Transition probability of a two-level atom interacting with a time-symmetric pulse

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We have studied a special class of time-symmetric pulses interacting with a two-level system. These pulses are built out of a class of asymmetric functions for which the analytic description of the interaction was possible. The time behavior of these pulses is given. We have evaluated the transition probability in a closed form. These pulses can be used to simulate the familiar Lorentzian and Gaussian pulse shapes remarkably well. Therefore, they can be used in computing the overall features of the interaction process. Comparisons of the transition amplitudes are made between the closed-form solution and numerically calculated Lorentzian and Gaussian cases.

I. INTRODUCTION

In many applications of laser spectroscopy an atom, subject to interaction with a resonant or quasiresonant EM field, may be considered as a two-level system, the two levels being those connected by the radiation field. The problem of finding transition probabilities is then reduced to the problem of solving a time-dependent Schrödinger equation for the two levels. This in turn reduces to a pair of coupled differential equations for the level amplitudes a_1 and a_2:

$$\frac{d}{dt}a_1=-i\beta f(t)e^{-i\alpha t}a_2 ,\qquad \frac{d}{dt}a_2=-i\beta f(t)e^{i\alpha t}a_1 . \tag{1}$$

In deriving (1), one assumes that the EM field has an optical frequency carrier ω_L which is detuned from the atomic transition ω_0 by a (small) quantity α. In (1) $f(t)$ represents the time-dependent, and slowly varying, field amplitude; β is a coupling coefficient. The rapidly varying terms oscillating at twice the optical frequencies have been eliminated (rotating-wave approximation).

Relaxation terms are not present in (1). Therefore Eqs. (1) will apply to those cases where the interaction between the field and the atom takes place in a time shorter than the lifetime of the two levels, thus restricting the field amplitude $f(t)$ to a class of functions which go to zero sufficiently fast as t approaches $\pm\infty$. For a thorough description of this problem the reader is referred to Allen and Eberly.[1]

Other problems in many areas of physics reduce to solving (1) subject to particular initial conditions. For instance, in experiments of laser spectroscopy[2] the laser field induces transitions between atomic or (quasi) molecular levels. Here several atomic levels are involved in the whole process. However, Vainshtein *et al.*[3] and Payne and Nayfeh[4] have shown that the essential physics is contained in the time evolution of a two-level system, thus bringing the problem back to that of solving (1).

Although simple in form, Eqs. (1) constitute a formidable problem. The general solution is known only when $\alpha=0$. In this case the simple transformation

$$z=\int_{-\infty}^{t} f(t')dt' \tag{2}$$

(where the integral is supposed to converge when evaluated from $-\infty$ to $+\infty$, as discussed above) maps the system (1) into the differential equation

$$a''_{1,2}+\beta^2 a_{1,2}=0 , \tag{3}$$

where primes denote differentiation with respect to z. Solutions of (3) are

$$a_{1,2}(z)=A_{1,2}\sin(\beta z)+B_{1,2}\cos(\beta z) \tag{4}$$

so that, transforming back to the t variable, and taking as initial conditions

$$a_1(-\infty)=1,\quad a_2(-\infty)=0 , \tag{5}$$

one gets

$$a_1(t)=\cos\left[\beta\int_{-\infty}^{t} f(t')dt'\right] ,\qquad a_2(t)=-i\sin\left[\beta\int_{-\infty}^{t} f(t')dt'\right] . \tag{6}$$

Equations (6) display the well-known feature[1] that, when $\beta\int_{-\infty}^{+\infty} f(t)dt$ is an integer multiple of π, the (upper) level, whose amplitude is a_2, is left after the interaction with the EM field in the same condition (i.e., with zero amplitude) as it was prior to the interaction.

When $\alpha\neq 0$, only a few particular cases can be solved analytically. To the authors' knowledge, these include the following.

(i) Constant envelope amplitude:

$$f(t)=\begin{cases}0, & t<\tau_i\\ f_0=\text{const}, & \tau_i<t<\tau_f\\ 0, & \tau_f<t .\end{cases} \tag{7}$$

this is the so-called "Rabi solution." Inserting (7) into (1), one shows that the solution is given by

$$a_{1,2}(t)\approx A_{1,2}\exp\left[i\frac{-\alpha+(\alpha^2+4\beta^2f_0^2)^{1/2}}{2}t\right]+B_{1,2}\exp\left[i\frac{-\alpha-(\alpha^2+4\beta^2f_0^2)^{1/2}}{2}t\right]. \quad (8)$$

The level amplitudes oscillate at the frequencies $[-\alpha\pm(\alpha^2+4\beta^2t_0^2)^{1/2}]/2$ (Rabi frequencies).

(ii) The solution for the pulse shape

$$f(t)=\text{sech}(t/2) \quad (9)$$

was obtained some 50 years ago by Rosen and Zener.[5] We omit the derivation of the solution here since it will be derived as a special case of a more general solution later on. The very interesting feature of this solution is that the upper-level population $|a_2(t)|^2$, assumed to be zero at $t=-\infty$, displays at $t=+\infty$ the following behavior:

$$|a_2(\infty)|^2=\text{sech}^2(\pi\alpha)\sin^2(\beta\pi), \quad (10)$$

$\beta\pi$ being the area $\beta\int_{-\infty}^{+\infty}f(t)dt$ of the pulse. Once again, as in the $\alpha=0$ case, one has the feature that the upper-level population is left zero after the passage of the pulse if the pulse area is an integral multiple of π. The only difference is that now $|a_2|^2$ cannot reach, at any time, the maximum allowed value of 1 because of the detuning α.

This remarkable fact led Rosen and Zener to formulate the hypothesis that such a behavior was characteristic for any kind of pulse; they conjectured that a general possible form for $|a_2|^2$ could have been

$$|a_2(\infty)|^2=|\mathscr{F}(\alpha)\sin A|^2, \quad (11)$$

where A is the area of the pulse and $\mathscr{F}(\alpha)$ is the Fourier transform of the pulse shape evaluated at $\omega=\alpha$. Equation (10) would be a special case of (11) for a hyperbolic-secant pulse shape. This conjecture was proven wrong by a number of authors.[6,7]

(iii) Quite recently, Bambini and Berman[8] have shown that the Rosen-Zener analytical solution can be generalized. They found that an analytical solution is possible for a whole class of functions $f(t)$ whose shape is given in parametric form,

$$f(z)=\frac{\sqrt{z(1-z)}}{\lambda z+\mu}, \qquad e^t=\frac{z^\mu}{(1-z)^{\lambda+\mu}}, \quad (12)$$

where λ and μ are real parameters, and z is a dummy variable ranging from 0 to 1 when t ranges from $-\infty$ to $+\infty$. It is easy to see that, when $\mu=\frac{1}{2}$ and $\lambda=0$, one gets from (12)

$$f(t)=\text{sech}(t),$$

thus showing that the Rosen-Zener pulse shape[9] belongs to the class of functions (12). Apart from the hyperbolic-secant case, all other pulses of (12) are asymmetric in time, i.e., for $\lambda>0$ the growth of the pulses is faster than their decay and, conversely, for $\lambda<0$ the decay is faster than their growth. The remarkable feature of these pulses is that they never leave the upper level unpopulated for any value of the area of the pulse,[8,9] contrary to the prediction of the Rosen-Zener conjecture.

With the advent of fast computer facilities, the problem of finding analytical solutions to (1) could seem to be overcome. However, this is not exactly the case. When α gets very large, numerical integration of (1) must be performed with special care due to the fast variations undergone by the coupling terms $e^{\pm i\alpha t}$. Thus, although possible in principle, numerical calculations may become costly, and, after all, they do not disclose anything else but numbers. Thus some speculations about analytical properties of (1) and its solutions still seem to be in order. Moreover, ordinary perturbation theory fails to predict transition probabilities due to the existence of an essential singularity of (1) at $t=\infty$.

In 1972—1973 Crothers[10] obtained a general formula for evaluating the transition probability of a two-level system under the action of a time-dependent coupling potential. His formula applies even if α is made time dependent and it changes sign at some time during the interaction.[11] More recently, Berman and Robinson[12] have shown that the asymptotic behavior of the transition probability is related to the residue of first-order poles of $f(t)$ in the complex t plane.

We want to show here another approach to the problem. Using a symmetrization procedure for pulses in (12), one can build up a class of functions (all with the same area and same peak amplitude) whose analytical solution can be written down. Although the transition probability will appear in the final results in terms of the hypergeometric function, its evaluation is much faster and more precise than direct numerical integration of (1), especially for larger values of α.

As we will see, these pulse shapes can be made similar to other pulse shapes (for instance, Lorentzian pulse shapes) which are frequently met when solving practical problems and may therefore serve as an indicator of the overall properties of these pulses. The pulse shapes we will obtain are a one-parameter family and may vary in a large range. The asymptotic behavior of their solutions will be compared to the asymptotic behavior obtained through numerical computations from other kinds of pulse shapes.

II. DERIVATION OF SYMMETRIC PULSES

First of all, we rederive here the pulse shape for which an analytical solution exists. System (1) can be written in the form of a single second-order differential equation for a_1,

$$\ddot{a}_1+\left[i\alpha-\frac{\dot{f}}{f}\right]\dot{a}_1+\beta^2f^2a_1=0. \quad (13)$$

A similar equation holds for a_2, the only difference being the replacement of α with $-\alpha$.

The most general transformation of the independent variable is

$$z=\int_{-\infty}^{t}\rho(t')f(t')dt' , \tag{14}$$

where $\rho(t)$ is an arbitrary function. The assumption is made that ρf is summable in the infinite interval $(-\infty,+\infty)$. Inserting (14) into (13) one finds

$$\rho^2 f^2 a_1'' + (i\alpha\rho+\dot\rho) f a_1' + \beta^2 f^2 a_1 = 0 , \tag{15}$$

where a prime indicates a derivative with respect to the new variable z and a dot a derivative with respect to the old variable t.

Equation (15) can be transformed into the hypergeometric equation

$$z(1-z)a_1'' + (Az+B)a_1' + Da_1 \tag{16}$$

if the following identifications are made:

$$\frac{\beta^2}{\rho^2}=\frac{D}{z(1-z)} , \tag{17}$$

$$\frac{(i\alpha\rho+\dot\rho)f}{\rho^2 f^2}=\frac{Az+B}{z(1-z)} . \tag{18}$$

Equation (17) can be satisfied if we set

$$\beta^2 = D , \tag{19}$$

$$\rho=\sqrt{z(1-z)} , \tag{20}$$

while Eq. (18) gives $f(z)$ after insertion of (20):

$$f(z)=\frac{i\alpha\sqrt{z(1-z)}}{(A+1)z+B-\frac{1}{2}} . \tag{21}$$

The hypergeometric equation has fixed singular points at $z=0$, $z=1$, and $z=\infty$. With the choices made, $t=-\infty$ corresponds to $z=0$ and $t=+\infty$ corresponds to $z=1$. The integral $\int_{-\infty}^{+\infty}\rho(t)f(t)dt$ is, therefore, equal to 1, as can be easily seen.

The arbitrary constants A and B are chosen in such a way as to obtain a real function $f(z)$. Thus we set

$$A+1=i\alpha\lambda , \tag{22}$$

$$B-\tfrac{1}{2}=i\alpha\mu , \tag{23}$$

where λ and μ are real parameters. Thus we find

$$f(z)=\frac{\sqrt{z(1-z)}}{\lambda z+\mu} . \tag{24}$$

The explicit time dependence of f cannot be written down. One can only find the relation between t and z by integrating the equation

$$\frac{dz}{dt}=\rho f\equiv\frac{z(1-z)}{\lambda z+\mu} ; \tag{25}$$

one finds

$$t=\ln\frac{z^{\mu}}{(1-z)^{\lambda+\mu}} . \tag{26}$$

The pulse shapes thus obtained in parametric form [Eqs. (24)–(26)] all have the same area π: indeed,

$$\int_{-\infty}^{\infty}f(t)dt=\int_0^1 f(z)\frac{dz}{z}=\int_0^1\frac{dz}{\sqrt{z(1-z)}}=\pi . \tag{27}$$

Their peak amplitude is not fixed, however. The maximum of $f(z)$ is attained at

$$z_{\max}=\frac{\mu}{2\mu+\lambda} \tag{28}$$

and has the value

$$f_{\max}=\frac{1}{2\sqrt{\mu(\mu+\lambda)}} . \tag{29}$$

The maximum of the pulse is not at $t=0$ unless $\lambda=0$. The problem of finding solutions for the level amplitudes can be solved if the pulse function $f(t)$ which appears in (1) or (13) belongs to the class of functions defined by (24)–(26). Indeed, for these functions, Eq. (13) maps into (16), and the hypergeometric function provides the analytical solution to our problem.

The general solution of Eq. (16) is[13] a linear combination of two independent solutions,

$$a_1(z)=A_1F(a,b;c;z) + B_1 z^{1-c}F(a-c+1,b-c+1;2-c;z) , \tag{30}$$

where $F(a,b,c,z)$ is the hypergeometric function whose parameters a,b,c are defined by the following relations:

$$a+b=-(1+A) ,\quad ab=-D ,\quad c=B . \tag{31}$$

The two constants A_1 and B_1 in (30) can be evaluated in terms of the initial conditions $a_{1,2}(z=0)$; one finds

$$a_1(z)=a_1(0)F(a,b;c;z) - a_2(0)\frac{i(-ab)^{1/2}}{1-c} \times z^{1-c}F(a-c+1,b-c+1;2-c;z) . \tag{32}$$

The nature of the solution is described in Ref. 8. We point out here the remarkable fact that these pulses do not allow for a zero transition probability except in the cases of resonant tuning ($\alpha=0$) or a time-symmetric pulse ($\lambda=0$).

III. SYMMETRIZATION OF PULSES

The question now arises whether the hyperbolic-secant pulse shape is an exceptional case, in the sense that it is the only pulse shape which allows for zero transition probability at $\alpha\neq 0$. We have indirect proofs that this is not the case. Indeed, numerical work[14] has shown that other pulse shapes, such as the Lorentzian pulse or the Gaussian pulse, share the same property with the hyperbolic-secant pulse, although the zero transition probability is attained for values of the pulse area which are not equally spaced, as is the case for the hyperbolic-secant pulse shape. Moreover, Robinson[7] has recently pointed out that, in general, the lack of zero transition probability is a direct consequence of the asymmetry of pulses and was not peculiar to pulses of the class described above.

It is of interest, therefore, to see the asymptotic behavior of the upper-level amplitude when the two-level system is subject to pulses of a time-symmetric nature. This can be done by symmetrizing the pulses of the family and looking for the transition probability they cause. To this end, we consider a pulse shape which is given by

$$f(t)=f(\mu,\lambda\,|\,z) \tag{33}$$

for $-\infty < t \le t_{max}$ (i.e., for $0 < z < z_{max}$) and

$$f(t)=f(-t+2t_{max}) \tag{34}$$

for $t_{max} \le t < +\infty$. These pulses are symmetric in time, although their maximum is not achieved at $t=0$. These pulses are not analytical functions of t, as in the previous case, because of the matching at $t=t_{max}$. However, both the function and its derivative are continuous through $t=t_{max}$ and no jump occurs in Eq. (13).

The area of the pulse is twice the area from $-\infty$ to t_{max} because of symmetrization of the pulse. In terms of the variable z we find

$$\theta=2\beta\int_{-\infty}^{t_{max}} f(t)dt =4\beta\arcsin\sqrt{z_{max}}\ , \tag{35}$$

where the value z_{max} at which the function attains its maximum is given by

$$z_{max}=\frac{\mu}{2\mu+\lambda}\ . \tag{36}$$

In Fig. 1 a few examples of pulse shapes are reported. Here, each pulse is identified by the value of the parameter λ, and the other two parameters μ and β have been determined in such a way that the area θ is equal to π and the peak amplitude is equal to 1. Thus μ and β satisfy the relations

$$\beta f_{max}\equiv\frac{\beta}{2}\frac{1}{\sqrt{\mu(\mu+\lambda)}}=1\ , \quad 4\beta\arcsin\left[\frac{\mu}{2\mu+\lambda}\right]^{1/2}=\pi \tag{37}$$

for each value of λ.

Pulses with negative values of λ have sharp peaks and long tails, thus displaying a "Lorentzian-like" behavior.

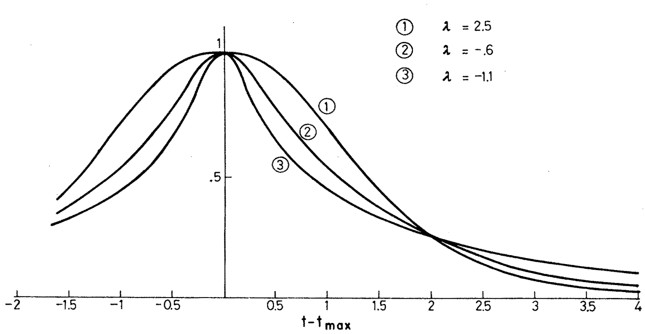

FIG. 1. Pulse shapes of three symmetrized pulses with $\lambda=2.5$, -0.6, and -1.1. Areas of the pulses are normalized to π and the peak amplitudes are set to 1.0.

On the other hand, pulses with positive values of λ have a flat maximum but decrease faster to zero in their tails. Thus they show a "Gaussian-like" behavior.

Now we address the problem of finding the asymptotic behavior of the two-level system interacting with a potential whose time behavior is given by any member of our one-parameter family. We assume that the system at $t=-\infty$ has the initial conditions

$$a_1(-\infty)=1\ , \quad a_2(-\infty)=0 \tag{38}$$

and we look for a_1 and a_2 at $t=+\infty$. To this end, we first write the general solution for $a_1(z)$ and $a_2(z)$ in the first half of the interaction, i.e., from $z=0$ to $z=z_{max}$. The solution for a_1 is given by (30),

$$a_1(z)=A_1F(a,b;c;z)+B_1z^{1-c}F(a-c+1,b-c+1;2-c;z)\ . \tag{39}$$

The parameters a, b, and c which enter the hypergeometric function can be evaluated solving (22), (23), and (31). We find

$$a=-\frac{i\alpha\lambda}{2}+\left[\beta^2-\frac{\alpha^2\lambda^2}{4}\right]^{1/2}, \quad b=-\frac{i\alpha\lambda}{2}-\left[\beta^2-\frac{\alpha^2\lambda^2}{4}\right]^{1/2}, \quad c=\tfrac{1}{2}+i\alpha\mu\ . \tag{40}$$

The general solution for a_2 can be easily found if we note that changing the sign of α in (1) is the same as interchanging a_1 and a_2. Thus

$$a_2(z)=A_2F(\bar a,\bar b;\bar c;z)+B_2z^{1-\bar c}F(\bar a-\bar c+1,\bar b-\bar c+1;2-\bar c;z)\ , \tag{41}$$

where $\bar a,\bar b,\bar c$ are obtained from a, b, and c of (40) by changing α to $-\alpha$.

The four constants of integration A_1, B_1, A_2, and B_2 which appear in (39) and (41) cannot be independent; they are linked by system (1), which determines two of them in terms of the remaining two. Inserting (39) and (41) into (1), and using the initial conditions (38), we find

$$A_1=1\ , \quad B_1=0\ , \quad A_2=0\ , \quad B_2=-\frac{i\beta}{1-\bar c}\ . \tag{42}$$

Details of this calculation are shown in Appendix A. At the pulse maximum, a_1 and a_2 have the value

$$a_1=F(a,b;c;z_{max})\ , \quad a_2=-\frac{i\beta}{1-\bar c}z_{max}^{1-\bar c}F(\bar a-\bar c+1,\bar b-\bar c+1;2-\bar c;z_{max})\ . \tag{43}$$

When $t > t_{max}$ the level amplitudes a_1 and a_2 still satisfy system (1), but $f(t)$ must be evaluated from the symmetry condition

$$f(t)=f(-t+2t_{max}) . \quad (44)$$

Thus the change of variables

$$\tau=2t_{max}-t , \quad (45)$$

$$b_1=a_2e^{i\alpha t_{max}} , \quad (46)$$

$$b_2=a_2e^{-i\alpha t_{max}} , \quad (47)$$

brings system (1) into the system

$$\frac{db_1}{d\tau}=i\beta f(\tau)e^{i\alpha\tau}b_2 ,$$
$$\frac{db_2}{d\tau}=i\beta f(\tau)e^{-i\alpha\tau}b_1 . \quad (48)$$

Comparing (48) with (1), we see that b_1 will have the same solutions as a_2 and b_2 the same solutions as a_1. Thus we obtain

$$b_2(\tau)=\nu_1F(a,b;c;z)$$
$$+\nu_2z^{1-c}F(a-c+1,b-c+1;2-c;z) ,$$
$$b_1(\tau)=\nu_3F(\bar{a},\bar{b};\bar{c};z)$$
$$+\nu_4z^{1-\bar{c}}F(\bar{a}-\bar{c}+1,\bar{b}-\bar{c}+1;2-\bar{c};z) . \quad (49)$$

The relation between ν_1,ν_2 and ν_3,ν_4 is

$$\nu_3=-\frac{1}{(ab)^{1/2}}(1-c)\nu_2 ,$$
$$\nu_4=-\frac{1}{c}(ab)^{1/2}\nu_1 \quad (50)$$

(see Appendix A). Thus we can match the solutions at $t=t_{max}$, i.e., at $z=z_{max}$. We denote by $\mathscr{F}_1$ and $\mathscr{F}_2$ the two quantities

$$\mathscr{F}_1=F(a,b;c;z_{max}) , \quad (51)$$

$$\mathscr{F}_2=\frac{\beta}{1-\bar{c}}z_{max}^{1-\bar{c}}F(\bar{a}-\bar{c}+1,\bar{b}-\bar{c}+1,2-\bar{c};z_{max}) , \quad (52)$$

and setting

$$x=\nu_1,\ \ y=\frac{1}{(ab)^{1/2}}(1-c)\nu_2 \quad (53)$$

and noting that

$$F(\bar{a},\bar{b};\bar{c};z_{max})=\mathscr{F}_1^* ,$$
$$\frac{\beta}{1-c}z_{max}^{1-c}F(a-c+1,b-c+1;2-c;z_{max})=\mathscr{F}_2^* , \quad (54)$$

we obtain the system

$$i\mathscr{F}_2x+\mathscr{F}_1^*y=e^{i\alpha t_{max}}\mathscr{F}_1 ,$$
$$\mathscr{F}_1x+i\mathscr{F}_2^*y=-ie^{-i\alpha t_{max}}\mathscr{F}_2 . \quad (55)$$

Since system (1) is unitary, $|a_1|^2+|a_2|^2=|a_1(-\infty)|^2+|a_2(-\infty)|^2=1$. Thus, because of (43)

$$|\mathscr{F}_1|^2+|\mathscr{F}_2|^2=1 ,$$

and the solution of (55) is

$$x=-2i\,\mathrm{Re}(\mathscr{F}_1^*\mathscr{F}_2e^{-i\alpha t_{max}}) , \quad (56)$$

$$y=e^{i\alpha t_{max}}\mathscr{F}_1^2-e^{-i\alpha t_{max}}\mathscr{F}_2^2 . \quad (57)$$

We can easily evaluate the asymptotic behavior of the two-level system under the action of a symmetrized pulse. The value of the upper-level amplitude at $t=+\infty$ can be obtained from the value of $b_2(\tau)$ evaluated at $t=-\infty$, or

TABLE I. Symmetrized pulse shapes with unit peak amplitude and area equal to π.

$t-t_{max}$	$\lambda=-1.1$	$\lambda=-0.8$	$\lambda=-0.6$	$\lambda=-0.45$
0.0	1.0000	1.0000	1.0000	1.0000
0.2	0.8720	0.9179	0.9440	0.9591
0.4	0.7151	0.7869	0.8363	0.8694
0.6	0.5978	0.6705	0.7257	0.7660
0.8	0.5107	0.5754	0.6266	0.6658
1.0	0.4435	0.4979	0.5413	0.5751
1.2	0.3899	0.4340	0.4686	0.4952
1.4	0.3459	0.3805	0.4066	0.4258
1.6	0.3089	0.3352	0.3534	0.3659
1.8	0.2774	0.2963	0.3078	0.3144
2.0	0.2502	0.2626	0.2683	0.2701
3.0	0.1550	0.1474	0.1369	0.1264
4.0	0.0992	0.0844	0.0704	0.0592
5.0	0.0642	0.0486	0.0363	0.0277
6.0	0.0418	0.0280	0.0187	0.0130
7.0	0.0273	0.0162	0.0097	0.0061
8.0	0.0178	0.0093	0.0050	0.0028
9.0	0.0116	0.0054	0.0026	0.0013
10.0	0.0076	0.0031	0.0013	0.0006
11.0	0.0050	0.0018	0.0007	0.0003

TABLE II. Symmetrized pulse shapes, with unit peak amplitude and area equal to π.

$t-t_{max}$	$\lambda=2.5$	$\lambda=1.5$	$\lambda=0.8$	$\lambda=0.5$
0.0	1.0000	1.0000	1.0000	1.0000
0.2	0.9894	0.9885	0.9871	0.9859
0.4	0.9542	0.9513	0.9465	0.9423
0.6	0.8921	0.8870	0.8786	0.8714
0.8	0.8059	0.7994	0.7889	0.7802
1.0	0.7038	0.6973	0.6870	0.6786
1.2	0.5963	0.5911	0.5830	0.5763
1.4	0.4930	0.4898	0.4847	0.4805
1.6	0.4004	0.3391	0.3871	0.3952
1.8	0.3211	0.3215	0.3219	0.3221
2.0	0.2554	0.2569	0.2591	0.2608
2.2	0.2020	0.2042	0.2077	0.2103
2.4	0.1592	0.1618	0.1659	0.1691
2.6	0.1252	0.1280	0.1323	0.1358
2.8	0.0984	0.1011	0.1053	0.1088
3.0	0.0072	0.0798	0.0838	0.0872
4.0	0.0229	0.0243	0.02662	0.02864
5.0	0.00677	0.00738	0.00844	0.00939
6.0	0.00201	0.00225	0.002676	0.003078
7.0	0.0059	0.000685	0.000845	0.001012

$z=0$. Since $F(a,b;c;z)|_{z=0}=1$, we obtain from (47) and (49)

$$a_2(\infty)=b_2(-\infty)e^{i\alpha t_{max}}=xe^{i\alpha t_{max}} . \quad (58)$$

It is interesting to notice that x is imaginary for any pulse shape of our family. Its square modulus gives the transition probability for system (1).

IV. EVALUATION OF TRANSITION PROBABILITIES AND DISCUSSION

Numerical evaluation of (56) requires the evaluation of the hypergeometric functions $\mathscr{F}_1$ and $\mathscr{F}_2$. This can be done by means of the series

$$F(a,b;c;z)=1+\frac{ab}{c}z+\frac{a(a+1)(b+1)b}{c(c+1)}\frac{z^2}{2!}+\cdots$$

which is absolutely convergent when $\text{Re}(c-a-b)>0$, a condition which is satisfied in both cases.

In Tables I and II we report numerical values of symmetrized pulses for $\lambda=-0.45$, -0.6, -0.8, -1.1 and 0.5, 0.8, 1.5, 2.5. The values of μ and β have been chosen in order to keep the area of the pulses equal to π and their peak amplitude equal to 1. Pulses with $\lambda<0$ display a sharper peak than the $\lambda=0$ pulse [the hyperbolic-secant pulse (HSP)] and a longer tail. For this reason they will be called Lorentzian-like pulses, although they do not approach the Lorentzian for any finite or infinite value of λ. On the other hand, pulses with $\lambda>0$ display a flat peak at their maximum but go to zero faster than the HSP. As a prototype of these kinds of pulses one may choose the Gaussian function normalized to have unit peak amplitude and an area equal to π. Thus, one may call these pulses Gaussian-like pulses.

In Fig. 2 the pulse shapes with $\lambda=-1.1$ and -0.6 are shown by the solid and dashed-dotted lines respectively; in the same figure the Lorentzian pulse is also displayed. Figure 3, on the other hand, shows the pulse shape with $\lambda=2.5$, and the Gaussian pulse is also reported for comparison. At exact resonance, $\alpha=0$, the only relevant feature of the pulse is its area; pulses having the same area determine the same transition probability. When $\alpha\neq0$ however, different segments of the pulse are sampled by the two-level system with different phases. One may expect that the Lorentzian-like pulses will behave in much the same way as the proper Lorentzian pulse. This has been checked and verified. In Table III we report transition amplitudes for pulses with $\lambda=-1.1$ and $\alpha=0.2$, at various values of the pulse area. We have chosen to report the quantity x of the formula (56). The phase factor $e^{i\alpha t_{max}}$ which appears there is a mere consequence of the fact that the pulse maximum is not located at $t=0$. This factor, however, does not enter the transition-probability formula, which in any case is given by $|x|^2$. For com-

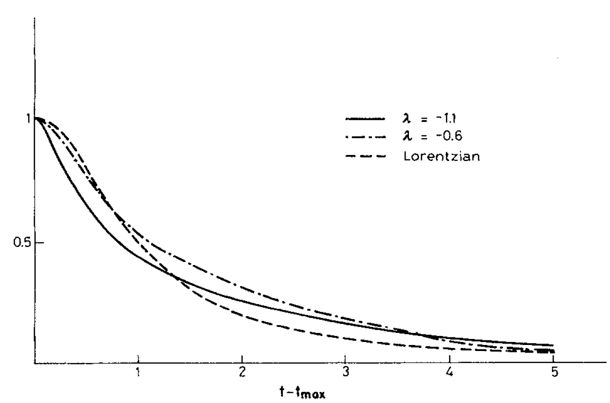

FIG. 2. Comparison between the line shapes of the Lorentzian pulse and two simulating symmetrized pulses with $\lambda=-0.6$ and -1.1. Areas and peak amplitudes are equal.

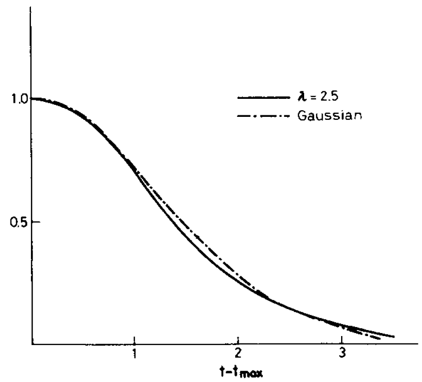

FIG. 3. Comparison between the line shapes of the Gaussian pulse and simulating symmetrized pulse with $\lambda=2.5$. Areas and peak amplitudes are equal.

parison, transition amplitudes are reported for the Lorentzian pulses with the same area and detuning. These have been evaluated as explained in Appendix B. We note the transition-probability amplitudes are imaginary, a feature which is common to any time-symmetric pulse with its maximum located at $t=0$.

Table III shows that transition amplitudes for the two cases are indeed similar, though not the same. In both cases, for instance, the first nonzero values of the area for which the transition amplitude vanishes is greater than π, and actually the two values do not differ more than a few percent. For a HSP this would be equal to π, independent of the determining α.

In Table IV transition amplitudes are reported for a symmetrized pulse with $\lambda=1.5$ (a Gaussian-like pulse) and, for comparison, a Gaussian pulse. In both cases, $\alpha=0.2$. Here again the results in the two cases are similar.

Whether these results may be useful in calculating transition probabilities for actual pulse shapes is a questionable matter. It is, however, much faster to fit an actual pulse shape to a symmetrized pulse and then to evaluate transition probabilities for the latter one than to perform direct numerical integrations over very long time intervals with the time pulse shape. It is also quite clear that a Lorentzian or a Gaussian pulse shape may often represent

TABLE III. Transition-probability amplitude for Lorentzian and Lorentzian-like pulses. Detuning α has been chosen to be 0.2.

Area $(\times\pi^{-1})$	Imx (Lorentzian)	Imx (Symmetric pulse, $\lambda=-1.1$)
0.2	−0.487	−0.502
0.4	−0.822	−0.820
0.6	−0.902	−0.840
0.8	−0.706	−0.558
1.0	−0.302	−0.085
1.2	0.176	+0.399
1.4	0.574	+0.711
1.6	0.770	+0.737

TABLE IV. Transition-probability amplitude for Gaussian and Gaussian-like pulse shapes. Detuning α has been chosen to be 0.2.

Area $(\times\pi^{-1})$	Imx (Gaussian pulse)	Imx (Gaussian-like pulse, $\lambda=5$)
0.2	−0.569	−0.566
0.4	−0.920	−0.915
0.6	−0.916	−0.913
0.8	−0.560	−0.561
1.0	+0.014	+0.006
1.2	+0.586	+0.574
1.4	+0.935	+0.923

only an approximation to the pulse shape one has to deal with.

On the other hand, the symmetrized pulses allow for a fast numerical evaluation of transition amplitudes, something which may be useful if many calculations have to be performed for different physical situations. The matter is also complicated by the fact that transition amplitudes appear to depend critically on the pulse shapes.

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APPENDIX A

The level amplitude a_1 satisfies the hypergeometric equation

$$z(1-z)a_1''+[c-(a+b+1)z]a_a'-aba_1=0\ , \qquad \text{(A1)}$$

where a, b, and c are parameters given by Eqs. (40). The general solution is

$$a_1(z)=A_1F(a,b;c;z)$$
$$+B_1z^{1-c}F(a-c+1,b-c+1;2-c;z)\ , \qquad \text{(A2)}$$

where A_1 and B_1 are constants.

The other level amplitude a_2 satisfies the same equation except that a, b, and c are replaced by their complex conjugates $\bar{a}$, $\bar{b}$, and $\bar{c}$. It is easy to see that changing α to $-\alpha$ is equivalent to exchanging a_1 and a_2 in (1). Hence a_2 is given by

$$a_2(z)=A_2F(\bar{a},\bar{b};\bar{c};z)$$
$$+\beta_2z^{1-\bar{c}}F(\bar{a}-\bar{c},\bar{b}-\bar{c}+1;2-\bar{c};z)\ . \qquad \text{(A3)}$$

We point out the relations

$$c-a=\bar{a}-\bar{c}+1,\quad c-b=\bar{b}-\bar{c}+1\ ,$$
$$c=1-\bar{c},\quad 1-c+a+b=\bar{c}-\bar{a}-\bar{b}\ . \qquad \text{(A4)}$$

The four coefficients A_1, A_2, B_1, and B_2 are not independent because (1) must be satisfied. Indeed, using (1), (24), (26), and (40), we can write

$$a_2(z)=z^c(1-z)^{1+a+b-c}\frac{i}{(-ab)^{1/2}}a_1'(z)\ . \quad \text{(A5)}$$

The derivative can be evaluated using the relations

$$\frac{d}{dz}F(a,b;c;z)=\frac{ab}{c}F(a+1,b+1;c+1;z)\ , \quad \text{(A6)}$$

$$\frac{d}{dz}[z^{c-1}F(a,b;c;z)]=(c-1)z^{c-2}F(a,b;c-1;z)\ , \quad \text{(A7)}$$

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z)\ . \quad \text{(A8)}$$

Equation (A5) gives us

$$a_1(z)=A_1\frac{(ab)^{1/2}}{c}z^{1-\bar{c}}F(\bar{a}-\bar{c}+1,\bar{b}-\bar{c}+1;2-\bar{c};z)$$
$$+B_1(1-c)\frac{1}{(ab)^{1/2}}F(\bar{a},\bar{b};\bar{c};z)\ . \quad \text{(A9)}$$

Comparing (A3) and (A9) we find

$$B_2=A_1\frac{(ab)^{1/2}}{c}=-\frac{i\beta}{1-\bar{c}}A_1\ ,$$
$$A_2=B_1\frac{1-c}{(ab)^{1/2}}=i\frac{\bar{c}}{\beta}B_1\ . \quad \text{(A10)}$$

APPENDIX B

Numerical integration of system (1), starting from its initial conditions

$$a_1=1\ ,\qquad a_2=0 \quad \text{(B1)}$$

at $t=-\infty$, may lead to numerical errors, since $t=-\infty$ is a singular point for the system. These difficulties can be avoided if we are dealing with time-symmetric pulse shapes such that

$$f(-t)=f(t)\ . \quad \text{(B2)}$$

Indeed, if (B2) is satisfied, then the solutions of the system (1) display several symmetries and we can evaluate the transition amplitudes by means of a single numerical integration from $t=0$ to $+\infty$. This can be done as follows. System (1) possesses two linearly independent solutions, the so-called fundamental solutions, namely,

$$a_1^{(1)}=\rho_1(t)\ ,\qquad a_2^{(1)}=\rho_2(t) \quad \text{(B3)}$$

and

$$a_1^{(2)}=\sigma_1(t)\ ,\qquad a_2^{(2)}=\sigma_2(t)\ , \quad \text{(B4)}$$

with

$$\rho_1(0)=1\ ,\qquad \rho_2(0)=0 \quad \text{(B5)}$$

and

$$\sigma_1(0)=0\ ,\qquad \sigma_2(0)=1\ . \quad \text{(B6)}$$

We can relate the ρ solution to the σ solution by noting that $-a_2^*, a_1^*$ also satisfy the system (1). Thus

$$\sigma_1(t)=-\rho_2^*(t)\ ,\qquad \sigma_2(t)=\rho_1^*(t)\ . \quad \text{(B7)}$$

This is true for any pulse shape. The general solution of system (1) is therefore given by

$$a_1(t)=\lambda_1\rho_1(t)+\lambda_2\sigma_1(t)\equiv\lambda_1\rho_1(t)-\lambda_2\rho_2^*(t)\ ,$$
$$a_2(t)=\lambda_1\rho_2(t)+\lambda_2\sigma_2(t)\equiv\lambda_1\rho_2(t)+\lambda_2\rho_1^*(t)\ , \quad \text{(B8)}$$

where ρ_1,ρ_2 satisfy the condition (B3). We can now use the symmetry properties (B2) to establish the relations

$$\rho_1(-t)=\rho_1^*(t)\ ,\qquad \rho_2(-t)=\rho_2^*(t) \quad \text{(B9)}$$

which, along with (B7), enable us to completely solve our problem in terms of the functions $\rho_1(t)$ and $\rho_2(t)$ to be known in the range $t=0-+\infty$.

Denoting by R_1,R_2 the (complex) values of $\rho_1(t)$ and $\rho_2(t)$ at $t=+\infty$, we have, using (B8) and (B9),

$$a_1(-\infty)=\lambda_1R_1^*-\lambda_2R_2\ ,\qquad a_2(-\infty)=\lambda_1R_2^*+\lambda_2R_1\ , \quad \text{(B10)}$$

and we choose λ_1,λ_2 in order to have $a_1(-\infty)=1$, $a_2(-\infty)=0$. This gives

$$\lambda_1=R_1\ ,\qquad \lambda_2=-R_2^* \quad \text{(B11)}$$

since $|\rho_1|^2+|\rho_2|^2=1$. Substituting (B11) into (B8) we find the asymptotic values of $a_1(+\infty)$ and $a_2(+\infty)$ as

$$a_1(\infty)=R_1^2+R_2^{*2}\ ,\qquad a_2(\infty)=R_1R_2-R_1^*R_2^*\ . \quad \text{(B12)}$$

Equations (B12) show that the asymptotic amplitude at $t=+\infty$ of the upper level a_2 is purely imaginary if a_1 is chosen to be 1 at $t=-\infty$.

We have numerically integrated system (1) for Lorentzian pulse shapes $f(t)=1/(1+t^2)$ and for Gaussian pulse shapes $f(t)=\exp(-t^2/\pi)$ with the detuning parameter α set equal to 0.2. Initial conditions have been chosen at $t=0$ as $a_1=1$ and $a_2=0$ (the ρ solution), and we have integrated the system up to values of t where the two amplitudes do not change appreciably. The step of integration has been chosen in such a way to keep the norm $|a_1|^2+|a_2|^2$ equal to unity within the required accuracy. Then the asymptotic values of $\rho_1(t)$ and $\rho_2(t)$ so obtained have been inserted into (B12) in order to evaluate $a_1(+\infty)$ and $a_2(+\infty)$. In the text the imaginary part of the transition amplitudes, rather than the transition probabilities, are reported.

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