (2.9) and (2.13) into (3.1), it appears to be considerably easier to first slightly rewrite this Hamiltonian. Let us partially integrate the potential energy term of the string. Using the boundary condition (2.3) at $x=\Lambda$, this leads to

$$H=\tfrac{1}{2}m\dot{\zeta}^2+\tfrac{1}{2}b\zeta^2+\tfrac{1}{2}B(\zeta-z_0)^2-\tfrac{1}{2}\rho c^2 z_0 z_{,x}(0) + \tfrac{1}{2}\rho\int_{+0}^{\Lambda}(z_{,t}^2-c^2 z z_{,xx})dx \ . \tag{4.1}$$

Next using the "$x=0$" boundary equation (2.4) to eliminate $z_{,x}(0)$ yields

$$H=\tfrac{1}{2}m\dot{\zeta}^2+\tfrac{1}{2}b\zeta^2+\tfrac{1}{2}B\zeta(\zeta-z_0) + \tfrac{1}{2}\rho\int_{+0}^{\Lambda}(z_{,t}^2-c^2 z z_{,xx})dx \ . \tag{4.2}$$

Finally, using the "$x=0$" boundary equation (2.5) to eliminate z_0 and, moreover, employing the wave equation (2.2) on $x\in(0,\Lambda)$ for $z_{,xx}$, one obtains

$$H=\tfrac{1}{2}m(\dot{\zeta}^2-\zeta\ddot{\zeta})+\tfrac{1}{2}\rho\int_{+0}^{\Lambda}(z_{,t}^2-z z_{,tt})dx \ . \tag{4.3}$$

Now inserting (2.9) and (2.13) at $t=0$, one readily finds

$$H=\tfrac{1}{2}\sum_k \vartheta_{kk}(p_k^2+\omega_k^2 q_k^2) \ , \tag{4.4}$$

where use has been made of (3.12) and (3.20), and of (3.10) and (3.11) for ϑ_{kl}. The Hamiltonian (4.4) clearly represents a sum of independent harmonic oscillators with so-called normal mode frequences ω_k. Of course, this is as it should be if (2.9) and (2.13) indeed exactly solve the dynamics. The fundamental commutator pertaining to (4.4) is found in the standard manner from (3.14) and (3.21), on the basis of the only nonzero commutators

$$[\zeta,m\dot{\zeta}]=i\hbar,\ \ [z(x),\rho\dot{z}(x')]=i\hbar\delta(x-x') \ , \tag{4.5}$$

to be

$$[q_k,p_l]=i\hbar\vartheta_{kl}^{-1} \ . \tag{4.6}$$

It is recalled that ϑ_{kl} is diagonal. In view of the definitions (3.12) and (3.20), one may ultimately present the result also in terms of the a_k and a_k^* as

$$H=\sum_k \hbar\omega_k\vartheta_{kk}a_k^* a_k \ , \tag{4.7}$$

$$[a_k,a_l^*]=\vartheta_{kl}^{-1} \ , \tag{4.8}$$

any other commutator being zero, as usual. It is now in fact a textbook exercise to construct the complete set of harmonic oscillator states per modes ω_k for the present system. In principle, from these or any linear combination of them (for instance, coherent states, see, e.g., Refs. 29—31) anything can be computed. See also Appendix B, where we calculate a transition element, and Appendix C where coherent states are considered. In view of earlier work, which as explained in the Introduction can be considered as a major motivation for the current investigation, it is of particular interest to study the quantal fluctuations of the particle in an eigenstate of (4.7). Special attention will be given to be ground state, which is not only typical but will also be the zero-temperature final state for the oscillator, independent of the initial conditions, in the limit $\Lambda\to\infty$ where this subsystem behaves dissipatively for all times. See also Sec. V. In Appendix D the case of the thermal string is considered. The ground state is defined by $\vartheta_{kk}^{1/2}a_k\,|\,0\rangle$ for all k, and expectation values in it will from here on be indicated by a subscript zero. Using (2.13) for the position fluctuations and the kinetic momentum fluctuations, one easily gets, respectively,

$$\langle\zeta^2\rangle_0=\tfrac{1}{2}\hbar\sum_k \vartheta_{kk}^{-1}\omega_k^{-1}\Phi_k^0(0)^2 \ ,$$
$$\langle p^2\rangle_0=\tfrac{1}{2}\hbar m^2\sum_k \vartheta_{kk}^{-1}\omega_k\Phi_k^0(0)^2 \ , \tag{4.9}$$

where $p\equiv m\dot{\zeta}$. By means of (2.14), (2.10), and (2.12) it is straightforward to arrive at the explicit formulas[32]

$$\langle\zeta^2\rangle_0=\frac{\hbar}{\Lambda}\sum_k \vartheta_{kk}^{-1}\frac{4\lambda^2\omega_k(1-\Omega^2\tau^2)^2}{(\Omega^2-\omega_k^2)^2+4\lambda^2\omega_k^2(1-\omega_k^2\tau^2)^2} \ , \tag{4.10}$$

$$\langle p^2\rangle_0=\frac{\hbar}{\Lambda}m^2\sum_k \vartheta_{kk}^{-1}\frac{4\lambda^2\omega_k^3(1-\Omega^2\tau^2)^2}{(\Omega^2-\omega_k^2)^2+4\lambda^2\omega_k^2(1-\omega_k^2\tau^2)^2} \ . \tag{4.11}$$

These expressions are exact for any size Λ of the system. Notice that neither $\langle\xi^2\rangle_0$ nor $\langle p^2\rangle_0$ involve infrared problems ($\omega_k \downarrow 0$) as $\Omega\neq 0$. Further, as long as $\tau\neq 0$, (4.10) converges like ω_k^{-4} and (4.11) does so like ω_k^{-2} when $\omega_k\to\infty$. Only if $\tau=0$, the momentum fluctuations (4.11) show an ultraviolet logarithmic divergence since in that case the summand goes like ω_k^{-1} when $\omega_k\to\infty$. This very special case for the present model, however, precisely represents the situation considered earlier in Refs. 11 and 12. Physically, with $m\neq 0$ and Ω finite, the case $\tau=0$ can be realized only by $B=\infty$, which defines "strong" (or "rigid," "stiff") coupling between particle and transmission line field. See Fig. 1. See also Ref. 22.

V. INFINITE-SYSTEM LIMIT

Let us consider the interesting limit of infinite system size $\Lambda\to\infty$. In this case, where the string extends to infinity, the frequency spectrum becomes continuous and sums over discrete ω_k can be replaced by integrals over continuous ω. This replacement can be done perfectly simply in the present treatment. There are no subtleties involved and, hence, there is no reason to go into the complex frequency plane for the purpose (see, e.g., Refs. 17, 20, and 33). From (2.11) and (2.12), with $1=\Delta k$, one infers that

$$\frac{c}{\Lambda}\Delta k=\frac{1}{\pi}\Delta\omega_k\left[1-\frac{c}{\Lambda}[\varphi_k'+O(\Delta\omega_k)]\right] . \tag{5.1}$$

But according to (2.12) $\varphi_k'\equiv\partial\varphi_k/\partial\omega_k$, and higher derivatives, will be a function of order Λ^0. Hence, as $\Lambda\to\infty$ one has

$$\frac{c}{\Lambda}\sum_{k=0}^{\infty}\to\frac{1}{\pi}\int_0^{\infty}d\omega . \tag{5.2}$$

Consider now the dynamics of the particle as given by (3.27), which is correct subject to the initial condition $z(x,0)=\dot z(x,0)=0$ along the string. By means of (5.2) the propagator (matrix element) (3.28) becomes a fairly simple integral which may be calculated explicitly using standard contour integration in the complex ω-plane. Factorizing the denominator in (3.28), and taking $t>0$, the relevant poles are easily seen to follow from the zeros of

$$\chi^{-1}(\omega)=2i\lambda\tau^2\omega^3-\omega^2-2i\lambda\omega+\Omega^2 , \tag{5.3}$$

where $\chi(\omega)$ is just the usual response function (see, e.g., Refs. 22, 23, and 34) for the classical dynamical system

$$2\lambda\tau^2\dddot{\xi}+\ddot\xi+2\lambda\dot\xi+\Omega^2\xi=0 . \tag{5.4}$$

This exact result for the dissipative behavior of the oscillator in the limit $\Lambda\to\infty$ could have been obtained easily in the alternative way outlined in the Introduction. Namely, if the string is at complete rest at $t=0$ [as it is with (3.27)] and if its length is infinite, then—classically and at zero temperature—there can only be outgoing waves *à la* d'Alembert in the system, which stem from the excitation of the oscillator. But hence the field along the string is of the type $z(x-ct)$ only, such that $z_{,x}=-(1/c)z_{,t}$. Using this relation in the boundary condition (2.7) at $x=0$, recalling that $z_{,t}(0,t)\equiv\dot z_0(t)$, and noticing from (2.4) that $\xi(t)$ must obey the same differential equation as $z_0(t)$, one immediately arrives again at (5.4). The result (5.4) is exact for the present model for all time $t\in(0,\infty)$ only if $\Lambda=\infty$. Its crucial feature is $\tau\neq 0$. As has been noted at the end of Sec. IV, it is precisely *this* feature which provides a sufficient intrinsic frequency cutoff in order to yield a finite result in (4.11).[35]

In the physically significant case where τ is very small compared to any other time scale in the system (i.e., Ω^{-1}, λ^{-1}), the analysis of the zeros of (5.3) simplifies enormously. It is an elementary exercise, keeping only leading terms in τ^2 everywhere, to show that $\chi^{-1}(\omega)$ from (5.3) can be factorized as[36]

$$\chi^{-1}(\omega)\cong(2i\lambda\tau^2\omega-1)(\omega-\bar\omega+i\lambda)(\omega+\bar\omega+i\lambda) , \tag{5.5}$$

where $\bar\omega\equiv(\Omega^2-\lambda^2)^{1/2}$ represents the standard reduced classical frequency. According to (5.5) the dynamics of the particle is now clearly governed by three simple exponentials, one of which is purely and rapidly decaying. Instead of calculating (3.28)—in the limit $\Lambda\to\infty$ and with the normalization adapted to the approximation involved in (5.5)— one can, of course, also determine the three constants of integration involved in solving (5.4) in the approximation (5.5) by means of the initial conditions, i.e., $\xi(0)$, $\dot\xi(0)$, and (2.5), which relates $\ddot\xi(0)$ to $\xi(0)$. Finally, one can similarly determine $A(t)$ directly from the three requirements $A(0)=0$, $\dot A(0)=1$, and $\ddot A(0)=0$. Keeping again only leading orders of τ^2 everywhere, one finds

$$A(t)\cong 8\lambda^3\tau^4[e^{-t/2\lambda\tau^2}-e^{-\lambda t}\cos(\bar\omega t)] +(1/\bar\omega)e^{-\lambda t}\sin(\bar\omega t) . \tag{5.6}$$

Let us consider the oscillator's quantum fluctuations (4.10) and (4.11) in the limit $\Lambda\to\infty$. Using (5.2) the sums again become integrals, which can be calculated to give a finite result since $\tau\neq 0$, as noted previously. We once more discuss the important limit $\tau\to 0$. In the approximation (5.5) the integral (4.10) for the position fluctuations can be evaluated as

$$\langle\xi^2\rangle_0=\frac{4\hbar\lambda^2}{\pi\rho c}(1-\Omega^2\tau^2)^2\int_0^{\infty}d\omega\,\omega\,|\chi(\omega)|^2 \tag{5.7}$$

$$\cong\frac{2\hbar\lambda}{\pi m}(1-\Omega^2\tau^2)\times\int_0^{\infty}d\omega\,\omega/\{(1+4\lambda^2\tau^4\omega^2)\times[(\Omega^2-\omega^2)^2+4\lambda^2\omega^2]\} . \tag{5.8}$$

Then setting $s\equiv\omega^2$ and writing the integrand as two obvious fractions yields[37]

$$\langle\xi^2\rangle_0\cong\frac{\hbar\lambda}{\pi m}(1-\Omega^2\tau^2)\times\int_0^{\infty}ds\left[\frac{1-4\lambda^2\tau^4 s}{(s-\Omega^2)^2+4\lambda^2 s}+\frac{16\lambda^4\tau^8}{1+4\lambda^2\tau^4 s}\right] . \tag{5.9}$$

In the usual manner, proceeding with the first fraction (see, e.g., Ref. 38) and carefully evaluating the pertinent logarithmic integrals, one obtains

$$\langle \xi^2 \rangle_0 \cong \frac{\hbar\lambda}{\pi m}(1-\Omega^2\tau^2)\left[\frac{1}{2\lambda\varpi}\operatorname{arccot}\left[\frac{\varpi^2-\lambda^2}{2\lambda\varpi}\right] + 4\lambda^2\tau^4\ln(4\lambda^2\tau^4\Omega^2)\right] . \quad (5.10)$$

This reduces to the by now well-known result if $\tau=0$, as it should (see, e.g., Refs. 11, 12, 15, 22, 23 and 39). The integral (4.11) for the momentum fluctuations can be handled in the approximation (5.5) as

$$\langle p^2 \rangle_0 = \frac{4\hbar\lambda^2}{\pi\rho c}(1-\Omega^2\tau^2)^2 \int_0^\infty d\omega\,\omega^3 \,|\chi(\omega)|^2 \quad (5.11)$$

$$\simeq \frac{\hbar\lambda}{\pi m}(1-\Omega^2\tau^2)\int_0^\infty ds\, s\left[\frac{1-4\lambda^2\tau^4 s}{(s-\Omega^2)^2+4\lambda^2 s} + \frac{16\lambda^4\tau^8}{1+4\lambda^2\tau^4 s}\right] . \quad (5.12)$$

The linearly diverging parts of the two fractions precisely cancel and one is left with

$$\langle p^2 \rangle_0 \cong \frac{\hbar\lambda}{\pi m}(1-\Omega^2\tau^2)\int_0^\infty ds\left[\frac{4\lambda^2\tau^4\Omega^4+s}{(s-\Omega^2)^2+4\lambda^2 s} - \frac{4\lambda^2\tau^4}{1+4\lambda^2\tau^4 s}\right] . \quad (5.13)$$

This form can now be treated in the usual manner, as with (5.9). The result is

$$\langle p^2 \rangle_0 \cong \frac{\hbar\lambda}{\pi m}(1-\Omega^2\tau^2)\left[\frac{\varpi^2-\lambda^2}{2\lambda\varpi}\operatorname{arccot}\left[\frac{\varpi^2-\lambda^2}{2\lambda\varpi}\right] - \ln(4\lambda^2\tau^4\Omega^2)\right] , \quad (5.14)$$

which clearly exhibits the appearance of the logarithmic divergence when $\tau\to 0$. From $|\chi(\omega)|^2$ it is clear that the effective cutoff frequency may be given as $\omega_c=(2\lambda\tau^2)^{-1}$.

VI. FINAL REMARKS

The dynamics of the system described by the Lagrangian (2.1) has been solved exactly for arbitrary finite size Λ by means of the Lagrange formalism, employing techniques of spectral decomposition and determining the pertinent phase shifts. Then, in Sec. III A it has been shown that the model is free of self-accelerating or runaway modes. Section III B provided a discussion of the nature of the nonorthogonality of the eigenfunctions. This feature was clearly shown to arise from the flexible ($B<\infty$) coupling of the particle to the continuum transmission line. Only if this coupling were rigid, so that τ as defined in (2.8) were zero, according to (2.14) that $\Phi_k^0(0)=\Phi_k(0)$, and by (3.10) the mass m can then be included as a singularity at $x=0$ in the mass density ρ of the string (see also Refs. 11, 12, 15, 22, and 27). Of course, the nonorthogonality feature also vanishes from the formulas if $m=0$, i.e., if the particle were totally absent. In Sec. III C a number of completeness relations for the obtained set of eigenfunctions were derived, while the dynamical initial-value problem has been solved explicitly in Sec. III D.

In Sec. IV we have considered the exact quantum-mechanical spectrum of the system. As expected, it consists of a complete set of harmonic oscillator eigenstates per classical natural mode ω_k. The particle's quantal noise has been discussed as a function of the coupling strength between the particle and the string. The previously observed (see, e.g., Refs. 11, 12, 22, and 23) logarithmic ultraviolet divergence for the momentum fluctuations has been shown, most clearly in (5.14), to originate in the strong (or rigid) coupling limit (Hooke's constant $B\to\infty$, i.e., $\tau\to 0$). It may further be of interest in this context to note that the present mechanical model depicted in Fig. 1 has an electrical analog where, essentially, self-inductances take the place of masses and capacitors play the role of springs. See, e.g., Refs. 11 and 16. This analogy is relevant, for instance, with regard to Josephson and superconducting quantum interference device (SQUID) circuit investigations (see, e.g., Refs. 23, 40, and 41).

Finally, Sec. V contains considerations regarding the limit of a (semi-) infinitely extended string. The problem of a two-sided string extending from $x=-\Lambda(\to-\infty)$ to $x=\Lambda(\to\infty)$ has been discussed in Ref. 12 and essentially leads to a redefinition of the friction coefficient λ (i.e., the oscillator's dissipation rate) by a factor of 2. In the limit $\Lambda\to\infty$ spectral sums become integrals over a continuous frequency spectrum in a very simple manner. Several formulas, e.g., for the propagators (3.24)—(3.26) and the quantum noise (4.9)—(4.11), were shown in their relation to the exact classical response function in this limit. The particle behaves dissipatively: there are two damping terms in its equation of motion, namely, $2\lambda\dot{\xi}$ and $2\lambda\tau^2\dddot{\xi}$. The case where τ is much less than $1/\lambda$ and $1/\Omega$ has been treated explicitly in some more detail. A basic result of this paper is the quantitative recognition of an intrinsic Lorentzian cutoff factor with finite cutoff frequency[42]

$$\omega_c=(2\lambda\tau^2)^{-1} , \quad (6.1)$$

which effectively suppresses the quantum-mechanical infinities found in earlier work.

APPENDIX A: THE SINGULAR NATURE OF η_{kl}

We prove by explicit construction that the matrix η_{kl} defined by (3.4) is singular. If η_{kl}^{-1} exists, then it is defined by

$$\sum_l \eta_{kl}\eta_{ln}^{-1}=\delta_{kn} . \quad (A1)$$

Using (3.10) for η_{kl}, one obtains

$$\sum_l [\vartheta_{kl}-m\Phi_k^0(0)\Phi_l^0(0)]\eta_{ln}^{-1}=\rho\delta_{kn} . \quad (A2)$$

Since by definition $\vartheta_{kl}=\vartheta_{kk}\delta_{kl}$, one immediately finds

$$\eta_{kn}^{-1}=\rho\vartheta_{kk}^{-1}\delta_{kn}+m\vartheta_{kk}^{-1}\Phi_k^0(0)\sum_l \eta_{ln}^{-1}\Phi_l^0(0) . \quad (A3)$$

Multiplying (A3) by $\Phi_k^0(0)$ and summing over k, one can solve for

$$\sum_l \eta_{ln}^{-1}\Phi_l^0(0)=\rho\vartheta_{nn}^{-1}\Phi_n^0(0)\bigg/\left[1-\sum_k m\vartheta_{kk}^{-1}\Phi_k^0(0)^2\right], \tag{A4}$$

which upon substitution into (A3) gives the result

$$\eta_{kl}^{-1}=\rho\vartheta_{kl}^{-1} +\rho m\vartheta_{kk}^{-1}\Phi_k^0(0)\vartheta_{ll}^{-1}\Phi_l^0(0)\bigg/\left[1-\sum_n m\vartheta_{nn}^{-1}\Phi_n^0(0)^2\right]. \tag{A5}$$

However, according to the completeness relation (3.19) the denominator in the off-diagonal part of η_{kl}^{-1} will always be exactly zero, which explicitly proves the assertion that η_{kl}^{-1} does not exist.

APPENDIX B: QUANTUM EXPONENTIAL DECAY

In this appendix we briefly present a calculation of a transition probability between oscillator states of the particle. It is basically an adapted version of the original calculation of Schwabl and Thirring.[18] These authors observed two difficulties in their modeling: (i) the absence of negative frequencies spoiled the purely exponential behavior and (ii) even after including those (in fact nonexistent) negative frequencies, their initial normalization $P(0)=1$ was spoiled [it approached unity only in the weak damping limit $\lambda\to 0$, such that $\bar\omega\equiv(\Omega^2-\lambda^2)^{1/2}\to\Omega$; see the footnote on p. 231 of Ref. 18]. The authors, however, apparently did not recognize a more substantial deficiency of their original formula (with only positive frequencies), namely, that it has a logarithmic ultraviolet divergence when the time $t\to 0$.

Following Ref. 18, let us define for the oscillator the annihilation operator

$$a(t)\equiv i(m/2\hbar\Omega)^{1/2}[\dot\xi(t)-i\Omega\xi(t)], \tag{B1}$$

which together with its conjugate creation operator $a^*(t)$ forms the usual algebra $[a(t),a^*(t)]=1$ on the basis of (4.5). We then define our initial state with probability 1 by operating once with $a^*(0)$ on the physical ground state $|0\rangle$, which as in Sec. IV is defined by $\vartheta_{kk}^{1/2}a_k|0\rangle=0$ for all $k=0,1,2,\ldots$. This creates the so-called "undressed" state of Ref. 18. Now one might ask for the probability that the particle remains in this first excited state until time t. In an obvious notation we have

$$P(t)\equiv|\langle 1,\xi_t|1,\xi_0\rangle|^2\equiv|\eta_0\langle 0|a(t)a^*(0)|0\rangle|^2. \tag{B2}$$

Here η_0 is a normalization factor at $t=0$, which is needed since the state $|1,\xi_0\rangle$ as defined above is not necessarily normalized to unity. This is so because, obviously, the ground state $|0\rangle$ of the Hamiltonian (4.7) does not coincide with the ground state of the Hamiltonian operator $H_0\equiv\hbar\Omega a^*(0)a(0)$. It is now straightforward to insert (2.13) into (B1), and to obtain

$$P(t)=\left|\frac{\eta_0}{4\Omega}\sum_k m\vartheta_{kk}^{-1}\omega_k^{-1}(\omega_k+\Omega)^2\Phi_k^0(0)^2e^{-i\omega_k t}\right|^2. \tag{B3}$$

In the limit $\Lambda\to\infty$ the sum becomes an integral according to (5.2). But because the sum, and hence the integral, contains essentially only positive frequencies, (B3) cannot lead to purely exponential behavior, as in Ref. 18. As usual, in the weak damping limit $\lambda\to 0$ the main contribution to (B3), due to the denominator arising from $\Phi_k^0(0)^2$, is recognized to come from $\omega_k\simeq\Omega$. Moreover, the factor $(\omega_k+\Omega)^2$ in the numerator effectively suppresses a resonance near $\omega_k=-\Omega$. Simply extending then the spectrum to all negative frequencies and collecting those contributions which are even in ω_k, using (3.26) one finds

$$P(t)=[\dot A(t)]^2+(1/4\Omega^2)[\ddot A(t)-\Omega^2A(t)]^2, \tag{B4}$$

where $\eta_0=1$ in view of the initial conditions $\dot A(0)=1$, $\ddot A(0)=A(0)=0$. In connection with the Introduction (see also Refs. 20 and 22) it is worthwhile to mention that these initial conditions cannot form a self-consistent set if $\tau=0$, i.e., in the case of perfectly rigid (or strong) interaction. Namely, considering $\Lambda=\infty$ for convenience, in that case $A(t)$ obeys (5.4) with $\tau=0$. But then (5.4) reduces to the simple textbook damped oscillator equation (e.g., Ref. 1), which is of the second order only. Hence $\ddot A(t\downarrow 0)=-2\lambda$, which is confirmed by an explicit calculation.[43] So, if and only if $\tau=0$ we should have set $\eta_0=(1+\lambda^2/\Omega^2)^{-1/2}$. These remarks merely show how unphysical it is to consider the rigid coupling case on its own. If $\tau>0$, no matter how small it may be, we just face a matter of time scales. Taking $\Lambda=\infty$, and on a time scale much longer than τ, i.e., when transient effects have disappeared, one may use the result (5.6) for $A(t)$ with $\tau=0$. Inserting this into (B4), one gets

$$P(t)\cong e^{-2\lambda\tau}\left[\left[1+\frac{1}{2}\frac{\lambda^2}{\Omega^2}\right]+\frac{1}{2}\frac{\lambda^2}{\Omega^2}\cos(2\bar\omega t)\right], \tag{B5}$$

disregarding terms of higher order in λ/Ω. Obviously, deriving the quantum-mechanical law of exponential decay is a nontrivial business (see also, e.g., Ref. 44).

Let us finally consider the initial-state normalization η_0 in some more detail in the general case. From (B3), apart from an irrelevant phase factor we have

$$\eta_0^{-1}=\frac{1}{4\Omega}\sum_k m\vartheta_{kk}^{-1}\omega_k^{-1}(\omega_k+\Omega)^2\Phi_k^0(0)^2. \tag{B6}$$

Using the completeness relation (3.19) and (4.9) for the ground-state fluctuations, one rewrites (B6) as

$$\eta_0^{-1}=\frac{1}{2}+\frac{1}{2m\hbar\Omega}(\langle p^2\rangle_0+m^2\Omega^2\langle\xi^2\rangle_0), \tag{B7}$$

which considerably facilitates the discussion. For instance, from (5.10) and (5.14) if $\lambda\downarrow 0$ with $\tau\neq 0$, i.e., in the weak damping limit, ultimately $\langle p^2\rangle_0\simeq m^2\Omega^2\langle\xi^2\rangle_0\simeq\frac{1}{2}m\hbar\Omega$ take on their standard free oscillator values, and η_0^{-1} neatly approaches unity. On the other hand, for arbitrary $\lambda\neq 0$ but if $\tau\downarrow 0$, i.e., in the strong coupling limit, $\langle p^2\rangle_0$ becomes the dominant quantity in (B7) since it grows logarithmically as $\ln\omega_c\sim-\ln\tau$ in that case. This is precisely the behavior of $P(t\downarrow 0)$ in Ref. 18,[45] which therefore is apparently due to the rigid coupling of the electron to the electromagnetic field in the Schwabl and Thirring model.

APPENDIX C: COHERENT-STATE EVOLUTION

It is a standard exercise to construct the coherent states for the quantum system of (4.7) and (4.8). See, e.g., Refs. 29—31. Per mode these states can be considered basically as eigenstates of the annihilation operator $\vartheta_{kk}^{1/2}a_k$. Therefore,

$$\vartheta_{kk}^{1/2}a_k \mid \{\alpha_l\}\rangle \equiv \alpha_k \mid \alpha_k\rangle \prod_{l\neq k} \mid \alpha_l\rangle \ , \tag{C1}$$

where $\{\alpha_k\}$ is a set of arbitrary complex numbers. Each $\mid\alpha_k\rangle$ can be written in terms of the number states of mode k, as

$$\mid \alpha_k\rangle = e^{-\mid\alpha_k\mid^2/2}\sum_{n_k}(n_k!)^{-1/2}\alpha_k^{n_k}\mid n_k\rangle \ . \tag{C2}$$

We define expectation values as, for example, $\langle z(x,t)\rangle \equiv \langle \alpha \mid z(x,t)\mid \alpha\rangle$, where we have set $\mid\alpha\rangle \equiv \mid\{\alpha_k\}\rangle$. From (2.9) and (C1) one has

$$\langle z(x,t)\rangle = \sum_k \left[\frac{\hbar}{2\omega_k\vartheta_{kk}}\right]^{1/2}(\alpha_k e^{-i\omega_k t}+\alpha_k^* e^{i\omega_k t})\Phi_k(x) \ , \tag{C3}$$

which appears to be as classical as possible. Let us now assume that for the string initially $\langle z(x,0)\rangle = \langle \dot z(x,0)\rangle = 0$, while for the particle $\langle\zeta(0)\rangle \equiv \zeta(0)$ and $\langle\dot\zeta(0)\rangle \equiv \dot\zeta(0)$. Using the completeness relations (3.18) and (3.19) it is easily verified that this amounts to the choice (see also, e.g., Ref. 17, Sec. 21)

$$\alpha_k = (2\hbar\omega_k\vartheta_{kk})^{-1/2}[\omega_k\zeta(0)+i\dot\zeta(0)]m\Phi_k^0(0) \ . \tag{C4}$$

Then using (2.13), (C1), (C4), and (3.26), one finds

$$\langle\zeta(t)\rangle = \dot A(t)\zeta(0)+A(t)\dot\zeta(0), \tag{C5}$$

which is as close to (3.27) as quantum theory allows. Let us now consider $\langle\zeta^2(t)\rangle \equiv \langle\alpha\mid\zeta^2(t)\mid\alpha\rangle$. Using (2.13), the commutator (4.8), and (C1), the result may be written in the comprehensive form

$$\langle\zeta^2(t)\rangle = \langle\zeta(t)\rangle^2 + \langle\zeta^2\rangle_0 \ , \tag{C6}$$

where $\langle\zeta^2\rangle_0$ has been identified with the ground-state noise (4.9). Similarly, one finds for the momentum fluctuations

$$\langle p^2(t)\rangle = \langle p(t)\rangle^2 + \langle p^2\rangle_0 \ , \tag{C7}$$

where $\langle p^2\rangle_0$ is also given in (4.9). Since $\langle\zeta(t)\rangle$ and $\langle p(t)\rangle = m\langle\dot\zeta(t)\rangle$ decay to zero when $\Lambda=\infty$ according to (5.4), (C6) and (C7) clearly show that the oscillator will necessarily always end up at least with the system's ground-state noise independent of its initial conditions, as was stated in the main text.

APPENDIX D: THERMAL STRING

Suppose that our system (2.1), consisting of the particle and the string, is embedded in an infinitely large thermostat. If $t<0$, the system is taken to be in thermal equilibrium with this super-reservoir, so that we may apply the standard methods of (quantum) statistical mechanics. At $t=0$ the string still is considered to be in this equilibrium state, but the particle is excited by some unspecified external mechanism. Evidently, this situation is of considerable experimental interest. Below we will answer the question: what will the dynamics of the particle be in this case? In view of the Hamiltonian (4.7) the thermal equilibrium state may be specified by (see, e.g., Refs. 11, 20, and 30)

$$\langle a_k\rangle = 0, \ \ \langle a_k a_l\rangle = 0, \ \ \langle a_k^* a_l\rangle = \vartheta_{kl}^{-1}N_k \ , \tag{D1}$$

$$N_k \equiv [\exp(\beta\hbar\omega_k)-1]^{-1}, \ \ \beta \equiv (k_B T)^{-1} \ . \tag{D2}$$

It is recalled that $\vartheta_{kl}=\vartheta_{kk}\delta_{kl}$. See (3.11). From (2.9) at $t=0$, and (D1), one obtains

$$\begin{aligned}
\langle z(x,0)\rangle &= \langle \dot z(x,0)\rangle \\
&= \langle z(x,0)\dot z(x',0)+\dot z(x',0)z(x,0)\rangle = 0 \ , \\
\langle z(x,0)z(x',0)\rangle \\
&= \hbar\sum_k (N_k+\tfrac{1}{2})\vartheta_{kk}^{-1}\omega_k^{-1}\Phi_k(x)\Phi_k(x') \ , \\
\langle \dot z(x,0)\dot z(x',0)\rangle &= \hbar\sum_k (N_k+\tfrac{1}{2})\vartheta_{kk}^{-1}\omega_k\Phi_k(x)\Phi_k(x') \ .
\end{aligned} \tag{D3}$$

Now, from (3.23) for $\zeta(t)$ and (D3), it is obvious that the particle's position varies simply as (3.27), or (C5), i.e., purely classical. This also holds for the momentum $p\equiv m\dot\zeta$. Next we consider the fluctuations. First square (3.23) and then average, using (D3). Further invoking (3.25) for $A(x,t)$ and $\dot A(x,t)$, evaluating the integrals over $x\in(0,\Lambda)$, using the definition (3.4) of η_{kl}, and expressing η_{kl} in terms of ϑ_{kl} by means of (3.10), one finds

$$\begin{aligned}
\langle\zeta^2(t)\rangle = {} & \dot A^2(t)\langle\zeta^2(0)\rangle + \dot A(t)A(t)\langle\zeta(0)\dot\zeta(0)+\dot\zeta(0)\zeta(0)\rangle + A^2(t)\langle\dot\zeta^2(0)\rangle \\
& + \hbar\sum_k (N_k+\tfrac{1}{2})\vartheta_{kk}^{-1}\omega_k^{-1}\{[\dot A(t)-\cos(\omega_k t)]^2+\omega_k^2[A(t)-\omega_k^{-1}\sin(\omega_k t)]^2\}\Phi_k^0(0)^2 \ .
\end{aligned} \tag{D4}$$

The same calculation for $\langle\dot\zeta^2(t)\rangle$ yields

$$\begin{aligned}
\langle\dot\zeta^2(t)\rangle = {} & \ddot A^2(t)\langle\zeta^2(0)\rangle + \ddot A(t)\dot A(t)\langle\zeta(0)\dot\zeta(0)+\dot\zeta(0)\zeta(0)\rangle + \dot A^2(t)\langle\dot\zeta^2(0)\rangle \\
& + \hbar\sum_k (N_k+\tfrac{1}{2})\vartheta_{kk}^{-1}\omega_k^{-1}\{[\ddot A(t)+\omega_k\sin(\omega_k t)]^2+\omega_k^2[\dot A(t)-\cos(\omega_k t)]^2\}\Phi_k^0(0)^2 \ .
\end{aligned} \tag{D5}$$

It is noted in passing that, by explicit calculation, it is easily shown that the thermal string does not add to the dynamical cross correlations $\frac{1}{2}\langle \zeta(t)\dot{\zeta}(t)+\dot{\zeta}(t)\zeta(t)\rangle$. This should be contrasted with phenomenological theories (see, e.g., Ref. 11). Moreover, it is oncemore emphasized that the present formulas are still exact for the model (2.1) for any value of the parameters specifying it. The limit $\Lambda\to\infty$ can be taken again according to Sec. V. In the final steady state $t\to\infty$, which exists in fact only in the limit $\Lambda\to\infty$, (D4) and (D5) give

$$\langle \zeta^2(\infty)\rangle = \hbar\sum_k (N_k+\tfrac{1}{2})\vartheta_{kk}^{-1}\omega_k^{-1}\Phi_k^0(0)^2 ,$$
$$\langle p^2(\infty)\rangle = \hbar m^2\sum_k (N_k+\tfrac{1}{2})\vartheta_{kk}^{-1}\omega_k\Phi_k^0(0)^2 , \qquad \text{(D6)}$$

which precisely reduce to $\langle \zeta^2\rangle_0$ and $\langle p^2\rangle_0$ as given in (4.9) in the main text at zero temperature $T=0$, where $N_k=0$ according to (D2).

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[24]It may be noted that the present τ is closely related to τ_0 rather than τ of Ref. 22.

[25]Using (2.4) one explicitly needs the phase shifts (2.12) in addition.

[26]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. II, p. 1345.

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[28]In Ref. 27, formula (13), should read $\Theta_{kl}=\omega_l^2\vartheta_{kl}$. The ensuing diagonalization in that paper is still correct, but of little relevance to the original problem. Further, the result (2.18) of Ref. 22, although applying in fact to a somewhat different, longitudinal model, coincides with the above (3.11) if we now take $b=B$, while in Ref. 22 we set $m_0=0$ and read τ for τ_0.

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[32]These results coincide with $\langle U^2\rangle_0$ and $\langle P^2\rangle_0$ of Ref. 22 if one sets $\tau=0$ there and reads τ for τ_0, while presently we take $\Omega^2\tau^2=\frac{1}{2}$, i.e., $b=B$. See Fig. 1.

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[35]It is also useful to notice that the coupling constant τ and the friction constant λ are clearly playing separate roles. For instance, strong coupling ($\Omega\tau$ and $\lambda\tau\ll 1$) can coexist with weak damping ($\lambda\ll\Omega$).

[36]In this approximation the equation of motion is, of course, not exactly (5.4) anymore, but rather the following: $2\lambda\tau^2\dddot{\zeta}+(1+4\lambda^2\tau^2)\ddot{\zeta}+2\lambda(1+\Omega^2\tau^2)\dot{\zeta}+\Omega^2\zeta=0$.

[37]Everywhere in the following only the leading orders of τ^2 are kept in order not to obscure the formula by largely irrelevant details.

[38]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965), Sec. 2.103.

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[42]A slightly more general formula may be gleaned directly from, e.g., (4.11), namely, $\omega_c=(2\lambda\tau^2)^{-1}(1-8\lambda^2\tau^2)^{1/2}$, which of course reduces to (6.1) if $\lambda\tau\to 0$.

[43]Mathematically, this comes about through sums or integrals of the type $\int_0^\infty d\omega\,\omega^{-1}\sin(\omega t)=\pi/2$, even if $t\downarrow 0$. The problem is a matter of interchanging the noncommutating limits $t\downarrow 0$ and $\tau\downarrow 0$.

[44]E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1961), Sec. 18.9.

[45]See especially Ref. 18, formula (3.21).