Transport phenomena in a completely ionized gas with large temperature gradients

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A solution to the Boltzmann equation is found that extends to large gradients and fields the classical Chapman-Enskog approximation developed by Spitzer and collaborators for electron transport in a fully ionized gas. The extended solution is used to calculate correction factors to the classical transport coefficients of a nonuniform plasma that depend on the temperature-gradient scale length as a parameter. These factors lead to inherently flux-limited heat flow with values in close agreement with Monte Carlo and numerical Fokker-Planck calculations.

I. INTRODUCTION

The Boltzmann equation with a Fokker-Planck collision term was solved in the first-order Chapman-Enskog approximation[1] by Spitzer and collaborators[2] to obtain the electrical and thermal transport coefficients of a completely ionized gas. The data from recent laser plasma experiments, however, can be reproduced in simulations only if a large reduction of the classical value of thermal conductivity is assumed.[3–7] Since thermal energy is transported through a region of large temperature and density gradients in these experiments, it is likely that the conditions for validity of the Chapman-Enskog method are violated. In this paper, a technique for extending Spitzer's calculation to large gradients and strong fields is developed. This technique is then used to calculate correction factors for the classical transport coefficients that depend on the temperature-gradient scale length as a parameter.

The correction factors are based on a solution to the Boltzmann equation that contains both diffusion and streaming components and thus yields inherently flux-limited transport. It is a universal practice in numerical computations to impose an external flux limit such as $q \le f_c q_0$, where $q_0 = nkT(kT/m)^{1/2}$, so that in large gradients the diffusion flux does not exceed the value for streaming particles in a neutral gas. The correct choice of the parameter f_c in a plasma has never been precisely defined, but heuristic arguments suggest values in the range $0.2 < f_c < 1$.[8] The technique described in this paper leads to naturally flux-limited transport and avoids the need for an *ad hoc* flux limiter.

The correction factors derived from the transport solution yield results in close agreement with numerical Fokker-Planck and Monte Carlo calculations. The technique described here allows diffusion expressions corrected for large gradients to be used in place of these more computationally difficult procedures, and the method can be extended to a multigroup treatment where required.

II. FORMULATION OF THE PROBLEM

The transport of electrons mediated by electron-electron and electron-ion collisions will be considered in one space dimension of a fully ionized gas. The electron distribution function $f(t,r,v,\mu)$ obeys the Boltzman equation

$$\frac{\partial f}{\partial t}+v\mu\frac{\partial f}{\partial r}-\frac{eE}{mv}\left[\mu v\frac{\partial f}{\partial v}+(1-\mu^2)\frac{\partial f}{\partial \mu}\right]=-K(f,f)-K(f,f_z)\ , \qquad (1)$$

where t is the time, r the spatial coordinate, v the magnitude of the electron velocity, μ the direction cosine with respect to the r axis, E the magnitude of the electric field in the r direction, and e and m are the charge and mass of the electron. Magnetic fields are assumed absent.

The Fokker-Planck collision term for electron-electron scattering is[2]

$$K(f,f)=\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2 f\left[\langle\Delta_\xi\rangle+\frac{1}{v}\langle\Delta_\eta^2\rangle-\frac{1}{2v^2}\frac{\partial}{\partial v}(v^2\langle\Delta_\xi^2\rangle)\right]\right]+\frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\left[f\sin\theta\left[\langle\Delta_\eta\rangle-\frac{1}{2v}\frac{\partial}{\partial\theta}\langle\Delta_\eta^2\rangle\right]\right]$$
$$-\frac{1}{2v^2}\frac{\partial}{\partial v}\left[v^2\langle\Delta_\xi^2\rangle\frac{\partial f}{\partial v}\right]-\frac{1}{2v^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\langle\Delta_\eta^2\rangle\frac{\partial f}{\partial\theta}\right]-\frac{1}{v^2\sin\theta}\frac{\partial}{\partial\theta}\left[f\sin\theta\langle\Delta_\xi\Delta_\eta\rangle+\frac{\partial}{\partial v}(fv\sin\theta\langle\Delta_\xi\Delta_\eta\rangle)\right], \qquad (2)$$

and for electrons scattered by stationary ions of charge Z,

$$K(f,f_z)=-\frac{1}{2v^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\langle\Delta_{\eta,z}^2\rangle\frac{\partial f}{\partial\theta}\right]. \qquad (3)$$

These expressions are based on spherical coordinates in velocity space oriented along the direction of $\vec{E}$. The rectangular velocity shifts averaged over all collisions, $\langle\Delta_\xi\rangle$, $\langle\Delta_\eta\rangle$, $\langle\Delta_\xi^2\rangle$, $\langle\Delta_\eta^2\rangle$, and $\langle\Delta_\xi\Delta_\eta\rangle$, are determined from Chandrasekhar's analysis of binary encounters as extended by Spitzer and others.[2,9,10]

Once the Boltzmann equation (1) is solved for f, the electrical current j and heat flux q in the r direction are found from

$$j=-e\int f\mu v\,d^3v\ ,\qquad q=\frac{m}{2}\int f\mu v^3 d^3v\ . \tag{4}$$

The problem is to obtain the distribution function f as a solution to Eq. (1) in terms of the temperature and density gradients and electric fields that may be present in the plasma.

III. FIRST-ORDER CHAPMAN-ENSKOG APPROXIMATION

The first-order Chapman-Enskog method can be applied to Eq. (1) in the case of slow time variations, small gradients, and weak fields by assuming a solution of the form $f=f_0(1+D\mu)$ and evaluating the small terms on the left-hand side with the equilibrium distribution f_0. Since $K(f,f)\simeq K(f_0,f_1)+K(f_1,f_0)$, Eq. (1) can then be solved for $f_1=f_0 D\mu$.

In the steady state Eq. (1) becomes

$$\left[\frac{eE'}{kT}+(x^2-2.5)\frac{1}{T}\frac{\partial T}{\partial r}\right]f_0\mu v=-K(f,f)-K(f,f_z)\ , \tag{5}$$

where

$$f_0=n_e\left[\frac{m}{2\pi kT}\right]^{3/2}e^{-x^2}\ ,\qquad x^2=\frac{mv^2}{2kT}\ . \tag{6}$$

In this expression, the pressure-gradient term has been absorbed into E' and is carried through as a field component,

$$E'=E+\frac{1}{en_e}\frac{\partial P_e}{\partial r}\ ,$$

where T is the electron temperature, n_e the electron number density, and $P_e=n_e kT$. For a distribution function of the form $f=f_0(1+D\mu)$, the ion scattering term given by Eq. (3) becomes[2]

$$K(f,f_z)=\frac{A_D}{2v^3}f_0 D\mu=\frac{v}{\lambda}(f-f_0)\ , \tag{7}$$

where $A_D=8\pi e^4 Z n_e m^{-2}\ln\Lambda$, and $\lambda=\lambda_s x^4$ is the mean free path for momentum-exchanging collisions, with

$$\lambda_s=\frac{(kT)^2}{\pi e^4 Z n_e \ln\Lambda}\ . \tag{8}$$

An important special case of Eq. (5) is that of the Lorentz plasma where electrons scatter off of fixed ions and electron-electron interactions are neglected. With Eq. (7) for the scattering term, Eq. (5) can be solved immediately to give

$$f=f_0\left[1-\lambda_s x^4\left[\frac{eE'}{kT}+(x^2-2.5)\frac{1}{T}\frac{\partial T}{\partial r}\right]\mu\right]\ . \tag{9}$$

The form of Eq. (9) reveals the limitations of the Chapman-Enskog solution. As long as the gradients and fields are small, f goes negative (unphysically) at velocities beyond the range of physical interest. For gradients such that the scale length $L=T(\partial T/\partial r)^{-1}$ is smaller than the scattering mean free path $\lambda_s x^4$ over a significant portion of the distribution function, the approximation fails completely.[6] The method also fails for fields strong enough that an electron gains energy on the order of kT over one scattering length.

For a distribution function of the form $f=f_0(1+D\mu)$, the electron scattering term given by Eq. (2) becomes[2]

$$K(f,f)=-\frac{v}{\lambda_s x^4}\frac{\Lambda}{2Z}(D''+PD'+Q_1 D-S_1)f_0\mu\ , \tag{10}$$

where

$$P(x)=-2x-\frac{1}{x}+\frac{2x^2}{\Lambda}\Phi'(x)\ ,$$

$$Q_1(x)=\frac{1}{x^2}-2\frac{\Phi(x)-2x^3\Phi'(x)}{\Lambda}\ ,$$

$$S_1(x)=\frac{16}{3\pi^{1/2}\Lambda}[xI_3(x)-1.2xI_5(x)-x^4 I_0(x)(1-1.2x^2)]-\frac{2}{\Lambda}\lambda_s x^4\frac{eE'}{kT}(1-1.2x^2)\ ,$$

$$I_n(x)=\int_0^x y^n D(y)e^{-y^2}dy\ ,$$

$$\Phi(x)=\frac{2}{\pi^{1/2}}\int_0^x e^{-y^2}dy\ ,$$

$$\Lambda=\Phi(x)-x\Phi'(x)\ ,$$

and the primes on D and Φ indicate derivatives with respect to x. When the scattering terms given by Eqs. (7) and (10) are added to Eq. (5), a differential equation for D is obtained,

$$D''+PD'+\left[Q_1-\frac{2Z}{\Lambda}\right]D=R_1+S_1\ , \tag{11}$$

where

$$R_1(x)=\lambda_s x^4\left[\frac{eE'}{kT}+(x^2-2.5)\frac{1}{T}\frac{\partial T}{\partial r}\right]\frac{2Z}{\Lambda}\ .$$

Equation (11) was solved numerically in Ref. 2 with D expressed in terms of two factors (ZD_E/A) and (ZD_T/B) which represent the effects on the distribution function of the electric field and the temperature gradient. The solution to Eq. (5) becomes

$$f=f_0\left\{1-\lambda_s\left[\left[\frac{ZD_E}{A}\right]\frac{eE'}{kT}-2\left[\frac{ZD_T}{B}\right]\frac{1}{T}\frac{\partial T}{\partial r}\right]\mu\right\}\ , \tag{12}$$

where (ZD_E/A) and (ZD_T/B) are tabulated as functions of x in Ref. 2. When $Z\to\infty$, $(ZD_E/A)\to x^4$, $2(ZD_T/B)\to x^4(2.5-x^2)$, and Eq. (12) reduces to the form for a Lorentz plasma. Equation (12) fails as a solution in the same way as Eq. (9) in the presence of large gradients and strong fields.

A property of the first-order Chapman-Enskog solution that will be important in what follows is that the right-hand side of Eq. (5) is a homogeneous function of degree 1 in f_1. This means that the Fokker-Planck collision terms reduce to a relaxation form, $K=-(f-f_0)/\tau$. In the Lorentz plasma, the scatterers are defined independently of D, and τ is given explicitly by Eq. (7). When electron scattering is added, the relaxation time depends on D which must be found from Eq. (11). In this case the right-hand side of Eq. (5) can be written

$$-K(f,f)-K(f,f_z)=\frac{v}{\lambda_s x^4}\frac{\Lambda}{2Z}\frac{R_1}{D}(f-f_0)\ , \qquad (13)$$

where the effective scattering length is absorbed in the form of D. It is apparent from Eq. (13) that the Fokker-Planck collision term, although rewritten in relaxation form, differs from the relaxation approximations considered by Krook.[11]

IV. EXTENSION TO LARGE GRADIENTS AND FIELDS

As observed in Sec. III, the Chapman-Enskog method is effective in conditions where the scattering mean free paths are short and transport proceeds by diffusion. However, in a region $L=T(\partial T/\partial r)^{-1}$ where the temperature gradient is large, there will be a significant number of long mean-free-path particles that are freely streaming. In this section, a method originally developed for radiation transport[12] is adapted to a solution of the Boltzmann equation that goes over from the classical Chapman-Enskog form at low particle energies to the streaming form at energies for which the electron mean free path exceeds either the temperature-gradient scale length or the distance over which an electron gains energy kT from the electric field. The electric fields are assumed smaller than the critical value for electron runaway.[13]

A. Lorentz plasma

To more clearly illustrate the principles involved, the method is developed first for the Lorentz plasma and then extended to the more complex problem considered by Spitzer.[2] The basic equation follows from Eq. (1), with the scattering term given by Eq. (7):

$$\frac{\partial f}{\partial t}+v\mu\frac{\partial f}{\partial r}-\frac{eE}{mv}\left[\mu v\frac{\partial f}{\partial v}+(1-\mu^2)\frac{\partial f}{\partial \mu}\right]=-\frac{v}{\lambda}(f-f_0)\ . \qquad (14)$$

The right-hand side is the Fokker-Planck collision term in the short mean-free-path limit where $f\simeq f_0(1+D\mu)$ and is also correct in the long mean-free-path limit where it approaches zero. The assumption is made that the relaxation form of the collision term with the relaxation time defined in terms of the scattering length holds approximately between these two limits.

The second major assumption, which is also the key assumption in the method of Ref. 12, is that the angular dependence is weak, and the distribution function can be expressed in the separable form

$$f=I(t,r,v)\psi(t,r,v,\mu)\ , \qquad (15)$$

where ψ is a slowly varying function normalized according to

$$2\pi\int_{-1}^{1}\psi\,d\mu=\alpha(t,r,v)\ . \qquad (16)$$

This approximation is valid in the limiting cases of long and short scattering mean free paths (unidirectional and isotropic distributions, respectively) and, like the first major assumption, is assumed to be sufficient inbetween. The reduced distribution function I and particle flux density J, defined by

$$\alpha I(t,r,v)=2\pi\int_{-1}^{1}f(t,r,v,\mu)d\mu\ ,$$
$$J(t,r,t)=2\pi v\int_{-1}^{1}f(t,r,v,\mu)\mu\,d\mu\ ,$$

obey the equation

$$\frac{\partial}{\partial t}(I\alpha)+\frac{\partial J}{\partial r}-\frac{eE}{mv^2}\frac{\partial}{\partial v}(vJ)=-\frac{v}{\lambda}(I\alpha-I_0)\ , \qquad (17)$$

where $I_0=4\pi f_0$. It will be convenient also to define a normalized particle flux F, where $J=vI\alpha F$, with

$$\alpha F=2\pi\int_{-1}^{1}\psi\mu\,d\mu\ . \qquad (18)$$

It will be apparent below that the definition Eq. (15) leads to a separation of the μ variable with α as the associated integration constant. For convenience of notation the parameter α, which is determined by the boundary condition on the particle flux J, will be absorbed into the function I (i.e., $I\alpha\to I$).

When the function f given by Eq. (15) is substituted into Eq. (14) one obtains

$$\psi\left[\frac{\partial I}{\partial t}+\frac{v}{\lambda}I+v\mu\frac{\partial I}{\partial r}-\frac{eE}{mv}v\mu\frac{\partial I}{\partial v}\right]+I\left[\frac{\partial \psi}{\partial t}+v\mu\frac{\partial \psi}{\partial r}-\frac{e}{mv}E\left[v\mu\frac{\partial \psi}{\partial v}+(1-\mu^2)\frac{\partial \psi}{\partial \mu}\right]\right]=\frac{v}{\lambda}\frac{I_0}{4\pi}\ . \qquad (19)$$

Since ψ is assumed to be a slowly varying function, the derivatives of ψ in Eq. (19) can be neglected compared to the other terms. Integration of Eq. (19) over the complete solid angle then gives

$$\frac{1}{v}\frac{\partial I}{\partial t}+\frac{1}{\lambda}I+F\left[\frac{\partial I}{\partial r}-\frac{eE}{mv}\frac{\partial I}{\partial v}\right]=\frac{1}{\lambda}\frac{I_0}{4\pi}\ ,$$

which can be used to evaluate the first two terms in Eq. (19) to yield

$$\psi\left[(F-\mu)\frac{\lambda}{I_0}\left[-\frac{\partial I}{\partial r}+\frac{eE}{kT}\frac{\partial I}{\partial v}\right]+1\right]=\frac{1}{4\pi}\ .$$

With the definition

$$R=-\frac{\lambda}{I_0}\left[\frac{\partial I}{\partial r}-\frac{eE}{kT}\frac{\partial I}{\partial v}\right], \tag{20}$$

ψ can be written

$$\psi=\frac{1}{4\pi}\frac{1}{G-\mu R},$$

where the function $G=1+FR$ is required by the normalization condition Eq. (16) to be $G=R\coth R$. This gives for the distribution function

$$f=\frac{I}{4\pi}\frac{1}{R(\coth R-\mu)} \tag{21}$$

and associated particle flux $J=vIF$, where

$$F=(R\coth R-1)R^{-1}. \tag{22}$$

The parameter α must be determined by the boundary condition on the flux J. In the long mean-free-path limit, $R\rightarrow\infty$, $\coth R\rightarrow 1$, and the particle flux is $J=vI$, corresponding to streaming from a distant point source I. For streaming from an infinite planar isotropic source at temperature T_0, the particle flux is $J=vI_0/2$. For large gradients and fields, $\lambda\gg L$, $R\rightarrow\infty$, and the distribution function Eq. (21) becomes $f\simeq(I/4\pi)[\,|R|(1-\mu)]^{-1}$, where the direction of the particle flow ($\mu=1$) is determined by the sign of R. This distribution has the properties of a δ function which properly describes streaming from a point source. At a position $r\ll\lambda$ from an infinite planar isotropic source, the boundary dominates, and λ must be replaced by r for $\mu\geq 0$. Under these conditions $R\rightarrow 0$ for $\mu\geq 0$, $R\rightarrow\infty$ for $\mu<0$, and Eq. (21) reduces to a half-isotropic distribution.

For small gradients and fields, $\lambda\ll L$, $R\rightarrow 0$, and the distribution function becomes $f\simeq(I/4\pi)(1+\mu R)$. In this limit it is clear both physically and from Eq. (17) that $I\rightarrow I_0$ and $R\rightarrow R_0$, where

$$R_0=-\lambda_s x^4\left[\frac{eE'}{kT}+(x^2-2.5)\frac{1}{T}\frac{\partial T}{\partial r}\right], \tag{23}$$

which leads to the Chapman-Enskog solution Eq. (9).

The distribution function Eq. (21) is positive definite for all particle energies and scale lengths and extends the basic Chapman-Enskog approximation to situations where both streaming and diffusion components are present. Figure 1(a) shows the distribution function for positive values of μ in a temperature gradient for which $\lambda_s/L=0.1$ with the electric field determined by quasineutrality. Figure 1(b) shows the velocity profile for $\mu=\pm 0.99$. For small velocities, the distribution is diffusive and exhibits the familiar properties of the Chapman-Enskog solution: a velocity distribution skewed in relation to f_0 by the temperature gradient and field to produce a flow of heat with zero net particle flow. For large positive velocities the distribution shows the presence of hot, long mean-free-path electrons streaming through the region L from the hot side. The distribution for large negative velocities is depopulated with respect to f_0, since it represents long mean-free-path electrons originating from the cold side of the gradient region.

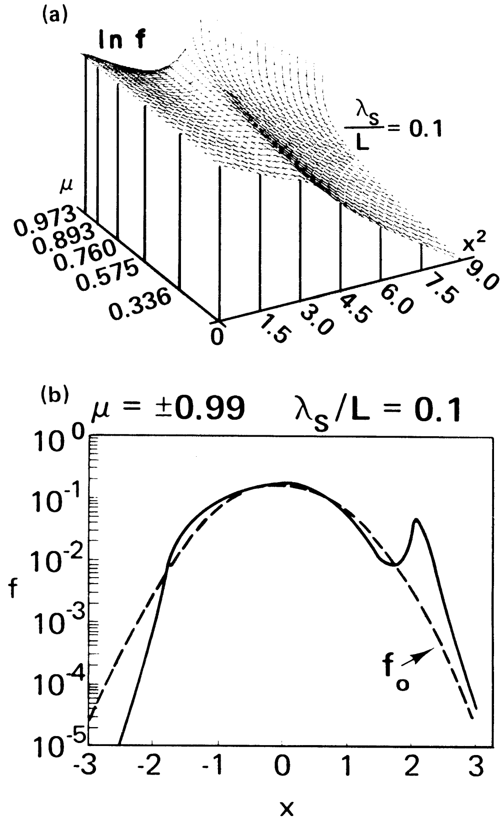

FIG. 1. Distribution function in a large temperature gradient. (a) Function for positive values of μ showing the streaming component along $\mu=1$. (b) Velocity profile for $\mu=\pm 0.99$ compared with the local Maxwellian f_0.

B. Scattering by electrons and fixed ions

For the problem considered by Spitzer,[2] the basic equation follows from Eq. (1), with the scattering term given by Eq. (13):

$$\frac{\partial f}{\partial t}+v\mu\frac{\partial f}{\partial r}-\frac{eE}{mv}\left[\mu v\frac{\partial f}{\partial v}+(1-\mu^2)\frac{\partial f}{\partial \mu}\right]=\frac{v}{\lambda_s x^4}\frac{\Lambda}{2Z}\frac{R_1}{D}(f-f_0). \tag{24}$$

The derivation proceeds in the same way as before to yield

$$\left[(F-\mu)\lambda_s x^4\left[\frac{2Z}{\Lambda}\right]\frac{D}{R_1}\frac{1}{I_0}\left[\frac{\partial I}{\partial r}-\frac{eE}{mv}\frac{\partial I}{\partial v}\right]+1\right]\psi=\frac{1}{4\pi},$$

which for

$$D_1=-D\left[\frac{2Z}{\Lambda}\right]\frac{R}{R_1},$$

$$D=-\lambda_s\left[\left[\frac{ZD_E}{A}\right]\frac{eE'}{kT}-2\left[\frac{ZD_T}{B}\right]\frac{1}{T}\frac{\partial T}{\partial r}\right], \tag{25}$$

leads to the solution

$$f=\frac{I}{4\pi}\frac{1}{D_1(\coth D_1-\mu)}. \tag{26}$$

The factors (ZD_E/A) and (ZD_T/B) in Eq. (25) are tabulated as functions of x in Ref. 2.

From small gradients and fields, the distribution given by Eq. (26) reduces to the Chapman-Enskog result, Eq. (12). For large gradients and fields, the long mean-free-path component is described either by a δ function in μ or a half-isotropic distribution, depending on the flux boundary condition. In between these two limits the solution is based on an approximate form of the fundamental equation that smoothly connects the two limits where it is strictly correct. The approximations inherent in the intermediate region are that (1) the Fokker-Planck collision term is represented by a relaxation form with a velocity-dependent relaxation time defined in terms of the scattering length, and (2) the distribution function is separable with the angular distribution represented by a slowly varying function.

The formulation is applicable to nonuniform plasmas. It follows from Eqs. (20) and (23) that the density gradient affects transport in the same way as the quasineutral field. It is convenient, therefore, to carry the pressure- or density-gradient term as a field component in E' as was done in Sec. III.

V. TRANSPORT COEFFICIENTS

The flow of electrical current and heat which is induced in the plasma by the presence of temperature gradients and electric fields is determined by Eq. (4). Since the distribution function given by Eqs. (25) and (26) properly accounts for strong gradients and fields, the resulting transport should be inherently flux limited.

Equation (4) can be written in terms of the particle flux J:

$$j=-e\int_0^\infty Jv^2dv\ ,$$
$$q=\frac{m}{2}\int_0^\infty Jv^4dv\ ,$$

where $J=vI_0\omega^2\Gamma D$, with $\Gamma(D_1)=(D_1\coth D_1-1)D_1^{-2}$. These expressions follow from Eq. (25) and $D_1\simeq\omega D$, where $\omega\equiv I/I_0$, a relation that will be justified below.

The electric current and heat flux become

$$j=\frac{4}{\pi^{1/2}}n_e\lambda_s\frac{e^2}{kT}\left[\frac{2kT}{m}\right]^{1/2}\times\int_0^\infty\Gamma\omega^2\left[\left[\frac{ZD_E}{A}\right]E'+\frac{k}{e}\left[-2\frac{ZD_T}{B}\right]\frac{\partial T}{\partial r}\right]x^3e^{-x^2}dx\ ,$$

$$q=-\frac{2}{\pi^{1/2}}n_e\lambda_s\frac{em}{kT}\left[\frac{2kT}{m}\right]^{3/2}\times\int_0^\infty\Gamma\omega^2\left[\left[\frac{ZD_E}{A}\right]E'+\frac{k}{e}\left[-2\frac{ZD_T}{B}\right]\frac{\partial T}{\partial r}\right]x^5e^{-x^2}dx\ ,$$

which, with $E'=E+(\partial P_e/\partial r)/en_e$, can be put into the simpler form

$$j=\sigma_L E\phi_E+\frac{\sigma_L}{en_e}\frac{\partial P_e}{\partial r}\phi_E+\alpha_L\frac{\partial T}{\partial r}\phi_T\ ,$$
$$q=-\beta_L E\psi_E-\frac{\beta_L}{en_e}\frac{\partial P_e}{\partial r}\psi_E-K_L\frac{\partial T}{\partial r}\psi_T\ , \tag{27}$$

where σ_L, α_L, β_L, and K_L are the transport coefficients in the Lorentz plasma,[2,9] and the correction factors are

$$\phi_E=\int_0^\infty\Gamma\omega^2\left[\frac{ZD_E}{A}\right]x^3e^{-x^2}dx\ ,$$
$$\phi_T=\tfrac{2}{3}\int_0^\infty\Gamma\omega^2\left[-2\frac{ZD_T}{B}\right]x^3e^{-x^2}dx\ ,$$
$$\psi_E=\tfrac{1}{4}\int_0^\infty\Gamma\omega^2\left[\frac{ZD_E}{A}\right]x^5e^{-x^2}dx\ ,$$
$$\psi_T=\tfrac{1}{10}\int_0^\infty\Gamma\omega^2\left[-2\frac{ZD_T}{B}\right]x^5e^{-x^2}dx\ . \tag{28}$$

For small values of D (small gradients and fields), the expressions (28) should approach the Spitzer values:[2,9] $\phi_E\rightarrow\gamma_E$, $\phi_T\rightarrow\gamma_T$, $\psi_E\rightarrow\delta_E$, and $\psi_T\rightarrow\delta_T$.

The problem of thermal transport under the condition of quasineutrality is of special interest owing to the anomalous results of the laser plasma experiments.[3-7] The quasineutrality condition $j=0$ determines the electric field,

$$\frac{eE'}{kT}=-\frac{3}{2}\frac{\phi_T}{\phi_E}\frac{1}{T}\frac{\partial T}{\partial r}=-(\tfrac{3}{2}-\delta)\frac{1}{T}\frac{\partial T}{\partial r}\ , \tag{29}$$

where δ measures the deviation of the field from the value in a classical Lorentz plasma. Under quasineutrality, D can be expressed in terms of the temperature-gradient scale length by

$$D=\frac{\lambda_s}{L}\left[(\delta-\tfrac{3}{2})\left[\frac{ZD_E}{A}\right]+\left[-2\frac{ZD_T}{B}\right]\right]\ ,$$

and the heat flux becomes

$$q=-\left[\psi_T-0.6\psi_E\frac{\phi_T}{\phi_E}\right]K_L\frac{\partial T}{\partial r}$$
$$=-g\left[Z,\frac{\lambda_s}{L}\right]K_L\frac{\partial T}{\partial r}\ . \tag{30}$$

The integrals in Eq. (28) can be evaluated numerically for different values of Z and λ_s/L to obtain correction factors for the classical transport coefficients which depend on the temperature gradient.

Although the quasineutral field is of significant magnitude only over the region of the gradient L, it is instructive to compare its value to the critical field for electron runaway,[13] $E_C=Ze\lambda^{-2}\log\Lambda$. This expression can be rewritten to yield

$$\frac{eE_c \lambda_s}{kT} = 4 ,$$

which, if the pressure-gradient term is assumed small, leads to $(\frac{3}{2} - \delta)\lambda_s / L \simeq 4$ for the corresponding value of the gradient parameter λ_s / L. Since δ typically ranges from 0.4 to 1.4, the quasineutral field in a steep gradient apparently never exceeds the critical value for electron runaway even for $\lambda_s / L \gtrsim 1$.

VI. COMPARISON WITH NUMERICAL SOLUTIONS

The solution represented by Eqs. (25) and (26) is strictly correct only in the short and long mean-free-path limits. In the intermediate region it is subject to two approximations as described at the end of Sec. IV. In particular, it is known that the relaxation form of the collision integral, Eq. (13), does not insure conservation of number of particles, momentum, and energy except in the limits. In order to check the validity of the method in the intermediate region, comparison is made to numerical solutions of Eq. (1) available in the literature. These solutions are generally of two types: numerical solutions of the multigroup Fokker-Planck equation[14–16] and Monte Carlo solutions.[17,18] There is some variation between the methods used for evaluating the collision integrals in these calculations and also in details of the models considered, but the results are generally consistent.

A. Model for evaluating the function I

In Eq. (26) the reduced distribution I is undetermined. In the short mean-free-path limit, the problem is local and $I \simeq I_0$, where $I_0 = 4\pi f_0$ is defined by a Maxwellian at the local temperature. For long scattering lengths, the problem is nonlocal, and I must be obtained from a solution to Eq. (17). In this section a simple model for evaluating I nonlocally is developed that is based on the temperature profile shown in Fig. 2. This profile is typical of the penetration of a thermal front into cold material and is representative of the models used in the numerical calculations.

In a quasisteady state, Eq. (17) (with α absorbed into I) becomes

$$I = I_0 - \left[-\frac{2Z}{\Lambda} \frac{D}{R_1} \frac{\lambda}{v} \right] \left[\frac{\partial J}{\partial r} - \frac{eE}{mv^2} \frac{\partial}{\partial v}(vJ) \right] ,$$

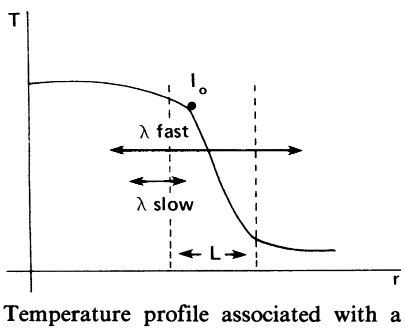

FIG. 2. Temperature profile associated with a heat front propagating into cold material, showing the relationship between $L \equiv T(\partial T / \partial r)^{-1}$ and λ in the model for $I = I_0 \omega$.

where the first quantity in large parentheses can be written $\lambda_{\rm eff} / v$. Since $I \to I_0$ in the limit as $\lambda_{\rm eff} \to 0$, J can be evaluated approximately by its local streaming value $J = vI_0$ to give

$$I = I_0 \left[1 - \frac{\lambda_{\rm eff}}{I_0} \left[\frac{\partial I_0}{\partial r} - \frac{eE}{mv} \frac{\partial I_0}{\partial v} - \frac{eE}{mv^2} 2I_0 \right] \right] , \tag{31}$$

which according to Eqs. (6) and (29) reduces to $I = I_0 \omega$, with

$$\omega = 1 + \frac{\lambda^*_{\rm eff}}{L} [(x^2 - 4 + \delta) + (1.5 - \delta) x^{-2}] . \tag{32}$$

In Eq. (32) $\lambda_{\rm eff}$ has been replaced by $\lambda^*_{\rm eff} \leq L$, so that $\lambda \partial / \partial r \leq L \partial / \partial r$ in accord with the form of the temperature profile in Fig. 2. In Eq. (31) the term $-eE\lambda / mv$ represents the change in velocity of an electron due to the influence of the field over a distance λ.

If Eq. (32) is used to incorporate approximately the nonlocal effects of the gradients and fields in the distribution I, Eq. (25) gives $D_1 \simeq \omega D$ where derivatives of ω are neglected. The particle current is then $J = vI_0 \omega^2 \Gamma D$ with $\Gamma = (\omega D \coth \omega D - 1)(\omega D)^{-2}$. The distribution also must be adjusted as described above to yield the half-isotropic flux boundary condition, since this is the one upon which the numerical models are based.

B. Results and comparison with other calculations

The correction factors given in Eq. (28) have been evaluated numerically under the condition of quasineutrality with $\partial P_e / \partial r = 0$, and the results are given for the Z values considered by Spitzer in Tables I–V. The quantity g appearing in Eq. (30) corresponds to Spitzer's $\epsilon \delta_T$. Also given in the tables are the ratio of the heat flux to the classical value, q / q_s, and the ratio of the heat flux to the nominal streaming value, q / q_0.

In Fig. 3 the results of Table V are seen to be in close agreement with a discrete ordinate solution of Eq. (1) for the Lorentz plasma.[15] In Fig. 4, which is adapted from Ref. 16, the results of Table III are compared to two solutions based on Legendre polynomials[14,16] and on a Monte Carlo calculation.[18] The Monte Carlo results, curve A, match the lower points on the dashed curves which are the results from Ref. 16. (The correspondence between the parameters used here and in Ref. 16 is $\bar{\lambda} / L_T = 1.85 \lambda_s / L$.) The lower points of the dashed curves correspond to the hot side of the gradient region and the

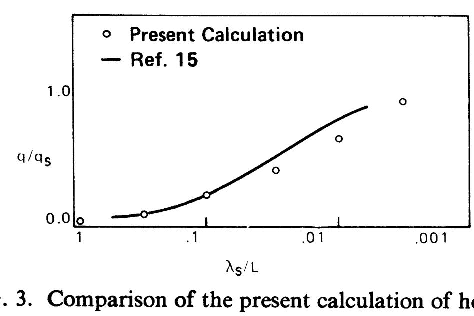

FIG. 3. Comparison of the present calculation of heat flux in steep gradients (O) with a numerical solution for a Lorentz plasma ($Z \to \infty$).

TABLE I. Transport correction factors and heat fluxes for different temperature-gradient scale lengths L. q_s is the classical heat flux and q_0 the nominal streaming value. $Z=1$ and Spitzer's $\epsilon\delta_T=0.094\,34$.

λ_S/L	ϕ_E	ϕ_T	ψ_E	ψ_T	g	q/q_s	q/q_0
0.0010	0.5819	0.269 1	0.4655	0.223 5	0.094 32	0.9998	0.0030
0.0032	0.5812	0.266 6	0.4635	0.219 1	0.091 56	0.9706	0.0094
0.010	0.5748	0.250 1	0.4492	0.196 1	0.078 78	0.8351	0.0251
0.032	0.5469	0.202 3	0.4011	0.144 2	0.055 18	0.5849	0.0564
0.10	0.4949	0.142 4	0.3309	0.092 92	0.035 77	0.3792	0.1142
0.32	0.4254	0.084 61	0.2534	0.051 47	0.021 23	0.2251	0.2169
1.00	0.3483	0.039 20	0.1820	0.023 94	0.011 65	0.1235	0.3718

TABLE II. Transport correction factors and heat fluxes for different temperature-gradient scale lengths L. q_s is the classical heat flux and q_0 the nominal streaming value. $Z=2$ and Spitzer's $\epsilon\delta_T=0.1461$.

λ_S/L	ϕ_E	ϕ_T	ψ_E	ψ_T	g	q/q_s	q/q_0
0.0010	0.6856	0.412 1	0.5822	0.354 6	0.144 6	0.990 1	0.0046
0.0032	0.6836	0.404 5	0.5770	0.342 0	0.137 1	0.938 8	0.0140
0.010	0.6700	0.366 7	0.5483	0.291 9	0.111 8	0.765 5	0.0357
0.032	0.6216	0.275 2	0.4687	0.197 5	0.073 00	0.499 6	0.0746
0.10	0.5398	0.172 3	0.3640	0.114 1	0.044 40	0.304 0	0.1417
0.32	0.4448	0.087 32	0.2640	0.056 53	0.025 43	0.174 1	0.2598
1.00	0.3446	0.027 17	0.1776	0.022 03	0.013 63	0.093 28	0.4350

TABLE III. Transport correction factors and heat fluxes for different temperature-gradient scale lengths L. q_s is the classical heat flux and q_0 the nominal streaming value. $Z=4$ and Spitzer's $\epsilon\delta_T=0.2057$.

λ_S/L	ϕ_E	ϕ_T	ψ_E	ψ_T	g	q/q_s	q/q_0
0.0010	0.7841	0.573 2	0.7030	0.511 3	0.202 9	0.986 5	0.0065
0.0032	0.7796	0.555 3	0.6917	0.482 9	0.187 2	0.910 3	0.0191
0.010	0.7551	0.483 7	0.6420	0.391 9	0.145 2	0.705 7	0.0463
0.032	0.6810	0.334 6	0.5249	0.244 0	0.089 32	0.434 2	0.0912
0.10	0.5668	0.182 3	0.3845	0.125 6	0.051 39	0.249 8	0.1640
0.32	0.4467	0.072 49	0.2641	0.054 30	0.028 58	0.139 0	0.2920
1.00	0.3239	−0.000 050	0.1638	0.014 55	0.014 56	0.070 80	0.4648

TABLE IV. Transport correction factors and heat fluxes for different temperature-gradient scale lengths L. q_s is the classical heat flux and q_0 the nominal streaming value. $Z=16$ and Spitzer's $\epsilon\delta_T=0.3130$.

λ_S/L	ϕ_E	ϕ_T	ψ_E	ψ_T	g	q/q_s	q/q_0
0.0010	0.9204	0.823 6	0.8824	0.778 1	0.304 3	0.972 2	0.0097
0.0032	0.9091	0.775 7	0.8554	0.706 1	0.268 2	0.856 7	0.0274
0.010	0.8615	0.627 3	0.7638	0.527 5	0.193 8	0.619 3	0.0619
0.032	0.7423	0.370 8	0.5845	0.285 3	0.110 1	0.351 8	0.1125
0.10	0.5760	0.141 3	0.3909	0.115 4	0.057 83	0.184 8	0.1846
0.32	0.4225	0.006 74	0.2461	0.032 10	0.029 74	0.095 02	0.3038
1.00	0.2851	−0.058 42	0.1408	−0.003 18	0.014 13	0.045 14	0.4510

TABLE V. Transport correction factors and heat fluxes for different temperature-gradient scale lengths L. q_s is the classical heat flux and q_0 the nominal streaming value. $Z=\infty$ and Spitzer's $\epsilon\delta_T=0.400$.

λ_S/L	ϕ_E	ϕ_T	ψ_E	ψ_T	g	q/q_s	q/q_0
0.0010	0.9971	0.983 2	0.9919	0.970 2	0.383 3	0.958 3	0.0122
0.0032	0.9784	0.898 9	0.9486	0.847 9	0.325 0	0.812 4	0.0332
0.010	0.9109	0.676 2	0.8229	0.588 5	0.222 0	0.554 9	0.0708
0.032	0.7603	0.340 8	0.6030	0.280 0	0.117 8	0.294 5	0.1203
0.10	0.5662	0.077 65	0.3830	0.089 10	0.057 58	0.144 0	0.1838
0.32	0.4018	−0.049 93	0.2324	0.010 79	0.028 12	0.070 30	0.2872
1.00	0.2678	−0.095 42	0.1316	−0.015 18	0.012 96	0.032 40	0.4136

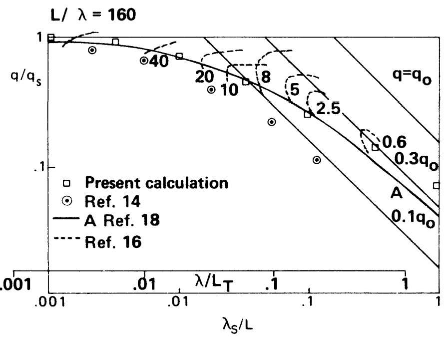

FIG. 4. Comparison of the present calculation (□) with two numerical solutions of the Fokker-Planck equations and a Monte Carlo solution. Curve A was calculated in Ref. 18, the dashed curves in Ref. 16 (curves labeled by slab thickness in terms of λ), and the points ⊙ in Ref. 14.

upper points to the cold side which is expected to be under the influence of nonlocal streaming effects. The calculations of Refs. 18 and 16 agree well with each other and with the calculations presented here. The Monte Carlo results of Ref. 17 are also consistent with a limit of $0.3q_0$. The results of Ref. 14, however, are considerably lower at $0.1q_0$.

In Fig. 5 the results of Ref. 16 are compared to the present calculation for two sets of boundary conditions. The half-isotropic boundary condition is appropriate for the hot side of the gradient region and leads to heat fluxes that match the lower points on the dashed curves. The streaming boundary condition is more appropriate for the cold side, as can be appreciated from Fig. 2. This is verified in Fig. 5 except for the curve labeled 0.6, which corresponds to a slab only $0.6\bar{\lambda}$ thick. In this case the cold side is still strongly under the influence of the hot boundary and should correspond more to the half-isotropic boundary condition.

VII. EFFECT OF STREAMING INSTABILITIES

The calculations presented in Tables I–V show a distribution function that is classical for gentle gradients, but as the gradient steepens, long mean-free-path particles begin to stream through the region of the gradient, forming a bump on the distribution for μ values near one. A distribution of this form is unstable to the formation of plasma waves which grow at the expense of the energy of particles in the bump. The evolution of the bump on tail instability was studied computationally in Ref. 19.

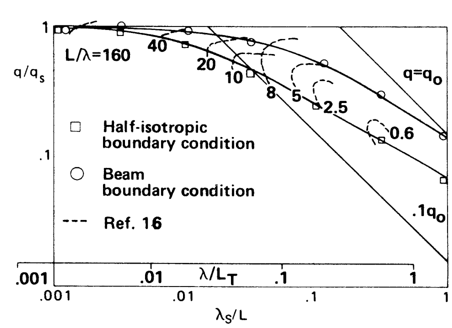

FIG. 5. Comparison of the present calculation using half-isotropic (□) and streaming (○) boundary conditions with results presented in Ref. 16. Lower points on the dashed curves correspond to the hot side of the gradient region and the upper points to the cold side.

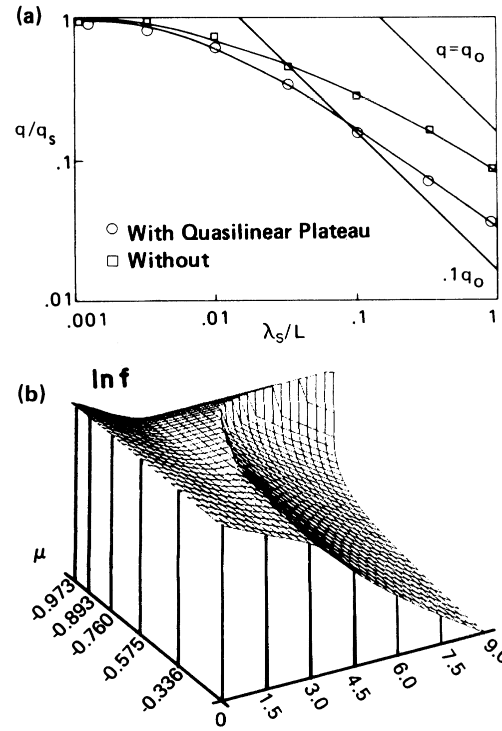

FIG. 6. Effect of streaming instabilities. (a) Heat-flow values computed with (○) and without (□) quasilinear plateau. (b) Distribution function for $\lambda_s/L=0.1$ showing the quasilinear plateau. [Compare with Fig. 1(a).]

The distribution in the streaming region is essentially one dimensional, as shown in Fig. 1(a). From quasilinear theory in one dimension,[20] it is known that the bump will flatten until $\partial f/\partial x \leq 0$, thus forming the "quasilinear plateau." This effect is particularly evident in Figs. 2 and 4 of Ref. 19. In order to see how much of an effect this instability might have on thermal transport in steep gradients, calculations have been performed with the distribution function Eq. (26) modified as described in Ref. 20. The transport results are shown in Fig. 6(a), with the distribution corresponding to $\lambda_s/L=0.1$ shown in Fig. 6(b). The heat fluxes are reduced by the streaming instability by approximately a factor of 2.

VIII. SUMMARY AND CONCLUSIONS

A solution to the Boltzmann equation has been found that extends the first-order Chapman-Enskog approximation to large gradients and fields. The solution is strictly correct in the limits of long and short scattering lengths. In between these two limits the solution is subject to two approximations: (1) The Fokker-Planck collision term is represented by a relaxation form with a velocity-dependent relaxation time defined in terms of the scattering length and (2) the distribution function is separable with the angular distribution represented by a slowly varying function.

The transport solution has been used to calculate correction factors for the classical transport coefficients in a nonuniform plasma that depend on the temperature-gradient scale length as a parameter. These factors lead to heat-flow values in close agreement with Monte Carlo and numerical Fokker-Planck calculations over a large range of temperature gradients.

Since the transport solution properly accounts for both diffusion and streaming components, it leads to transport coefficients that are inherently flux limited. The correction factors derived on the basic of this solution allow the simpler diffusion expressions to be used in numerical computations without the need to invoke an *ad hoc* flux limiter. The transport solution also provides a basis for investigating analytically the physical principles involved in more complex transport problems, notably the suprathermal component and its effect on the thermal flux through the quasineutral field and the density gradient.

Neither the solution derived here nor the more elaborate numerical solutions yields heat-flow values in steep gradients as small as those inferred from laser plasma experiments,[7] $f_c \sim 0.03-0.06$. The results shown in Fig. 4 indicate that factors other than the small gradient approximation in the classical calculation are responsible for the discrepancy.

ACKNOWLEDGMENTS

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