Higher-order bifurcations in a bistable system with delay

J. Y. Gao,* L. M. Narducci, H. Sadiky, M. Squicciarini, and J. M. Yuan
Department of Physics and Atmospheric Science, Drexel University, Philadelphia, Pennsylvania 19104
(Received 1 March 1984)

The output intensity of a hybrid bistable system with a delay in the feedback loop is known to undergo regular and irregular self-pulsing. With the help of high-resolution power spectra, we have analyzed numerically the transition of this system into chaos for several different values of the delay time, and found evidence for the existence of an infinite sequence of period-doubling bifurcations, followed by a reverse sequence in which the periodic components of the spectrum are superimposed to a continuous noisy structure. We have studied the effect of external Gaussian random noise and verified, as expected, the occurrence of bifurcation gaps. In the vicinity of the self-pulsing threshold, more than three independent frequency components coexist without evidence for the appearance of a strange attractor.

I. INTRODUCTION

The appearance of pulsations in the output of a passive optical system driven by a cw external source has long been recognized as a manifestation of instabilities which are usually the result of the interplay between nonlinearities and feedback.[1] When the delay associated with the feedback loop of a bistable optical system becomes sufficiently long,[2] the system can develop instabilities of a more complex nature than the ones that are responsible for the switching of the time-independent output signal between the low- and high-transmission states.

In this case, as Ikeda demonstrated for the first time, the output intensity can even develop irregular pulsations which are referred to, rather appropriately, as optical chaos. The experimental verifications of this effect by Gibbs, Hopf and collaborators,[3] and the additional observations reported by Okada and Takizawa,[4] and more recently, by Chrostowski and collaborators[5] have stimulated theoretical efforts to understand the phenomenology of the bifurcations and the structure of the chaotic regime.

Ikeda's original investigations of optical chaos[2(a)] focused on the behavior of a ring cavity containing a collection of absorbing two level systems under conditions such that the round-trip time of light through the interferometer is much longer than the atomic relaxation times. The obvious problems that would arise with the practical realization of a sufficiently long interferometric structure were removed by the clever observation[2,3] that the behavior of a long-ring cavity system could be simulated by a relative unsophisticated hybrid bistable device. The mathematical description of this system can be summarized by the following simple-looking delayed differential equation:

$$dV(t)/dt + V(t) = X(t-T) \tag{1}$$

for the feedback voltage $V(t)$ applied to the modulator. In Eq. (1), T is the delay time of the feedback loop, and X is a properly scaled output intensity. The time t and the delay T are both measured in units of the system's natural relaxation time (the reciprocal of its bandwidth). Thus, the feedback voltage signal at time t is controlled by the intensity of the light that has emerged from the modulator at the earlier time $t-T$. In turn, the output intensity X is related to the input intensity Y and to the total voltage, $\theta + V(t)$, applied to the modulator by the equation

$$X(t) = \tfrac{1}{2} Y\{1 - K\cos[\theta + V(t)]\} \tag{2}$$

which is characteristic of the electro- (or acousto-) optic system used in the hybrid device. In Eq. (2) θ denotes a fixed bias voltage applied to the modulator, in units of the half-wave voltage, and K is the modulation depth of the device.

Strictly speaking, a close similarity between the behavior of the hybrid device described by Eqs. (1) and (2), and the ring cavity system is to be expected only in the limit of very long delays. In fact, the original experimental demonstration of the effect[3] was carried out with a value of T of about 40. On the other hand, interesting effects were shown to persist[4] even for considerably smaller values of 1. Hence, in addition to the original motivation that prompted the first experimental investigation, it soon became clear that additional reasons existed for carrying out more detailed studies of the hybrid device.

Previous work[6] demonstrated the existence of a wide domain in control parameter space where unstable behavior was to be expected. Gao and collaborators explained the significant qualitative differences between the long- and short-delay observations[3,4] in terms of the behavior of the eigenvalues of the linearized equations, and exhibited the existence of period-doubling bifurcations with the help of numerical solutions of Eqs. (1) and (2) and of the spectral analysis of the output intensity.

The main objective of the present investigation is to clarify the following three questions that, in our opinion, have not been fully answered.

(i) Does the route to chaos involve an infinite sequence of period-doubling bifurcations for arbitrary values of the delay time?

(ii) What is the effect of external noise on the bifurcation sequence? (i.e., is the origin of the experimentally observed bifurcation gap intrinsic to the system, or can it be

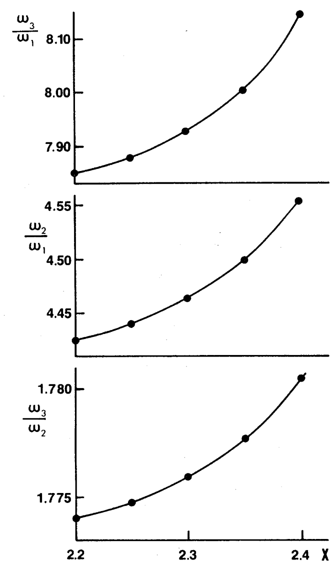

FIG. 1. The frequencies of several spectral components which appear in the output power spectrum slightly above the self-pulsing threshold have ratios that vary monotonically as functions of X. This behavior supports the conjecture concerning the existence of several independent and incommensurate frequencies. These data correspond to $T=0.5$.

ascribed to external perturbations?).

(iii) Can three or more independent frequencies coexist without the system developing a strange attractor?

In part, evidence for an affirmative answer to question (ii) was provided in Ref. 7 by varying the noise level relative to the input intensity signal. Independently this question has also received attention by Vallee *et al.*[8] who have studied, both theoretically and experimentally, how additive and multiplicative noise can produce bifurcation gaps.

The results of our study give strong support to the notion that the route to chaos involves an infinite period-doubling sequence for a wide range of values of the delay time T. The injection of Gaussian noise into the system, not surprisingly, is responsible for the disappearance of some of the fine features that are associated with higher bifurcations and thus provides a plausible explanation for the observed bifurcation gap. Finally, three or more seemingly independent frequencies appear to coexist under

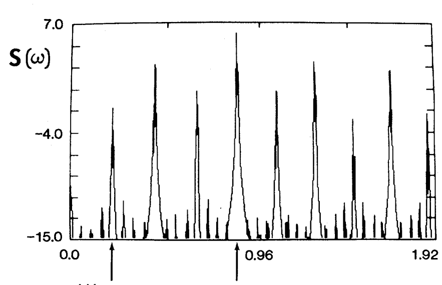

FIG. 2. Spectrum $S(\omega)$ of a $4P$ solution corresponding to $T=3.0$ and $X=1.87$. The subharmonic component $\omega/4$ and its multiples are clearly visible.

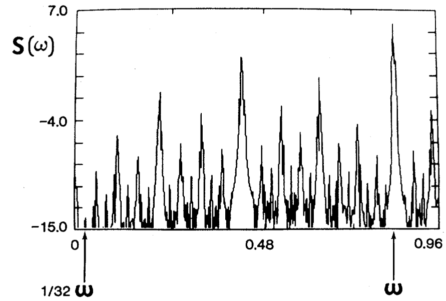

FIG. 3. Spectrum $S(\omega)$ of a $32P$ solution. The subharmonic component $\omega/32$ is barely visible on the far left of the spectrum, but the combination frequencies which result from the beating of this component with the other spectral lines are clearly identifiable. Note that the frequency scale of this picture has been expanded by a factor of 2 relative to that of Fig. 2.

conditions which favor the occurrence of highly regular pulsations in the output intensity.

II. THE BIFURCATION SEQUENCE INTO CHAOS

The delayed differential equation (1) and the system response function (2) are governed by the following four independent parameters: the input intensity Y, the delay time T of the feedback loop, the modulation depth K of the electro-optic device, and the bias voltage θ. We have selected K and θ, equal to the values measured in the experiments described in Ref. 7 ($K=0.96$, $\theta=-\pi$), and determined the locus of points in the X-T plane where the system is unstable against arbitrarily small fluctuations. Although the output intensity X is actually a dependent variable, the relation between Y and X in the lower branch of the state equation is single valued; in the following, we shall consider X as an independent parameter, for convenience.

The simple steps required to construct the instability domain have already been described in a previous publication, and the instability domain itself is displayed in Fig. 2 of Ref. 6(a). Although the present values of K and θ are somewhat different from those used previously, the quali-

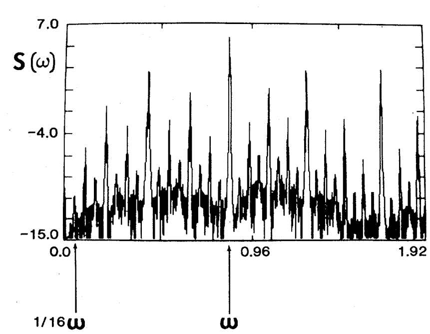

FIG. 4. Periodic oscillations with a noisy component are labeled by the symbol 2^nN, where 2^n refers to the type of periodicity displayed by the coherent component of the output signal. The spectrum $S(\omega)$ of this figure refers to a $16N$ solution, corresponding to $T=3$ and $X=1.8752$.

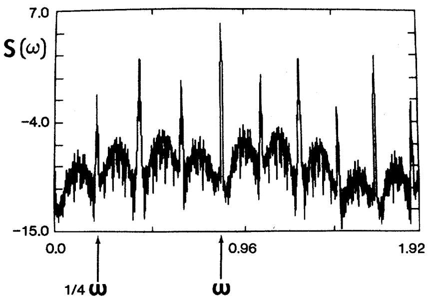

FIG. 5. Spectrum $S(\omega)$ of a $4N$ solution for $T=3$ and $X=1.876$.

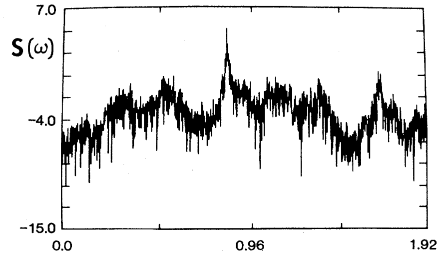

FIG. 7. Spectrum $S(\omega)$ of a $1N$ solution for $T=3$ and $X=1.9$.

tative shape of the instability domain in the X-T plane remains the same, and will not be reproduced here.[9]

Except for a narrow range of delay times in the neighborhood of the left edge of the instability boundary, the system develops a chaotic behavior for all values of T, when the level of the incident intensity becomes sufficiently large. In this study, we have focused our attention on three values of the delay, namely $T=0.5$, 3.0, and 10.0, and carried out a systematic study of the behavior of the time-dependent output intensity for increasing values of the injected signal. The solution of the delayed equation was obtained by integrating Eq. (1) formally, and then solving numerically for the time-dependent voltage $V(t)$ using knowledge of the earlier history of this variable.

As already observed [Ref. 6(b)], the power spectrum of the output intensity provides a convenient way to monitor in detail the behavior of the system. In our work, we have calculated the position of each spectral line and its integrated area. The peak location was determined by the formula

$$\bar{\omega}=\sum_i \omega_i S_i / \sum_i S_i \ , \qquad (3)$$

where the sums extend to all points of the discrete spectrum lying above a chosen discrimination level, and where S_i denotes the value taken by the power spectrum at frequency ω_i. Appropriate precautions have been built into our analyzing routine to prevent errors due to poor resolution or noisy data. Ambiguities have usually been resolved by direct visual inspection of the graphical output. The area of each spectral line, defined by the quantity S_i, is very useful in determining bifurcation thresholds, because it approaches zero smoothly in the neighborhood of the transition point.

Before summarizing the results of one of our scans, we must point out that the qualitative behavior of the spectra on the way to chaos remains very nearly the same, regardless of the chosen value of the delay. For values of the incident intensity slightly above the instability threshold, the spectrum displays a single large peak, with a frequency which is approximately equal to the imaginary part of the only unstable eigenvalue of the linearized equations, and several of its harmonic components; additional smaller features appear at frequencies which are surprisingly close to the imaginary parts of the stable eigenvalues. These spectral components appear to represent independent oscillations in the sense that their frequencies are not commensurate with each other or with the frequency of the main peak. Figure 1 provides support for this observation by displaying the smooth variation of several ratios of these frequencies upon varying the transmitted intensity.[10] Combination frequencies are also observable at the same time, as already shown in Fig. 2 of Ref. 6(b). The smaller features of the spectrum, which are readily observable just above the self-pulsing threshold, gradually become less important as the system approaches its first period-doubling bifurcation and practically disappear beyond it, apparently because their frequencies become multiples of the fundamental frequency of oscillation.

On increasing the incident intensity, the spectrum undergoes a regular sequence of period-doubling bifurcations. For a value of the delay equal to 0.5, we have observed self-pulsing oscillations of the $8P$ type (i.e., displaying the subharmonic component $\omega/8$), while for $T=3.0$, and $T=10.0$, we have been able to identify solu-

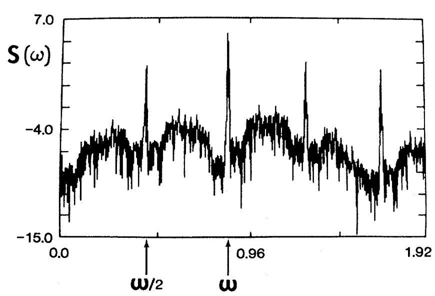

FIG. 6. Spectrum $S(\omega)$ of a $2N$ solution for $T=3$ and $X=1.879$.

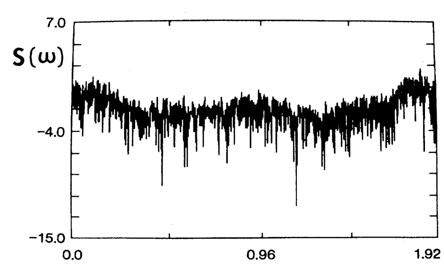

FIG. 8. Chaotic spectrum $S(\omega)$ for $T=3$ and $X=2.2$.

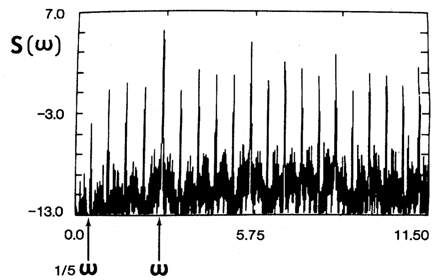

FIG. 9. Spectrum $S(\omega)$ of a periodicity window in chaos. This solution displays a subharmonic component $\omega/5$ for $T=0.5$ and $X=2.515$.

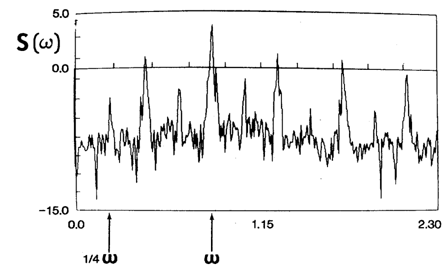

FIG. 10. Spectrum $S(\omega)$ of an $8P$ solution in the presence of Gaussian random noise with a standard deviation equal to 5% of the average output signal takes on the appearance of a $4P$ solution riding on top of a flat, noisy background.

tions as complicated as $32P$. Figures 2 and 3 show examples of $4P$ and $32P$ solutions. The available evidence points rather convincingly to the existence of an infinite cascade of period-doubling bifurcations for delays equal to 3.0 and 10.0, respectively. For $T=0.5$, the evidence is less compelling, but, in our opinion, still consistent with the same conclusion.

The emergence of a chaotic component in the output pulsations is made apparent by the continuous noisy background that develops under the coherent component of the spectrum. As the continuous background grows with increasing values of the incident intensity, the subharmonic components with the smallest frequencies gradually disappear through an inverse sequence of bifurcations. A particularly striking example is shown in Figs. 4—7. Eventually, for sufficiently large values of the driving field intensity, fully developed chaos takes over. The spectrum at this point shows no evidence of the preceding periodic structure except for a slight modulation of the power density (Fig. 8).

As one approaches the upper boundary of the instability domain, this scenario is apparently reproduced in reverse, at least judging from selected (but not detailed) observations for larger values of the driving field. The fully chaotic domain is not completely unstructured as evidenced by the appearance of periodicity windows. An example is shown in Fig. 9, where the subharmonic component $\omega/5$ is a prominent feature.

In order to study the effect of added noise on the system, we have used a Gaussian random number generator whose mean and standard deviation could be adjusted as needed. At each step in the numerical solution of the delayed differential equation, we have added a noise disturbance to each calculated value of the feedback voltage $V(t)$, and then constructed the power spectrum of the resulting time series in the usual way. The most obvious effect of the added noise is the appearance of a continuous background in the calculated spectra and the disappearance of the smallest subharmonic components (Fig. 10). It is interesting to observe that the injected noise produces a very different background structure from what is observed, for example, in Figs. 5—7.

In conclusion, our results support the existence of a unique route to chaos for the so-called Ikeda model for essentially any value of the delay time in the feedback loop. The first appearance of self-pulsing oscillations is followed by an apparently infinite sequence of period-doubling bifurcations, and subsequently by the gradual emergence of a chaotic component in the power spectrum of the output intensity. A reverse sequence of bifurcations, where the coherent subharmonic components gradually disappear as the continuous noise background becomes progressively larger, heralds the appearance of fully developed chaos. External noise can be a reasonable cause for the occurrence of bifurcation gaps; perhaps, external noise may be recognizable because of its uniform and unstructured spectral character. The occurrence of a large (perhaps even infinite) number of seemingly independent spectral components in the spectrum of periodic solutions slightly above threshold for self-pulsing may provide an interesting starting point for a reanalysis of some of the popular interpretations of the Ruelle-Takens scenario.

ACKNOWLEDGMENTS

This grant was partially supported by the U. S. Army Research Office under Contract No. DAAG-29-82-K0021 and by a Grant from the research laboratories of the Martin-Marietta Corporation. We are especially grateful to Professors J. Yorke and E. Ott for useful and illuminating comments.

*Permanent address: Physics Department, Jilin University, Changchun, Jilin, People's Republic of China.

[1]There is an extensive literature on the subject of self-pulsing in optical bistability. For a comprehensive review, see, for example, L. A. Lugiato, in *Progress in Optics,* edited by E. Wolf (North-Holland, Amsterdam, 1984), Vol. XXI, p. 104. See also *Optical Bistability,* edited by C. M. Bowden, M. Cliftan, and H. R. Robl (Plenum, New York, 1981).

[2](a) K. Ikeda, Opt. Comm. **30**, 257 (1979); (b) K. Ikeda, H. Daido, and O. Akimoto, Phys. Rev. Lett. **45**, 709 (1980).

[3](a) H. M. Gibbs, F. A. Hopf, D. L. Kaplan, and R. L. Shoemaker, Phys. Rev. Lett. **46**, 474 (1981); (b) F. A. Hopf, D. L. Kaplan, H. M. Gibbs, and R. L. Shoemaker, Phys. Rev. A **25**, 2172 (1982).

[4]M. Okada and K. Takizawa, J. Quantum Electron. **QE17**, 2135 (1981).

[5](a) J. Chrostowski, C. Delisle, and R. Tremblay, Can. J. Phys. **61**, 188 (1983); (b) J. Chrostowski, R. Vallee, and C. Delisle, Can. J. Phys. **61**, 1143 (1983).

[6](a) J. Y. Gao, J. M. Yuan, and L. M. Narducci, Opt. Commun. **44**, 201 (1983); (b) J. Y. Gao, L. M. Narducci, L. S. Schulman, M. Squicciarini, and J. M. Yuan, Phys. Rev. A **28**, 2910 (1983).

[7](a) M. W. Derstine, H. M. Gibbs, F. A. Hopf, and D. L. Kaplan, Phys. Rev. A **26**, 3720 (1982); (b) M. W. Derstine, H. M. Gibbs, F. A. Hopf, and D. L. Kaplan, Phys. Rev. A **27**, 3200 (1983).

[8]R. Vallee, C. Delisle, and J. Chrostowski (private communication). We are grateful to Dr. Chrostowski for making this information available to us prior to publication.

[9]Note, however, that the vertical axis of Fig. 2, Ref. 6(a) should have been labeled by X.

[10]We are indebted to Professor J. Yorke for useful suggestions and clarifying discussions on this point.