Time-dependent invariants and the Feynman propagator

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A class of recently discussed time-dependent classical Lagrangians possessing invariants is considered from a quantum-mechanical point of view. Quantum mechanics is introduced directly through the Feynman propagator defined as a path integral involving the classical action. It is shown, without carrying out explicit path integration, that the propagator for these time-dependent problems is related to the propagator of associated time-independent problems. The expansion of the propagator in terms of the eigenfunctions of the invariant operator and the quantum superposition principle follow naturally in our scheme. The theory is applied to obtain explicitly exact propagators for some illustrative examples.

I. INTRODUCTION

There has been considerable interest, in recent years, in the existence of exact invariants for certain time-dependent systems.[1-7] These invariants have invoked attention partly because of their relation to certain pairs of nonlinear equations of motion called Ermakov systems[8-15] and partly because of their utility in solving a class of time-dependent quantum-mechanical problems.[3,4,16-18] A well-known example of a system possessing an exact invariant is the harmonic oscillator with a time-dependent frequency described by the equation of motion

$$\ddot{x}+\omega^2(t)x=0 \,. \tag{1.1}$$

Lewis[1] has shown that the quantity

$$I(t)=\tfrac{1}{2}[(\dot{x}\rho-x\dot{\rho})^2+\omega_0^2(x/\rho)^2] \tag{1.2}$$

is an invariant for the problem where $x(t)$ satisfies (1.1) and $\rho(t)$ obeys the auxiliary equation

$$\ddot{\rho}+\omega^2(t)\rho=\omega_0^2/\rho^3 \,. \tag{1.3}$$

The equations (1.1) and (1.3) together are known as Ermakov pairs.[8] The invariant of (1.2) was first derived by Ermakov[8] by eliminating $\omega^2(t)$ between these two equations. Incidently, Lewis[1] derived the invariant by assuming, *ab initio*, a quadratic form in x and p for I and evaluating the coefficients from the invariance condition.

An alternative derivation of the invariant (1.2) and the auxiliary equation (1.3) has been given by Lutzky.[19,20] This method is based on an application of Noether's theorem to the Lagrangian

$$L=\tfrac{1}{2}[\dot{x}^2-\omega^2(t)x^2] \,. \tag{1.4}$$

This approach has been greatly exploited by Ray and Reid[9-15] to investigate the possibility of adding very general potential terms to the basic Lagrangian (1.4) so that the resulting system admits an invariant. The form of the invariant and the auxiliary equation would thus be different in each case. A useful generalization of (1.4) is the Lagrangian[15]

$$L=\tfrac{1}{2}[\dot{x}^2-\omega^2(t)x^2]-\rho^{-2}F(x/\rho) \,, \tag{1.5}$$

where F is an arbitrary function of its argument while ρ satisfies Eq. (1.3). The equation of motion and the invariant I read as

$$\ddot{x}+\omega^2(t)x+\rho^{-3}\frac{\partial F}{\partial y}=0 \,, \tag{1.6}$$

$$I(t)=\tfrac{1}{2}[(\dot{x}\rho-x\dot{\rho})^2+\omega_0^2(x/\rho)^2]+F(x/\rho) \,, \tag{1.7}$$

where

$$y=x/\rho \,.$$

The importance of these invariants is classical mechanics and the associated nonlinear superposition law[12,21-23] has been adequately discussed. Further it turns out that the classical invariant also becomes the quantum invariant when the canonical momentum p is replaced by the quantum-mechanical operator $(\hbar/i)(\partial/\partial x)$ with the auxiliary function ρ remaining a c-number. Lewis[2] and Lewis and Riesenfeld[3] first exploited the invariant operators to solve quantum-mechanical problems. In particular, they[2,3] have derived a simple relation between eigenstates of $\hat{I}$ and solutions of the time-dependent Schrödinger equation and have applied it to the case of a quantal oscillator with time-dependent frequency. Recently, Hartley and Ray[16-18] have applied this technique to derive a quantum-mechanical superposition law for the system described by (1.5)—(1.7).

The aim of the present paper is somewhat different. We believe that given the classical Lagrangian, quantum-mechanical considerations may be more directly introduced via the Feynman propagator.[24] This has the added advantage that quantum superposition is already built into it. In this context, Khandekar and Lawande[25,26] have obtained Feynman propagators for some time-dependent Lagrangians possessing invariants by an explicit path-integration technique. Employing Lewis-Riesenfeld theory they have also shown that the propagators for such problems admit expansions in terms of the eigenfunctions of the invariant operator.

In the present paper, we show, without employing explicit path integration, that the Feynman propagator for time-dependent Lagrangians of the type (1.5) is related to the propagator of an associated time-independent problem. The latter propagator may be explicitly found in a number of instances as discussed in Sec. III. Moreover, the expansion of the propagator in terms of the eigenfunctions of the invariant operator and hence the quantum superposition principle follow readily. Similar considerations also apply at least formally to more general velocity-dependent Lagrangains admitting invariants. The quantum mechanics of such Lagrangians with arbitrary velocity dependence (beyond linear or quadratic dependence) may, however, lead to difficulties of physical interpretation.

II. FEYNMAN PROPAGATOR

A. Derivation of the propagator

We first obtain the Feynman propagator for the time-dependent Lagrangian (1.5). The propagator $K(x'',t'';x',t')$ is the quantum-mechanical amplitude for finding a particle at the position x'' at time t'' if the particle had been at x' at an earlier time t'. It is defined as the path integral[24]

$$K(x'',t'';x',t')=\int \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} L\,dt\right]\mathscr{D}x(t)\ , \quad (2.1)$$

where L is the classical Lagrangian of the particle while $\mathscr{D}x(t)$ is the usual Feynman path differential measure implying that integrations are over all possible particle paths starting at $x(t')=x'$ and terminating at $x(t'')=x''$.

Next, we use the auxiliary equation (1.3) to eliminate $\omega^2(t)$ in the Lagrangian (1.5), so that we have

$$L=\frac{d}{dt}\left[\frac{\dot{\rho}x^2}{2\rho}\right]+L_0\ , \quad (2.2)$$

where

$$L_0=\frac{1}{2}\rho^2\left[\frac{d}{dt}\left[\frac{x}{\rho}\right]\right]^2-\frac{\omega_0^2}{2}\rho^{-2}\left[\frac{x}{\rho}\right]^2 -\rho^{-2}F(x/\rho)\ . \quad (2.3)$$

When form (2.2) is inserted in (2.1), the propagator reads as

$$K(x'',t'';x',t')=K_0\exp\left[\frac{i}{2\hbar}\left[\frac{\dot{\rho}''x''^2}{\rho''}-\frac{\dot{\rho}'x'^2}{\rho'}\right]\right]\ , \quad (2.4)$$

where K_0 is the new propagator involving the new Lagrangian L_0:

$$K_0=\int \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} L_0 dt\right]\mathscr{D}x(t)\ . \quad (2.5)$$

In Eq. (2.4) and subsequently, the prime and double prime indicate that the quantities are evaluated at $t=t'$ and at $t=t''$, respectively. Next, we introduce a new parameter τ related to time t by

$$\tau(t)=\int^t \rho^{-2}(s)ds \quad (2.6)$$

so that the action integral in (2.4) takes the form

$$\int_{t'}^{t''} L_0 dt=\int_{\tau'}^{\tau''}\bar{L}_0 d\tau\ , \quad (2.7)$$

where the new Lagrangian $\bar{L}_0$ has the form

$$\bar{L}_0\equiv\bar{L}_0\left[y,\frac{dy}{d\tau}\right]=\frac{1}{2}\left[\frac{dy}{d\tau}\right]^2-\frac{\omega_0^2y^2}{2}-F(y)\ , \quad (2.8)$$

where $y=x/\rho$. It is important to note that the parameter τ would in turn induce a transformation in the path differential measure $\mathscr{D}x(t)$. Such a transformation in the path differential measure has been considered by Fujiwara.[27] We use this approach to show in Appendix A that the required transformation is

$$\mathscr{D}x(t)=\frac{1}{\sqrt{\rho'\rho''}}\mathscr{D}y(\tau)\ . \quad (2.9)$$

It therefore follows from (2.4)—(2.9) that the Feynman propagator may be written in the neat form

$$K(x'',t'';x',t')=\frac{1}{\sqrt{\rho'\rho''}}\left\{\exp\left[\frac{i}{2\hbar}\left[\frac{\dot{\rho}''x''^2}{\rho''}-\frac{\dot{\rho}'x'^2}{\rho'}\right]\right]\right\}K_0(y'',\tau'';y',\tau')\ , \quad (2.10)$$

where $y=x/\rho$ and τ is as in (2.6) and

$$K_0(y'',\tau'',y',\tau')=\int \exp\left[\frac{i}{\hbar}\int_{\tau'}^{\tau''}\bar{L}_0 d\tau\right]\mathscr{D}y(\tau) \quad (2.11)$$

is the propagator corresponding to the related Lagrangian $\bar{L}_0(y,dy/d\tau)$ which represents a harmonic oscillator with a constant frequency ω_0 with an additional potential $F(y)$.

The implication of this result is clear. The original time-dependent quantum-mechanical problem [posed through the classical Lagrangian (1.5)] is completely solved if the propagator for the related (time-independent) Lagrangian $\bar{L}_0(y,dy/d\tau)$ defined in (2.8) is obtained.

We mention that the above procedure can be applied to write the propagator for a more general Lagrangian of the form

$$L_a=\tfrac{1}{2}a(t)[\dot{x}^2-\omega^2(t)x^2]-P(x,t)\ . \quad (2.12)$$

Physically, $a(t)>0$ represents either a variable mass of the particle or a frictional force depending linearly on particle velocity.

Application of Noether's theorem to (2.12) yields that the admissible form for $P(x,t)$ is given by

$$P(x,t)=\rho^{-2}F\left[\frac{\sqrt{a}\,x}{\rho}\right]\ , \quad (2.13)$$

where F is an arbitrary function of its argument and $\rho(t)$ satisfies the equation

$$\ddot{\rho}+\Omega^2(t)\rho=\frac{\omega_0^2}{\rho^3}\,, \tag{2.14}$$

where

$$\Omega^2(t)=\omega^2(t)+\frac{1}{4}\left[\frac{\dot{a}}{a}\right]^2-\frac{1}{2}\left[\frac{\ddot{a}}{a}\right]. \tag{2.15}$$

The invariant I and the classical equation of motion are given by

$$I=\tfrac{1}{2}Z^2+\tfrac{1}{2}\omega_0^2Y^2+F\,, \tag{2.16}$$

$$\frac{d}{dt}(a\dot{x})+\omega^2(t)ax+\sqrt{a}\,\rho^{-3}\frac{\partial F}{\partial y}=0\,, \tag{2.17}$$

where the variables Y and Z are defined as

$$\begin{aligned} Y&=\frac{\sqrt{a}\,x}{\rho}\,,\\ Z&=\sqrt{a}\left[\dot{x}\rho-\left[\dot{\rho}-\frac{\dot{a}}{2a}\rho\right]x\right]\\ &=\rho^2\frac{d}{dt}(\sqrt{a}\,x/\rho)\,. \end{aligned} \tag{2.18}$$

Alternatively, one can see that the Lagrangian (2.12) and (2.13) is related to the one in (1.5). A simple transformation

$$x=\frac{X}{\sqrt{a}} \tag{2.19}$$

applied to (2.12) yields

$$L_a=L(X,\dot{X},t)-\frac{1}{4}\frac{d}{dt}\left[\frac{\dot{a}}{a}X^2\right], \tag{2.20a}$$

where

$$L(X,\dot{X},t)=\tfrac{1}{2}[\dot{X}^2-\Omega^2(t)X^2]-\frac{1}{\rho^2}F(X)\,. \tag{2.20b}$$

This shows that $L(X,\dot{X},t)$ has the same form as the Lagrangian of Eq. (1.5) and that the two Lagrangians L_a and L differ in form only by an additional total time derivative of a function. It is, therefore, apparent that the auxiliary equation (2.14) and the expression for the invariant (2.16) may be formally obtained from the corresponding expressions (1.3) and (1.7) by merely replacing $\omega^2(t)$ by $\Omega^2(t)$ and x by $\sqrt{a}\,x$ in them. We use this fact to derive the Feynman propagator for L_a. The propagator reads as

$$\begin{aligned} K_a(x'',t'';x',t')&=\int\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}L_a dt\right]\mathscr{D}x(t)\\ &=(a'a'')^{1/4}\left\{\exp\left[-\frac{i}{4\hbar}\left[\frac{\dot{a}''}{a''}X''^2-\frac{\dot{a}'}{a'}X'^2\right]\right]\right\}\int\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}L(X,\dot{X},t)dt\right]\mathscr{D}X(t)\,. \end{aligned} \tag{2.21}$$

Here the path differential measure $\mathscr{D}X(t)$ in (2.21) corresponds to the free particle normalization with mass of the particle taken as unity. On the other hand, the measure $\mathscr{D}x(t)$ involves a normalization with respect to a particle of mass $a(t)$. As shown in Appendix B this leads to a factor $(a'a'')^{1/4}$ when we transform from $\mathscr{D}x(t)$ to $\mathscr{D}X(t)$ by the substitution $X=\sqrt{a}\,x$. The latter integral represents the propagator $K(X'',t'';X',t')$ for the Langrangian $L(X,\dot{X},t)$ which we have already evaluated in (2.10) and (2.11). We therefore obtain

$$K_a(x'',t'';x',t')=\frac{1}{\sqrt{\sigma'\sigma''}}\exp\left[\frac{i}{2\hbar}\left[\frac{a''\dot{\sigma}''x''^2}{\sigma''}-\frac{a'\dot{\sigma}'x'^2}{\sigma'}\right]\right]K_0(Y'',\tau'';Y',\tau')\,, \tag{2.22}$$

where τ is as in (2.6) and

$$\sigma=\frac{\rho}{\sqrt{a}} \tag{2.23}$$

and K_0 is as defined in (2.11) with y replaced by the new variable $Y=\sqrt{a}\,x/\rho$ therein.

We may also mention that more general classical Lagrangians involving velocity-dependent potentials have been considered by Ray and Reid.[14,15] Although these velocity-dependent systems are of great interest in classical mechanics, their physical interpretation in the quantum context is difficult and requires a much deeper study. However, we believe that quantum-mechanical considerations for such systems may be best introduced through the Feynman path integral approach. For this reason, we include here the formal expression of the propagator for such a Lagrangian.

Consider, for example, the system described by the Lagrangian

$$\mathscr{L}=\frac{a(t)}{2}[\dot{x}^2-\omega^2(t)x^2]-\rho^{-2}F\left[\frac{\sqrt{a}\,x}{\rho},\rho^2\frac{d}{dt}\left[\frac{\sqrt{a}\,x}{\rho}\right]\right], \tag{2.24}$$

which is a slight generalization of the system considered by Ray and Reid.[14] The auxiliary equation has the same form as (2.14) while the classical equation of motion and the invariant have the form

$$\frac{d}{dt}\left[a\dot{x}-\frac{1}{\sigma}\frac{\partial F}{\partial Z}\right]+\omega^2(t)ax+\frac{1}{\rho^2\sigma}\frac{\partial F}{\partial Y}-\frac{\dot{\sigma}}{\sigma^2}\frac{\partial F}{\partial Z}=0\,, \tag{2.25}$$

$$I(t)=\tfrac{1}{2}Z^2+\tfrac{1}{2}\omega_0^2Y^2+F(Y,Z)-Z\frac{\partial F}{\partial Z}\,, \qquad (2.26)$$

where Y,Z,σ are as defined in (2.18) and (2.23).

Following the approach described above, it is easy to derive the propagator

$$\tilde{K}(x'',t'';x',t')=\frac{1}{\sqrt{\sigma'\sigma''}} \times\exp\left[\frac{i}{2\hbar}\left[\frac{a''\dot{\sigma}''x''^2}{\sigma''}-\frac{a'\dot{\sigma}''x'^2}{\sigma'}\right]\right] \times\tilde{K}_0(Y'',\tau'';Y',\tau')\,, \qquad (2.27)$$

where

$$\tilde{K}_0(Y'',\tau'';Y',\tau')=\int \exp\left[\frac{i}{\hbar}\int \mathscr{L}_0\left[Y,\frac{dY}{d\tau}\right]d\tau\right]\mathscr{D}Y(\tau) \qquad (2.28)$$

is the propagator corresponding to the reduced Lagrangian

$$\mathscr{L}_0=\frac{1}{2}\left[\frac{dY}{d\tau}\right]^2-\frac{\omega_0^2Y^2}{2}-F\left[Y,\frac{dY}{d\tau}\right]. \qquad (2.29)$$

We might mention that evaluation of the associated propagators K_0 or $\tilde{K}_0$ is far from simple in a general case for arbitrary F. Nevertheless, since the major part involves the harmonic oscillator Lagrangian for which an exact propagator is available it is possible to develop perturbation expansions.[24] Alternatively if the classical equation of motion following from the reduced Lagrangian $\bar{L}_0$ (or $\mathscr{L}_0$) is solved, it is possible to adopt a semiclassical expansion[28] of K_0 (or $\tilde{K}_0$) involving expansion of the action around the classical path.

As a final comment, we see that the propagator depends on the auxiliary function $\rho(t)$ which obeys a nonlinear equation. This nonlinear equation may be solved either analytically[29] or numerically for a given $\omega^2(t)$ with appropriate initial conditions.

B. Expansion of the propagator

In the above derivation of the propagator, the auxiliary equation for $\rho(t)$ was used to remove from the Lagrangian a term which is a total time derivative of a function. This term when inserted in the action integral $\int_{t'}^{t''} L\,dt$ yields the exponential term in the propagator. The remaining action is then path integrated in the new variables to obtain the propagator K_0. We now discuss the role played by the invariant I in our scheme.

Consider, for example, the propagator of Eq. (2.10). Here $K_0(y'',\tau'';y',\tau')$ is a Feynman propagator in its own right obtained by a path integration of an action corresponding to $\bar{L}_0(y,dy/d\tau)$ of (2.8). The classical Hamiltonian corresponding to $\bar{L}_0$ is obtained by introducing the canonical momentum $p_y=dy/d\tau$ in (2.8):

$$\bar{H}_0=\tfrac{1}{2}p_y^2+\frac{\omega_0^2y^2}{2}+F(y)\,. \qquad (2.30)$$

On the other hand, the invariant I of (1.7) when written in terms of the new variables $y=x/\rho$ and τ, is identical to $\bar{H}_0$. We denote this energylike constant of motion by I_0.

The corresponding quantum Hamiltonian and the invariant operator are obtained by writing $p_y=-i\hbar\partial/\partial y$ in (2.30):

$$\hat{H}_0=\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2}+\frac{\omega_0^2y^2}{2}+F(y)\right]\equiv\hat{I}_0\,. \qquad (2.31)$$

The propagator $K_0(y'',\tau'';y',\tau')$ then represents the Green's function of the Schrödinger equation[24]

$$i\hbar\frac{\partial\psi(y,\tau)}{\partial\tau}=\hat{H}_0\psi(y,\tau)\,. \qquad (2.32)$$

Thus if the associated stationary problem

$$\hat{H}_0\phi_n(y)=\lambda_n\phi_n(y) \qquad (2.33)$$

has a complete set of normalized eigenfunctions $\phi_n(y)$ corresponding to eigenvalues λ_n, the propagator K_0 has the expansion[24]

$$K_0(y'',\tau'';y',\tau')=\sum_n e^{-(i/\hbar)\lambda_n(\tau''-\tau')}\phi_n^*(y')\phi_n(y'')\,. \qquad (2.34)$$

Here the eigenvalues λ_n may be both discrete and continuous. Equation (2.34) thus implies in general a summation over discrete eigenvalues and integration over continuous eigenvalues.

Since, as noted above, the quantum invariant $\hat{I}_0$ expressed in the variables y and τ is identical to the quantum Hamiltonian $\hat{H}_0$, $\phi_n(y)$ are also the eigenfunctions of the invariant operator $\hat{I}_0$. Finally, inserting the original variables x and t, we see that the propagator (2.10) has the following expansion in terms of the eigenfunctions of the invariant operator:

$$K(x'',t'';x',t')=\frac{1}{\sqrt{\rho'\rho''}}\exp\left[\frac{i}{2\hbar}\left[\frac{\dot{\rho}''x''^2}{\rho''}-\frac{\dot{\rho}'x'^2}{\rho'}\right]\right]\sum_n\exp\left[-\frac{i}{\hbar}\lambda_n\int_{t'}^{t''}\frac{dt}{\rho^2}\right]\phi_n\left[\frac{x''}{\rho''}\right]\phi_n^*\left[\frac{x'}{\rho'}\right]. \qquad (2.35)$$

It may be interesting to compare the present Lagrangian approach with that of Hartley and Ray[16,17] based on Lewis and Riesenfeld[3] theory. For a quantal system characterized by a time-dependent Hamiltonian $\hat{H}(t)$ and an invariant $\hat{I}(t)$ the general solution of the time-dependent Schrödinger equation is given by[3]

$$\Psi(x,t)=\sum_n c_n e^{i\alpha_n(t)}\Psi_n(x,t)\,. \qquad (2.36)$$

Here $\Psi_n(x,t)$ are the normalized eigenfunctions of the invariant operator defined by

$$\hat{I}\Psi_n(x,t)=\lambda_n\Psi_n(x,t),\ \ \lambda_n=\text{const}\ . \tag{2.37}$$

c_n are constants while the time-dependent phases $\alpha_n(t)$ are to be determined from the equation

$$\hbar\frac{d\alpha_n}{dt}=\left\langle\Psi_n\left|i\hbar\frac{\partial}{\partial t}-\hat{H}\right|\Psi_n\right\rangle . \tag{2.38}$$

Considering the quantal Hamiltonian corresponding to the Lagrangian (1.5), Hartley and Ray[16,17] perform the unitary transformation

$$\Psi_n'=U\Psi_n,\ \ U=\exp\left[-\frac{i\dot{\rho}x^2}{2\hbar\rho}\right], \tag{2.39}$$

which results in the transformation of the invariant $\hat{I}$ to $\hat{I}'=U\hat{I}U^\dagger$. Expressed in the new variable $y=x/\rho$, the transformed invariant $\hat{I}'$ and the corresponding normalized eigenfunctions $\phi_n(y)=\sqrt{\rho}\Psi_n'(x,t)$ correspond to our $\hat{I}_0$ and $\phi_n(y)$ given in (2.31) and (2.33). Further, the unitary transformation and the auxilary equation are employed[16] to arrive at the simple form of the phases

$$\alpha_n(t)=-\frac{\lambda_n}{\hbar}\int^t\frac{dt}{\rho^2} \tag{2.40}$$

which appear naturally in our expansion (2.34) for K_0.

In contrast, in the Feynman propagator formulation, the steps which are essentially equivalent to above are carried out classically on the Lagrangian resulting in a transformation of the path differential measure. The quantum-mechanical superposition principle manifests itself in the reduced propagator K_0.

Similar considerations also apply to the propagator K_a of (2.22) or $\tilde{K}$ of (2.27) corresponding to the velocity-dependent Lagrangian. Note that apart from a proper physical interpretation, the quantum Hamiltonian $\hat{\mathscr{H}}_0$ corresponding to the reduced Lagrangian $\mathscr{L}_0(Y,dY/d\tau)$ of (2.29) (which now contains arbitrary dependence on velocity) is still a linear operator and the superposition principle holds.

III. APPLICATIONS

We now discuss applications of our formulas (2.10) and (2.22) to obtain explicitly exact propagators for some time-dependent problems. Consider first the time-dependent harmonic oscillator for which $F=0$. Here $K_0(y'',\tau'';y',\tau')$ simply corresponds to a harmonic oscillator with constant frequency ω_0. An expression for K_0 is readily found from Feynman and Hibbs[24] which when inserted in (2.10) yields

$$K(x'',t'';x',t')=\left[\frac{\omega_0}{2\pi i\hbar\rho'\rho''\sin\phi(t'',t')}\right]^{1/2}\exp\left[\frac{i}{2\hbar}\left[\frac{\dot{\rho}''x''^2}{\rho''}-\frac{\dot{\rho}'x'^2}{\rho'}\right]\right]$$
$$\times\exp\left\{\frac{i\omega_0}{2\hbar\sin\phi(t'',t')}\left[\left[\frac{x''^2}{\rho''^2}+\frac{x'^2}{\rho'^2}\right]\cos\phi(t'',t')-2\frac{x''x'}{\rho''\rho'}\right]\right\}, \tag{3.1}$$

where

$$\phi(t'',t')=\omega_0(\tau''-\tau')=\omega_0\int_{t'}^{t''}\frac{dt}{\rho^2}\ . \tag{3.2}$$

A somewhat nontrivial example is that of a time-dependent harmonic oscillator with an additional inverse quadratic potential g/x^2 where g is a constant ($g>-\hbar^2/8$). In this case

$$F=g\Big/\left[\frac{x}{\rho}\right]^2 \tag{3.3}$$

and the propagator $K_0(y'',\tau'';y',\tau')$ resembles the radial propagator of a three-dimensional oscillator corresponding to an angular momentum $\beta-\frac{1}{2}$ where $\beta=\frac{1}{2}(1+8g/\hbar^2)^{1/2}$. The latter has been obtained by Peak and Inomata[30] which when inserted in (2.10) results in the following expression:

$$K(x'',t'';x',t')=\left[\frac{\omega_0}{i\hbar\rho'\rho''\sin\phi(t'',t')}\right](x''x')^{1/2}\exp\left[\frac{i}{2\hbar}\left[\frac{\dot{\rho}''x''^2}{\rho''}-\frac{\dot{\rho}'x'^2}{\rho'}\right]\right]$$
$$\times\exp\left[\frac{i\omega_0}{2\hbar}\left[\frac{x''^2}{\rho''^2}+\frac{x'^2}{\rho'^2}\right]\cot\phi(t'',t')\right]I_\beta\left[\frac{\omega_0x''x'}{i\hbar\rho''\rho'}\csc\phi(t'',t')\right]\ \ (0<x',x''<\infty)\ . \tag{3.4}$$

The propagators (3.1) and (3.4) have been obtained previously by Khandekar and Lawande[25] by carrying out an explicit path integration within the framework of Feynman's polygonal paths approach. In contrast, the present scheme yields a "back of an envelope" calculation of the same propagators based on the known results for a time-dependent oscillator. It is clear that the expansions of these propagators obtained in Ref. 25 also follow directly from the expansions of the related time-independent propagators. Note that potential $-\rho^{-2}F$ in these two examples does not depend on the auxiliary function $\rho(t)$. In general, however, the perturbative potential depends on ρ and is tailored according to the auxiliary equation for $\rho(t)$.

Another interesting case, where the present approach is applicable is the problem of a time-dependent quantal oscillator with linear damping and a perturbative force $f(t)$. The Lagrangian for this case has the form

$$L=a(t)\left[\frac{\dot{x}^2}{2}-\frac{\omega^2(t)x^2}{2}+f(t)x\right]. \quad (3.5)$$

For the special case when $a(t)=e^{\gamma t}$ the invariant for this problem and the exact propagator has been obtained by Khandekar and Lawande.[26] Also, for $a(t)=\int^t e^{\gamma(t')}dt'$, the invariant has been derived recently by Ray and Reid.[23] We rederive here the propagator using our scheme. The auxiliary equation for ρ is given by (2.14). It is clear that the "reduced Lagrangian" takes the form

$$\bar{L}_0\left[Y,\frac{dY}{d\tau},\tau\right]=\frac{1}{2}\left[\left[\frac{dY}{d\tau}\right]^2-\omega_0^2Y^2\right]+H(\tau)Y, \quad (3.6)$$

where

$$Y=\sqrt{a}\,x/\rho=\frac{x}{\sigma},$$
$$H(\tau)=\rho^2G(t)=\rho^3\sqrt{a(t)}f(t). \quad (3.7)$$

The Lagrangian $\bar{L}_0$ corresponds to a forced oscillator of constant frequency and force function $H(\tau)$. Note that this reduced Lagrangian $\bar{L}_0$ is not entirely time-independent because of the time-dependent force function $H(\tau)$. Although, it does not fall strictly in the general scheme of Sec. II, there is no difficulty of explicit evaluation. In fact, the propagator for this problem is available[24] and may be readily used to obtain

$$K(x'',t'',x',t')=\left[\frac{\omega_0}{2\pi i\hbar\sigma''\sigma'\sin\phi(t'',t')}\right]^{1/2}\exp\left[\frac{i}{2\hbar}\left[\frac{a''\dot{\sigma}''x''^2}{\sigma''}-\frac{a'\dot{\sigma}'x'^2}{\sigma'}\right]\right]$$
$$\times\exp\left\{\frac{i\omega_0}{2\hbar\sin\phi(t'',t')}\left[\left[\frac{x''^2}{\sigma''^2}+\frac{x'^2}{\sigma'^2}\right]\cos\phi(t'',t')-2\frac{x''x'}{\sigma''\sigma'}\right.\right.$$
$$+\frac{2x''}{\omega_0\sigma''}\int_{t'}^{t''}G(t)\sin\phi(t,t')dt+\frac{2x'}{\omega_0\sigma'}\int_{t'}^{t''}G(t)\sin\phi(t'',t)dt$$
$$\left.\left.-\frac{2}{\omega_0^2}\int_{t'}^{t''}\int_{t'}^{t}G(t)G(s)\sin\phi(t'',t)\sin\phi(s,t')ds\,dt\right]\right\} \quad (3.8)$$

where

$$G(t)=\sqrt{a(t)}f(t)\rho(t). \quad (3.9)$$

This expression agrees with the one derived by Khandekar and Lawande[26,31] by an explicit path integration of the Lagrangian (3.5) when $a=e^{\gamma t}$.

We may also add that the problem of a harmonically bound charged particle in an axial time-dependent magnetic field (with harmonic frequency varying with time) which has been considered by Lewis and Riesenfeld[3] also possesses a time-dependent invariant. The present technique is readily applicable even though the problem is two dimensional (planar motion). The propagator can be evaluated exactly with the knowledge of the standard propagator for a harmonic oscillator of a constant frequency.

APPENDIX A

In this appendix we outline Fujiwara's arguments to study how the transformation

$$d\tau=\frac{dt}{\rho^2} \quad (A1)$$

[cf. (2.6) in the text] induces a transformation in the path differential measure $\mathscr{D}x(t)$.

We recall that a finite approximation of the path integral involves a partitioning of the time interval into N subintervals of lengths $\Delta t_k=t_{k+1}-t_k$ $(k=0,1,\dots,N-1,\ t_0=t',t_N=t'')$ and a discretization of a path $x(t)$ by $x_k=x(t_k)$, $x_0=x(t')=x'$, $x_N=x(t'')=x''$. The path differential measure corresponds to the usual free particle normalization and has the form

$$\mathscr{D}x(t)\to(2\pi i\hbar\epsilon)^{-N/2}\prod_{k=1}^{N-1}dx_k \quad (A2)$$

where for simplicity, we have assumed subintervals of equal length ϵ.

Now for the new parameter τ with the abbreviations $\beta=1/\rho$, $\beta_k=\beta(t_k)$, we can write

$$\Delta\tau_k=\tau(t_{k+1})-\tau(t_k)=\int_{t_k}^{t_k+\epsilon}ds\,\beta^2(s)$$
$$=\epsilon\beta_k^2+\epsilon^2\beta_k\dot{\beta}_k+O(\epsilon^3)$$
$$=\epsilon\beta_k[\beta_k+\epsilon\dot{\beta}_k+O(\epsilon^2)]$$
$$=\epsilon\beta_k\beta_{k+1}[1+O(\epsilon^2)]. \quad (A3)$$

Hence it follows that

$$(2\pi i\hbar\epsilon)^{-N/2}=\prod_{k=0}^{N-1}\left[\left[\frac{\beta_k\beta_{k+1}}{2\pi i\hbar\Delta\tau_k}\right]^{1/2}[1+O(\epsilon^2)]\right]$$
$$=[1+O(\epsilon)]\left[\frac{\beta_N\beta_0}{2\pi i\hbar\Delta\tau_0}\right]^{1/2}$$
$$\times\prod_{k=1}^{N-1}\beta_k\left[\frac{1}{2\pi i\hbar\Delta\tau_k}\right]^{1/2} . \quad \text{(A4)}$$

Inserting $\beta_k=1/\rho_k$ in (A4) we can write

$$(2\pi i\hbar\epsilon)^{-N/2}\prod_{k=1}^{N-1}dx_k=\frac{1}{\sqrt{\rho_N\rho_0}}[1+O(\epsilon)]$$
$$\times\prod_{k=0}^{N-1}(2\pi i\hbar\Delta\tau_k)^{-1/2}$$
$$\times\prod_{k=1}^{N-1}d\left[\frac{x_k}{\rho_k}\right] \quad \text{(A5)}$$

which in the limit as $N\to\infty$ implies that

$$\mathscr{D}x(t)=\frac{1}{\sqrt{\rho''\rho'}}\mathscr{D}y(\tau) \quad \text{(A6)}$$

where $y=x/\rho$, $\rho''=\rho(t'')$, $\rho'=\rho(t')$. The equation (A6) is the same as (2.9) in the text.

APPENDIX B

Here, we show how the free-particle normalization used in (A2) alters when the mass $a(t)$ is varying with time. This normalization condition is

$$\int dx''\int\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}a(t)\frac{\dot{x}^2}{2}dt\right]\mathscr{D}x(t)=1 . \quad \text{(B1)}$$

In the Nth approximation of the path integral on the left-hand side of (B1), we use discretization with the measure

$$\mathscr{D}x(t)\to A_N\prod_{k=1}^{N-1}dx_k , \quad \text{(B2)}$$

where A_N is to be determined from condition (B1). Note that when $a(t)=1$, $A_N=(2\pi i\hbar\epsilon)^{-N/2}$ and the path differential measure is the same as in (A2). With this discretization the condition (B1) may be written as

$$\lim_{N\to\infty}I_N=1 , \quad \text{(B3)}$$

where

$$I_N=A_N\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[\frac{i}{\hbar}\sum_{k=0}^{N-1}a_k\frac{(x_{k+1}-x_k)^2}{\epsilon}\right]\prod_{k=1}^{N}dx_k$$
$$\epsilon=t_{k+1}-t_k \quad (k=0,1,\ldots,N-1) . \quad \text{(B4)}$$

If we introduce the transformation

$$ds=\frac{dt}{a(t)} \quad \text{(B5)}$$

then using the argument outlined in Appendix A above leading to the relation (A4) we have the identity

$$\frac{1+O(\epsilon)}{(2\pi i\hbar\epsilon)^{-N/2}}\frac{1}{(a_0a_N)^{1/4}}\left[\prod_{k=1}^{N-1}a_k^{-1/2}\right]\prod_{k=0}^{N-1}(2\pi i\hbar\Delta s_k)^{-1/2}=1 \quad (\Delta s_k=s_{k+1}-s_k) . \quad \text{(B6)}$$

We may therefore write

$$I_N=\frac{[1+O(\epsilon)]A_N}{(2\pi i\hbar\epsilon)^{-N/2}}\left[\prod_{k=1}^{N-1}a_k^{-1/2}\right]\frac{1}{(a_0a_N)^{1/4}}J_N , \quad \text{(B7)}$$

where

$$J_N=\prod_{k=0}^{N-1}(2\pi i\hbar\Delta s_k)^{-1/2}\int\cdots\int\exp\left[\frac{i}{\hbar}\sum_{k=0}^{N-1}(x_{k+1}-x_k)^2/\Delta s_k\right]\prod_{k=1}^{N}dx_k . \quad \text{(B8)}$$

It is clear that J_N has the form of the usual free particle normalization integral (with particle mass unity). Hence

$$\lim_{N\to\infty}J_N=1 \quad \text{(B9)}$$

and it follows from (B7) that

$$A_N=(2\pi i\hbar\epsilon)^{-N/2}(a_0a_N)^{1/4}\prod_{k=1}^{N-1}a_k^{1/2} . \quad \text{(B10)}$$

Hence

$$A_N\prod_{k=1}^{N-1}dx_k=(a_0a_N)^{1/4}(2\pi i\hbar\epsilon)^{-N/2}\prod_{k=1}^{N-1}dX_k . \quad \text{(B11)}$$

which in the limit $N\to\infty$ leads to

$$\mathscr{D}x(t)=(a'a'')^{1/4}\mathscr{D}X(t) \quad \text{(B12)}$$

with $X(t)=\sqrt{a(t)}x$. The transformation (B12) is used in Eq. (2.21) in the text.

[1]H. R. Lewis, Jr., Phys. Rev. Lett. 18, 510 (1967).
[2]H. R. Lewis, Jr., J. Math. Phys. 9, 1976 (1968).
[3]H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. 10, 1458 (1969).
[4]V. V. Dodonov, I. A. Malkin, and V. J. Manko, Physica (Utrecht) 72, 597 (1974).
[5]N. J. Gunther and P. G. L. Leach, J. Math. Phys. 18, 572 (1977).
[6]P. G. L. Leach, J. Math. Phys. 18, 1608 1902 (1977).
[7]W. Sarlet, J. Phys. A 11, 843 (1978).
[8]V. P. Ermakov, Univ. Izv. Kiev 20, 1 (1880).
[9]J. R. Ray and J. L. Reid, Phys. Lett. 71A, 317 (1979).
[10]J. R. Ray and J. L. Reid, J. Math. Phys. 20, 2054 (1979).
[11]J. R. Ray and J. L. Reid, Phys. Lett. 74A, 23 (1979).
[12]J. L. Reid and J. R. Ray, J. Math. Phys. 21, 1583 (1980).
[13]J. R. Ray, Phys. Lett. 78A, 4 (1980).
[14]J. R. Ray and J. L. Reid, J. Math. Phys. 22, 91 (1981).
[15]J. R. Ray, Prog. Theor. Phys. 65, 877 (1981).
[16]J. G. Hartley and J. R. Ray, Phys. Rev. A 24, 2873 (1981).
[17]J. R. Ray and J. G. Hartley, Phys. Lett. 88A, 125 (1982).
[18]J. G. Hartley and J. R. Ray, Phys. Rev. A 25, 2388 (1982).
[19]M. Lutzky, Phys. Lett. 68A, 3 (1978).
[20]M. Lutzky, Phys. Lett. 78A, 301 (1980).
[21]M. Lutzky, J. Math. Phys. 22, 2504 (1981).
[22]W. Sarlet and J. R. Ray, J. Math. Phys. 26, 1042 (1982).
[23]J. R. Ray and J. L. Reid, J. Math. Phys. 23, 1042 (1982).
[24]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[25]D. C. Khandekar and S. V. Lawande, J. Math. Phys. 16, 384 (1975).
[26]D. C. Khandekar and S. V. Lawande, J. Math. Phys. 20, 1870 (1979).
[27]I. Fujiwara, in *Many Body Problem,* edited by A. Cruz and T. W. Preist (Plenum, New York, 1969), p. 161.
[28]C. Dewitt, Ann. Phys. 97, 367 (1976).
[29]E. Pinney, Proc. Am. Math. Soc. 1, 681 (1950).
[30]D. Peak and A. Inomota, J. Math. Phys. 10, 1422 (1969).
[31]Note, however, that there are some misprints in Ref. 26. In particular, in the expression (80) for the propagator, the factors $e^{\gamma t''/2}$ and $e^{\gamma t'/2}$ associated with y' and y'' terms should not be there.