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Noise versus chaos in acousto-optic bistability

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We present a study of the evolution to chaos for the acousto-optic bistable device. We numerically solve the difference-differential equation describing the system which shows excellent agreement with the experiment. We analyze the influence of noise—additive and multiplicative—on the bifurcation sequence and on the onset of chaos. It is shown experimentally that both types of noise create a gap in bifurcation sequence. In addition we present a comparison between theory and experiment of the time evolution of the signal.
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I. INTRODUCTION

The chaotic or turbulent behavior seen in physical, chemical, or biological systems which are governed by deterministic equations has attracted intense interest recently.[1] It has been pointed out[2] that chaotic behavior can occur in an optical bistable system which can be described by a differential-difference equation. Since then, a period-doubling route to chaos has been demonstrated in a hybrid electro-optic[3] and acousto-optic[4] device with delay in the feedback loop, all-optical passive systems,[5,6] and single-mode lasers.[7] It is well known that such dynamical systems exhibiting a continuous instability as a function of the bifurcation parameter are extremely sensitive to small perturbations, particularly at the points close to the threshold of instability.

It has been shown that added noise can lead to a truncation of the sequence of periods. Experimentally, this question has been studied in various systems[8] by applying artificial and experimentally controlled broad-band noise to the control parameter. In the case of the electro-optic device, Derstine *et al.* studied the influence of the shot noise of their photomultiplier which is intensity dependent, and found some departure from the theoretical model[9] based on intensity-independent noise.

In optical bistability, however, noise may appear in both forms: intensity dependent (multiplicative) and intensity independent (additive). In this paper we will consider the two sources of noise influencing our system—intensity fluctuations of the laser and bias voltage fluctuations. They appear in either of the so-named multiplicative or additive form in the difference-differential equation describing the system.

In order to see the real influence of these two kinds of noise, we simulate them by adding noisy voltage with known characteristics in different places of the loop. In Sec. II we briefly describe the acousto-optic bistability. In Sec. III we describe the model used to simulate the effect of noise. In Sec. IV we analyze the time evolution of the noisy system in the chaotic region.

II. CHAOS IN ACOUSTO-OPTIC BISTABILITY

Figure 1 shows the experimental layout of our hybrid acousto-optic bistable device. The He-Ne laser diffracted light is detected by a photodiode and the signal is delayed by an amount of $\tau_D = 5\ \mu\text{sec} = 10\tau$ where τ is the response time of the system. This delay results from an intrinsic delay in the acousto-optic interaction and from the propagation time through several hundred meters of coaxial cable. The signal is fed to the rf generator (driver) which produces a voltage on the Bragg cell proportional to the feedback-signal amplitude, thus closing the loop. Further experimental details were published elsewhere.[4]

A Gaussian noise generator with a bandwidth of 5 MHz was used as a well-controlled source of noise. This noise was introduced by two methods. (1) additive—when fed into the amplifier producing a noisy offset in the loop and (2) multiplicative—when modulating the intensity of the laser by driving the second acousto-optic modulator operating in the linear mode (Fig. 1).

The experimental bifurcation sequence for both multiplicative and additive noise was obtained on the oscilloscope by means of an electronic window comparator. The results are illustrated in Fig. 2 for different noise levels. The relative width of the electronic window allowed us to obtain a well-defined branching structure while slowly varying μ (x axis). We could even observe the period-8 subharmonic at the lowest noise level. It is therefore nearly the null derivatives points of the signal that are plotted with the Z-axis input of the oscilloscope because the branches of the pitchfork bifurcations shown correspond to the regions of the most probable values of the signal.

We can easily observe (cf. Fig. 2) the progressive disappearance of the higher subharmonics with increasing noise. This general feature is discussed for the difference-equation case by Crutchfield *et al.*[10] In a similar manner we can say that the effect of noise in our system can be understood as a kind of dynamical average of the structure of attractors over a range of nearby parameters.

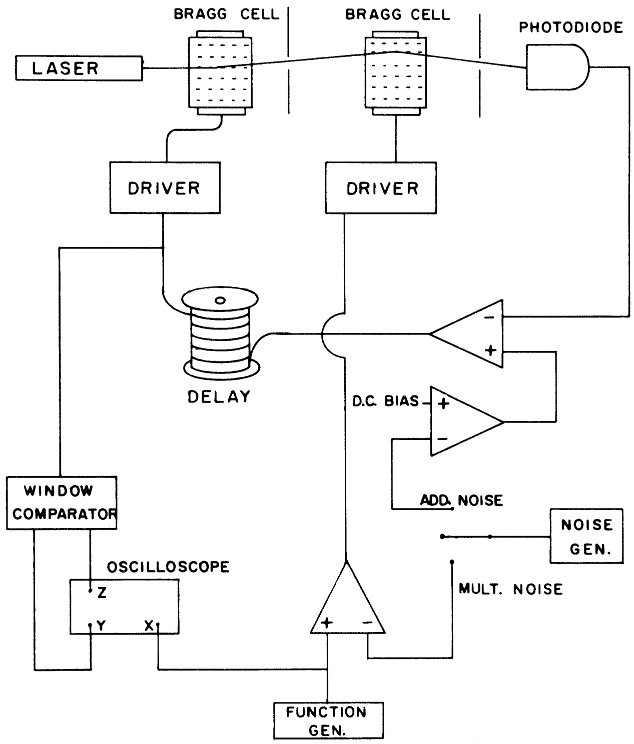

FIG. 1. Acousto-optic bistable device.

According to this view, the averaging of a periodic orbit with adjacent chaotic orbits tends to lower the transition to chaos. On the other hand, in the case of the transition from a periodic orbit to the next one, the averaging does not produce a shift of the bifurcation. In Fig. 3 we can see an enlargement of the bifurcation from the period-1 to the period-2 cycle for the following two cases: (1) when there is no noise added [Fig. 3(a)]; (2) with a multiplicative noise of amplitude $p=0.03$ [Fig. 3(b)]. It appears clearly that the bifurcation point remains globally unchanged even though a Fourier analysis of the signal reveals that the noise substantially reduces the slope of the growing of the period-2 frequency with the bifurcation parameter μ.

Moreover, we could observe, by studying the evolution of the Fourier components, that the disappearance of a periodic waveform for example, (8-P) was preceding that of its corresponding chaotic waveform (8-C) for both types of noise. Derstine *et al.* previously observed such a phenomenon, but for multiplicative noise only. The departure from the model developed by Crutchfield *et al.*,[9(a)] which predicts the simultaneous disappearance of 8-C and 8-P with increasing noise (symmetric gap), can thus be attributed to the fact that our hybrid system is described by a delay-differential equation instead of the nonlinear differential equation of the model.

III. NUMERICAL ANALYSIS OF THE STOCHASTIC EQUATION

The transient behavior of the noiseless system is given by the difference-differential equation[11]

$$\tau\frac{dX(t)}{dt}=-X(t)+\pi\{A-\mu\sin^2[X(t-\tau_D)-X_B]\}\ , \tag{1}$$

where X is the normalized voltage at the input of the acousto-optic driver; μ, proportional to the laser intensity, is the bifurcation parameter, and X_B and A are constants related to the voltage offset. The solution of Eq. (1) takes the form

$$X(t)=\int_{-\infty}^{t}e^{-(t-s)}F[X(s-\tau_D);\mu]ds\ , \tag{2}$$

where

$$F(X)=-\mu\pi\sin^2(X-X_B)+\pi A\ , \tag{3}$$

and in general requires numerical calculations. Solutions for the similar type of equations appearing in different examples of bistability have been presented recently.[12] Up until now, as far as we know, there was no attempt to record experimentally and compare with theory the transient behavior of the bistable system in the chaotic domain.

The deterministic evolution given by Eq. (1) is subject to at least two sources of noise in our experiment: the intensity of the laser undergoes fast δ-correlated fluctuations and the electrical part of the device also produces white noise. We model these by adding noise terms to the right-hand side of Eq. (1). The corresponding Langevin equation takes the form

$$\tau\frac{dX(t)}{dt}=-X(t)+\pi\{(A+q\xi_1)-(\mu+p\xi_2)\sin^2[X(t-\tau_D)-X_B]\}\ , \tag{4}$$

where ξ_1, ξ_2 are Gaussian random processes with zero mean, variance one, and a correlation function given by $G(t-t')=\delta(t-t')$; p and q control the amplitudes of the multiplicative and additive noise.

If the overall delay τ_D is small compared to the response time τ, the system will end up its evolution into the steady state governed by the initial conditions and parameter μ. In this case, it shows hysteresis and bistability in three modes:[13] input intensity variation, feedback gain, and modulator-bias voltage variation.

If the response time τ becomes much faster than the delay τ_D, the time evolution can be approximated by a difference equation. It shows the famous bifurcation sequence predicted by Feigenbaum. The difference equation correctly describes most of the basic features of the bifurcation sequence.

In Fig. 4 we present the bifurcation diagram for the noisy difference equation:

$$X(t)=\pi\{(A+q\xi_1)-(\mu+p\xi_2)\sin^2[X(t-\tau_D)-X_B]\} \tag{5}$$

for different additive $q\xi_1$ and multiplicative $p\xi_2$ noise levels. Comparison with the experimental results (Fig. 2) shows good qualitative agreement, particularly in the case of additive noise.

However, the stochastic-difference equation (5) can only give us a crude approximation of the real experiment. The differential term in Eq. (1) is responsible for interest-

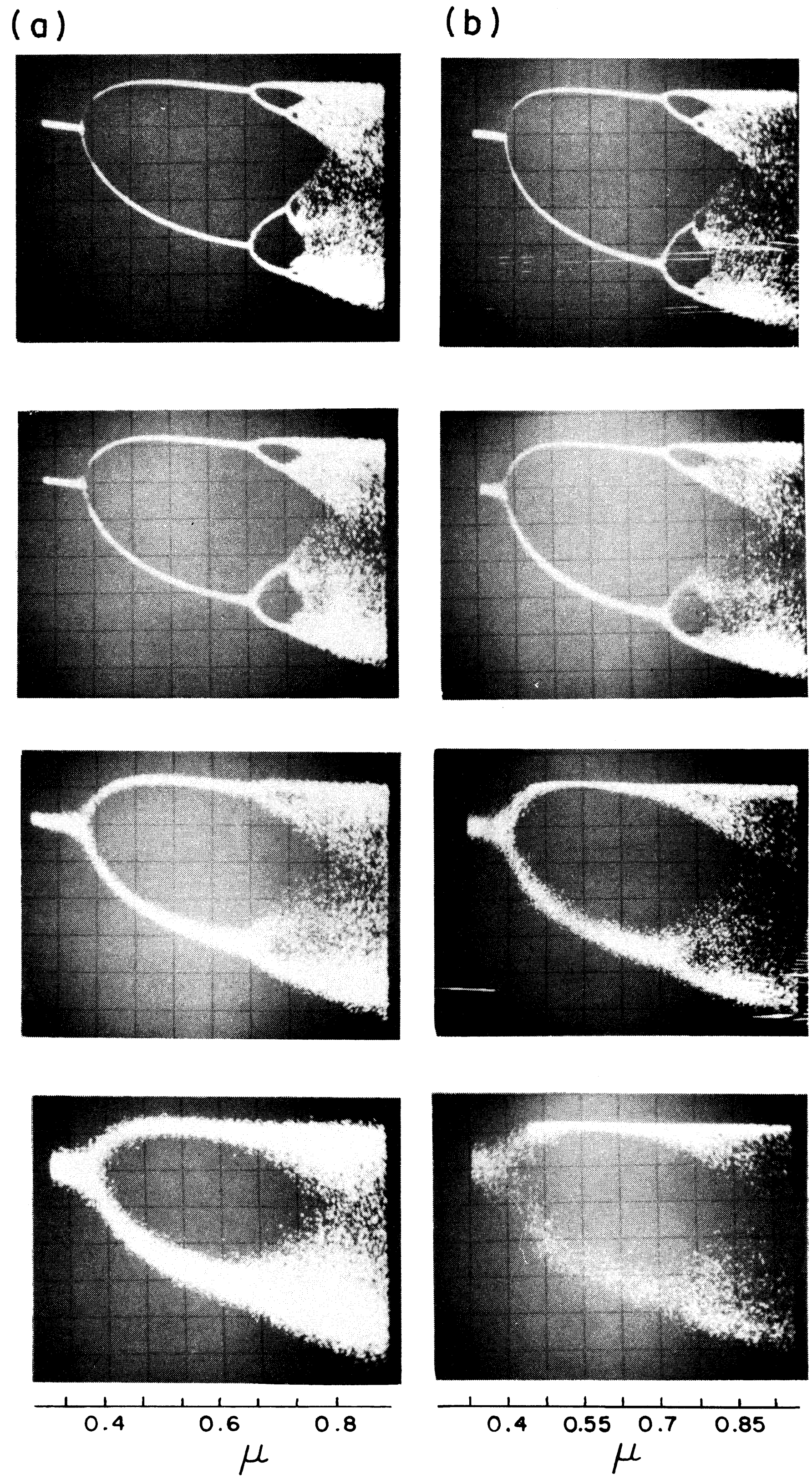

FIG. 2. Bifurcation diagram [of $\{X_n\}$ versus μ (the bifurcation parameter)] experimentally obtained for $\tau_D = 10\tau$. For the multiplicative noise the experimental error associated with the measure of the rms amplitude was of the order of 50%. For additive noise it was less than 10%. Figure (a) shows the effect of additive noise for different amplitudes q: (1) $q = 0.0004$, (2) $q = 0.001$, (3) 0.005, and (4) 0.008. Figure (b) shows the effect of multiplicative noise for different amplitudes p: (1) $p = 0$, (2) $p = 0.01$, (3) $p = 0.04$, and (4) $p = 0.15$. A residual, mainly additive, noise was always present with an amplitude of $q = 0.0004$.

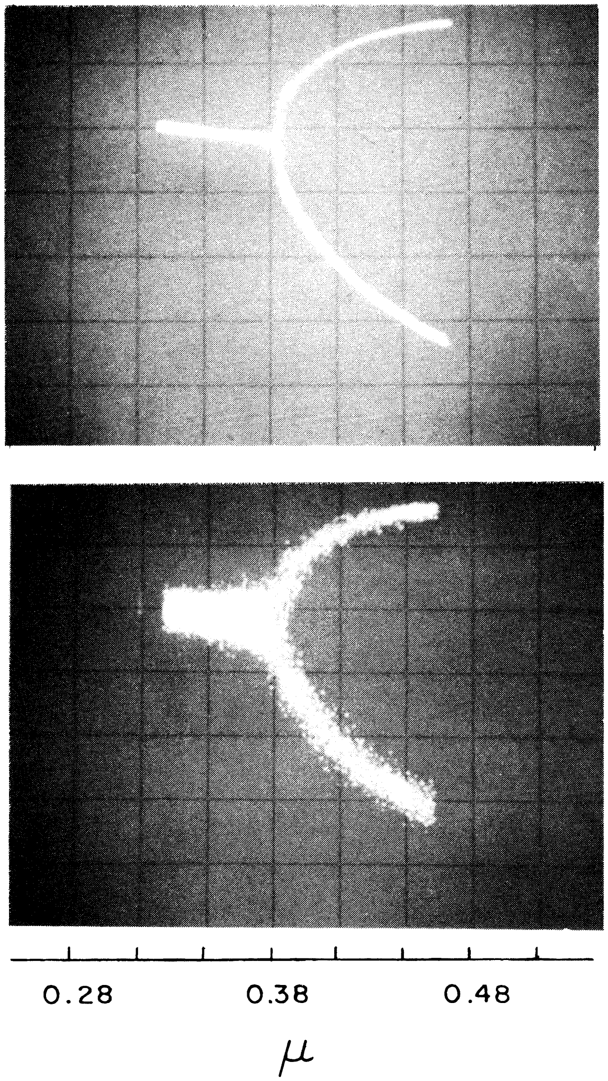

FIG. 3. Enlargement of the period-1 to period-2 bifurcation. Figure (a) corresponds to the situation when there is no noise added. Figure (b) is for a multiplicative noise of $p = 0.03$.

ing features that a simple difference equation cannot explain (frequency-locked oscillations, for example). Furthermore, in order to analyze the influence of noise on chaos on a definite time scale, one needs to consider the solution of the stochastic-differential-difference equation (4).

To solve the stochastic equation (4) it would require the simulation using Monte Carlo techniques. A discrete version of Eq. (4) can be obtained following, for example, a prescription of Sancho *et al.*[14] With this procedure, the resulting stochastic signal appears to be quite stable and correct in the periodic region and for low noise amplitude. However, in the chaotic region, as expected, the signal began to show numerical instabilities for very low noise level, resulting from the intrinsic error (of order $\Delta^{3/2}$) in the algorithm. In order to increase the precision we used a second method based on a Runge-Kutta model with error in order three (Δ^3). The results are in good agreement with those of the preceding method for the periodic region and did not seem to present numerical instabilities in the chaotic region when adding noise. The way we have modeled the noise in our system and the particular nature of the noise considered allows us to introduce noise in a simple manner.

IV. NOISY EVOLUTION TO CHAOS

The critical point in the analysis of a noisy chaotic signal is to distinguish between the effect of noise on the chaotic state itself and its effect on the stability of the numerical analysis. A simple way to realize this distinction is to study the probability density of the calculated signal. Moreover, such an analysis revealed that, as for the difference case, the global stability of a chaotic attractor is not perturbed very much by noise. We present in Fig. 5 the effect of noise ($q = 0.02$) on the probability density for the chaotic attractor at $\mu = 0.78$ from which we can conclude that the chaotic signal nearly behaves the same way, statistically speaking, in presence of noise.

On the other hand, the time evolution of the signal appears to be extremely sensitive to noise. It seems, therefore, that the exact trajectory of the signal is meaningless since it is indefinitely single valued or deterministic only in the ideal case when there is no perturbation on the system. In fact, the addition of noise at a very low level is sufficient to produce a multitude of progressively diverging trajectories all originating from the same initial conditions. Moreover, even without adding noise, the only uncertainty related to the numerical analysis was enough to cause the "deterioration" of the signal. While increasing the number of calculated points, and therefore the precision of the analysis, we could observe that the divergence of the trajectories was slightly postponed.

We present at Fig. 6 the superposition of chaotic trajectories calculated with a small additive noise ($q = 0.001$). This noise induces a progressive loss of the initial conditions so that all trajectories diverge more and more from one another with increasing time. Of course, this interesting feature is not really surprising since it is related to the intrinsic nature of chaos. However, one can be very surprised by the extreme sensitivity to noise of chaos for numerical and for experimental signal.

The experimental time evolution of chaos (for $\mu \approx 1$) starting from particular initial conditions was obtained by using a switching gate in the feedback loop of our system. The gate was alternately opened and closed by an external periodic signal, simultaneously triggering the oscilloscope. We could directly observe on the oscilloscope the superposition of many trajectories. The results are presented in Fig. 7 where we can see that for the intrinsic (unavoidable) noise level of the system ($q = 0.004$), trajectories diverge completely inside a time interval of about $20\tau_D$ which is of the order of 0.1 msec in our system. Comparison with numerical analysis shows good qualitative agreement. The experiment also demonstrates a great dependence of trajectories on fine change of the bifurcation parameter μ. It also appears that for a very small increase of μ, one could encounter frequency-locked, period-2 chaos, and some of the harmonics predicted by

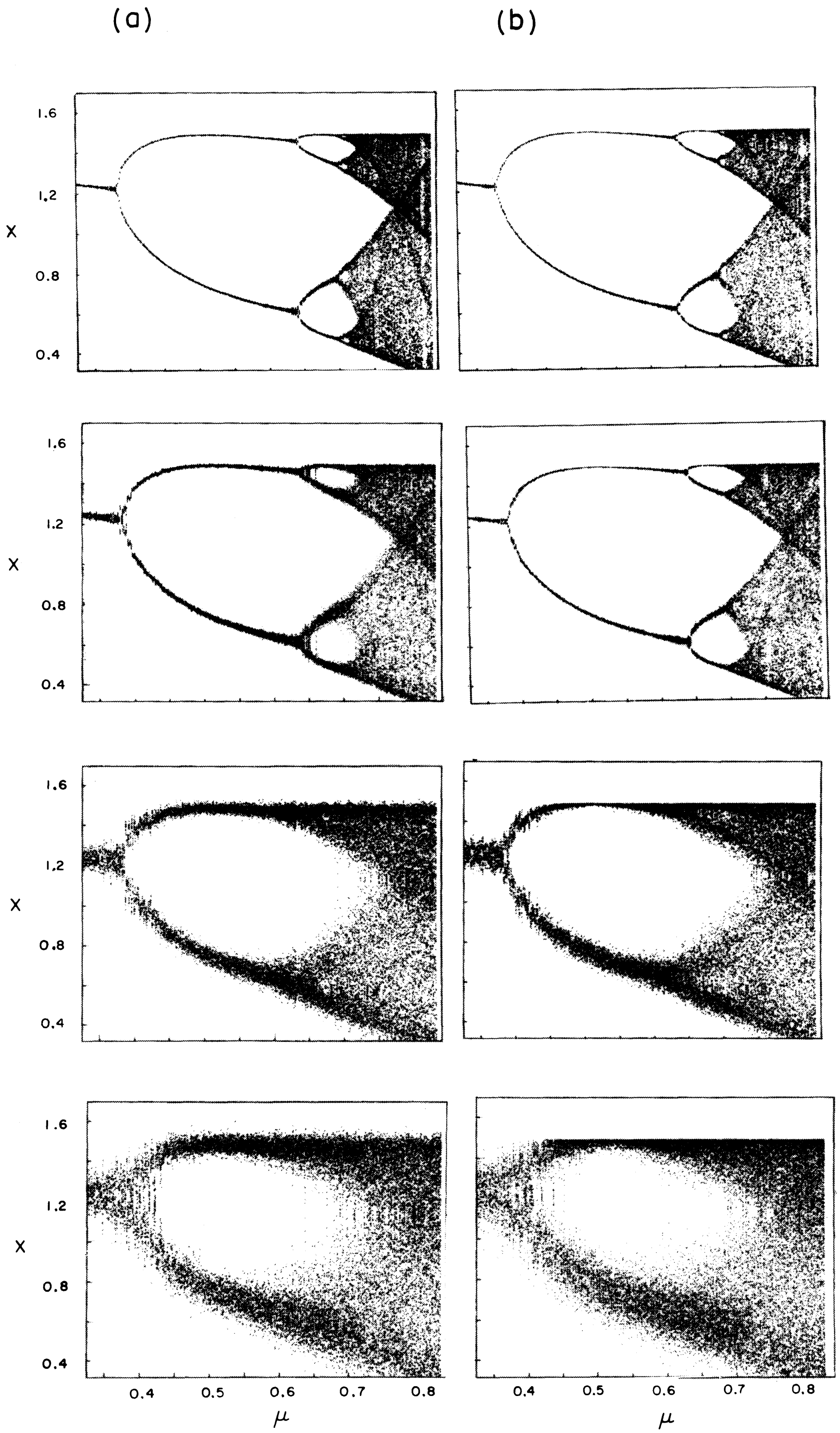

FIG. 4. Bifurcation diagram obtained from the difference equation (5). In (a) we show the effect of additive noise for (1) $q = 0.0003$, (2) $q = 0.0011$, (3) $q = 0.0045$, and (4) $q = 0.0083$. In (b) we consider the effect of multiplicative noise of amplitude: (1) $p = 0.0015$, (2) $p = 0.0025$, (3) $p = 0.02$, and (4) $p = 0.08$.

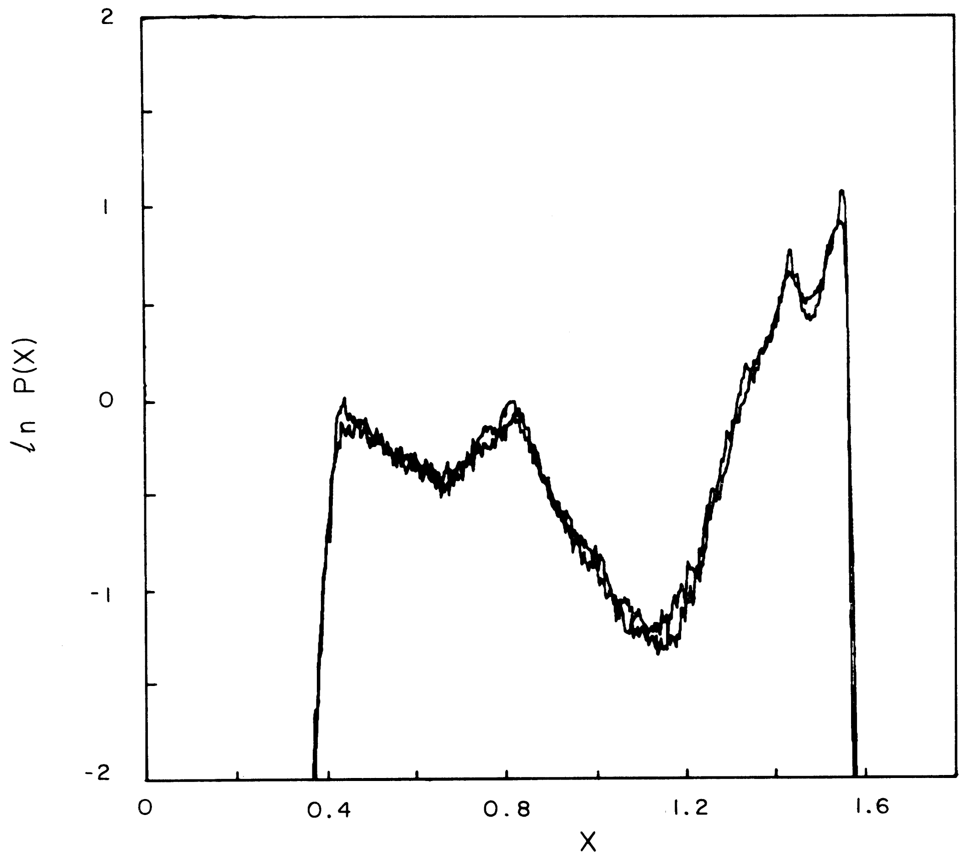

FIG. 5. Logarithmic plots of the probability density $P(X)$ for $\mu=0.78$. One of the curves shown corresponds to the noiseless case; the other is for an additive noise of amplitude $q=0.02$. $P(X)$ is a histogram of 500 000 points calculated from Eq. (4) and partitioned in 500 bins. The effect of noise is practically visible only near the peaks where it rounds off the curve.

Ikeda *et al.*[15] appearing and disappearing within a small interval of time.

V. CONCLUSION

We presented an experimental study of the acousto-optic bistable device in the periodic and in the chaotic regime with external noise. In the fully deterministic case, the agreement between theory and experiment is excellent. The noise was introduced into the system in two ways: by randomly varying the laser intensity (multiplicative noise) and by varying the dc offset voltage in the amplifier (additive noise). The bifurcation gap develops as the noise amplitude is increased. In addition, the "transient" chaotic signal experimentally obtained shows good agreement with the calculated one concerning the sensitivity to noise.

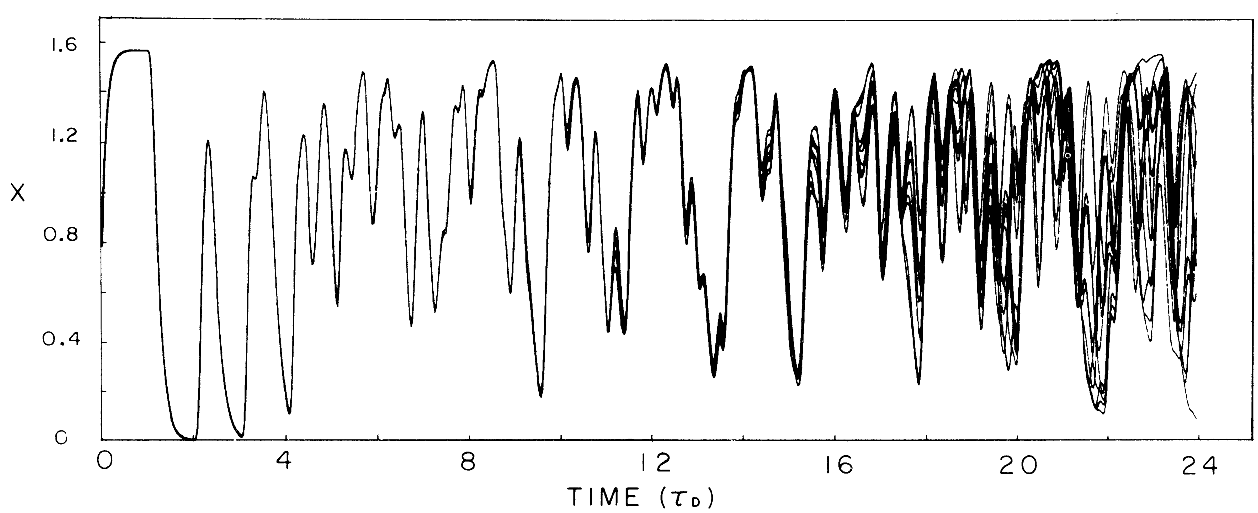

FIG. 6. Superposition of chaotic trajectories obtained numerically from Eq. (4) with $p=0$, $q=0.001$, and for $\mu=1$ and $\tau_D=10\tau$.

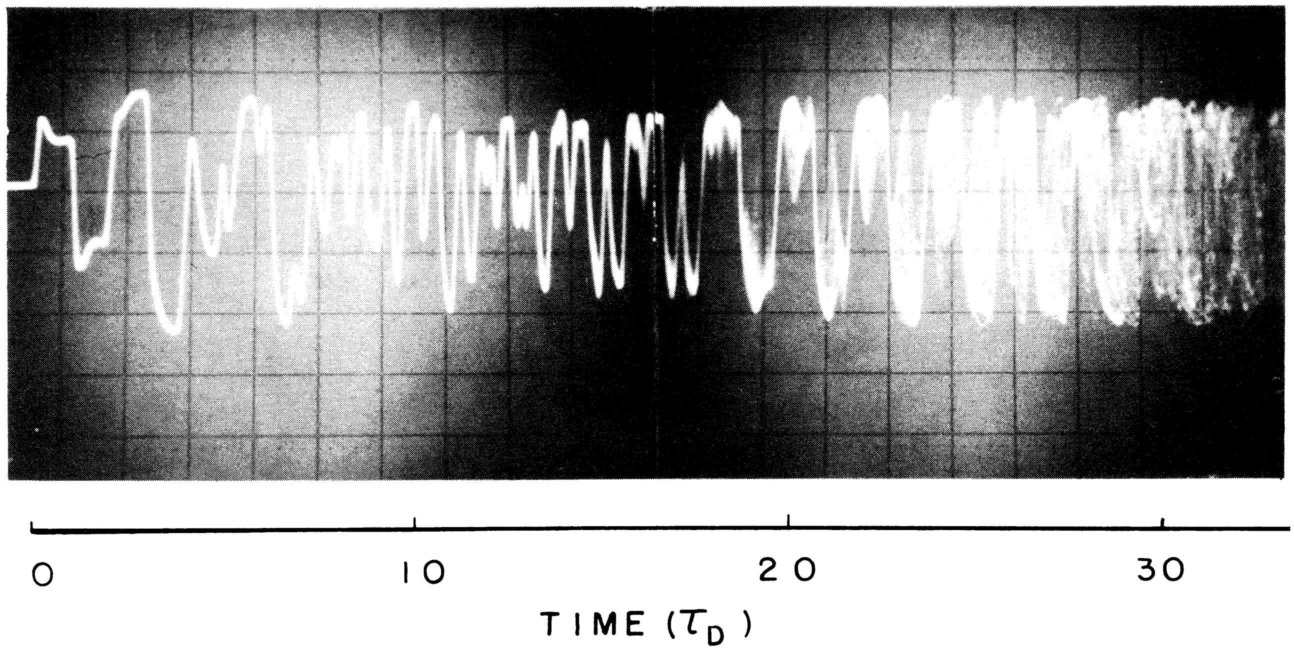

FIG. 7. Superposition of chaotic trajectories experimentally obtained, just after the feedback loop is closed, and for the background additive noise level of the system, $q=0.0004$.

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[1]See, for example, *Evolution of Order and Chaos in Physics, Chemistry, and Biology,* edited by H. Haken (Springer, Berlin, 1982); *Dynamical Systems and Chaos,* edited by L. Garrido, (Springer, Berlin, 1983); *Coherence and Quantum Optics V,* edited by L. Mandel and E. Wolf (Plenum, New York, in press).

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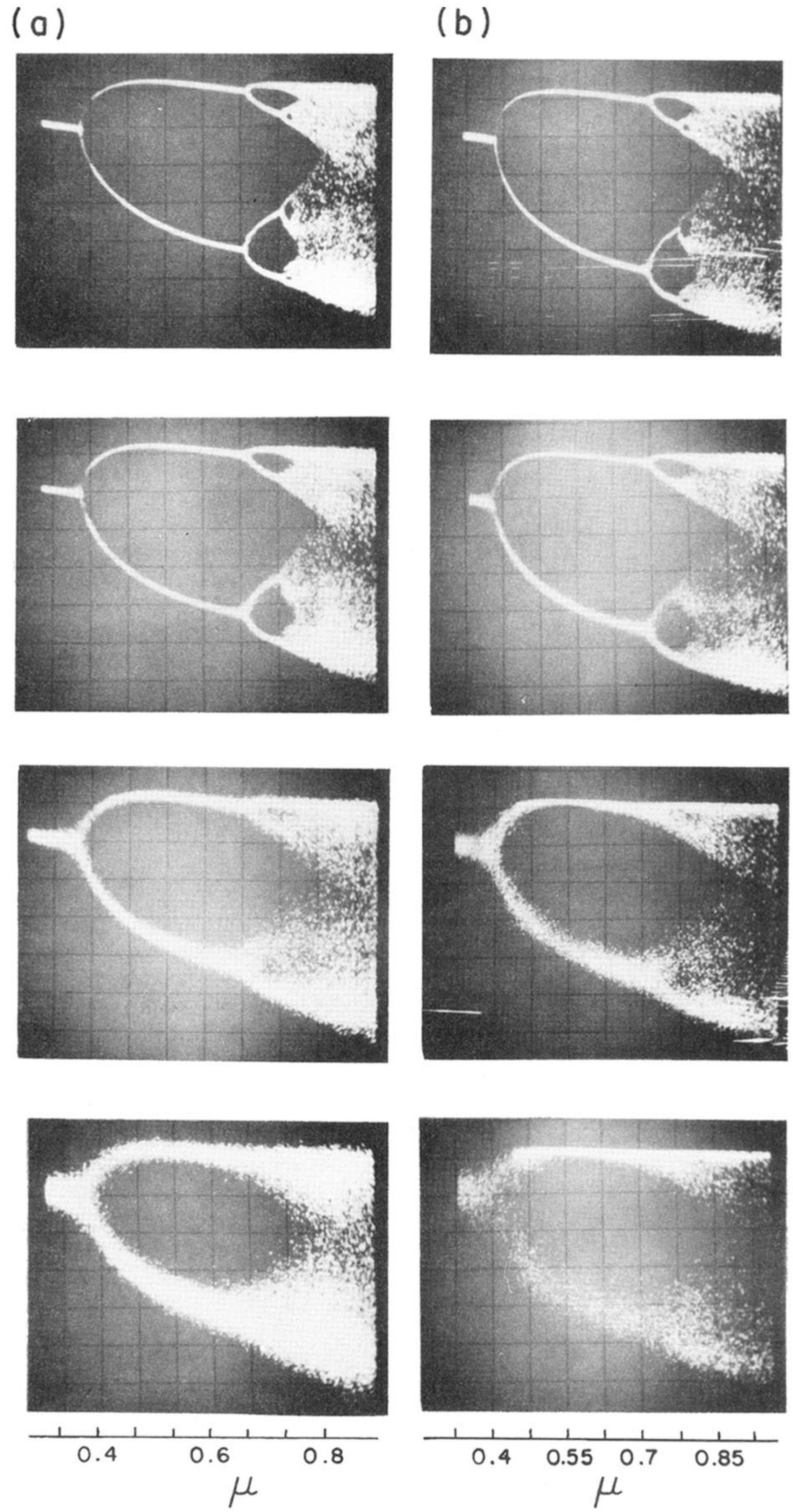

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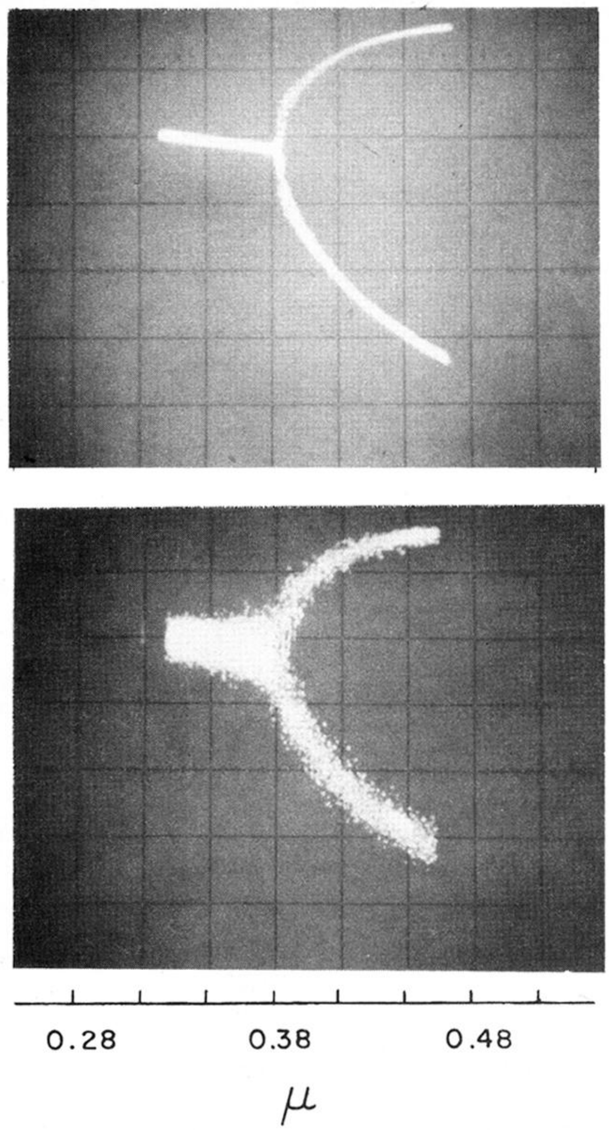

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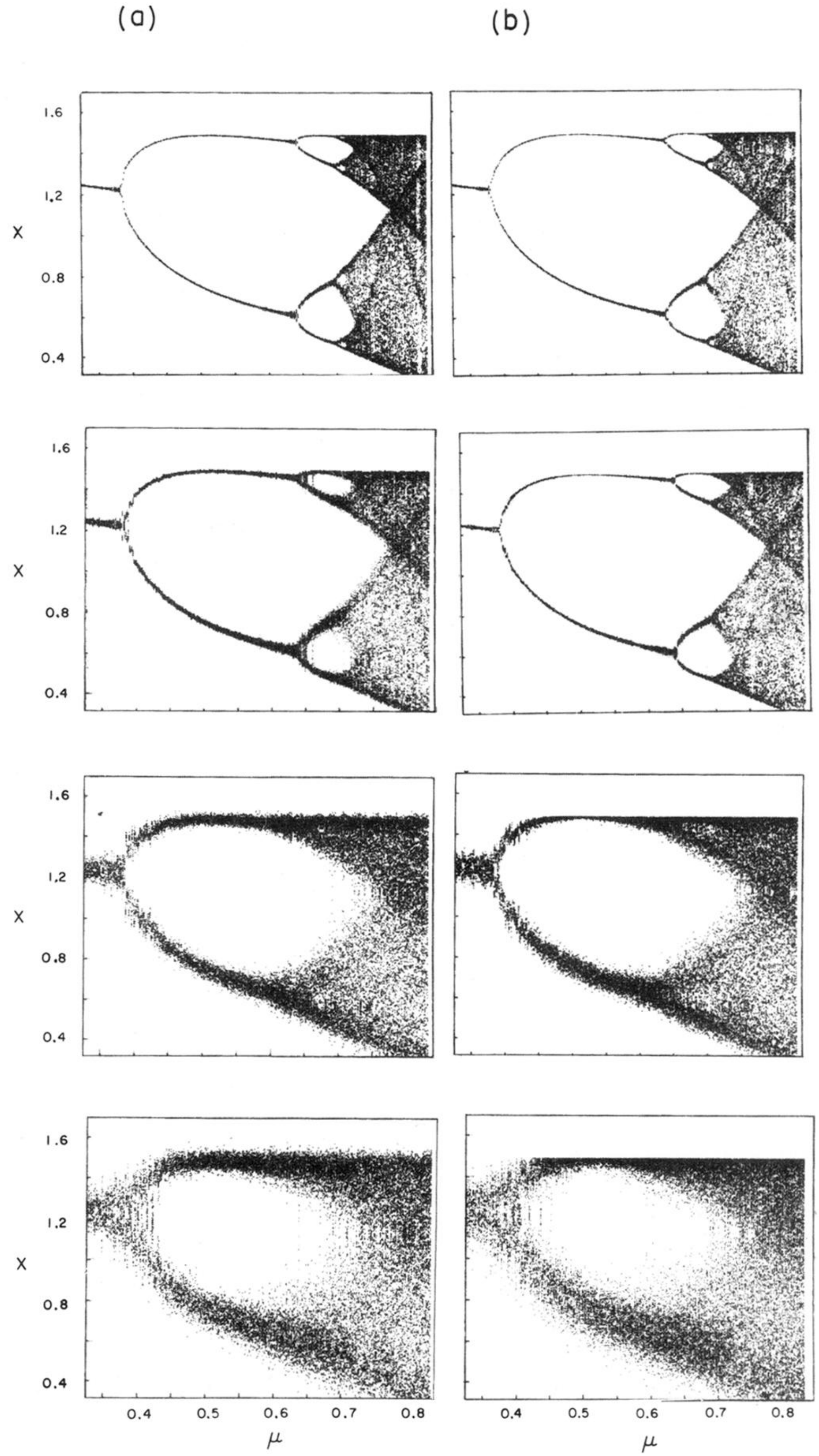

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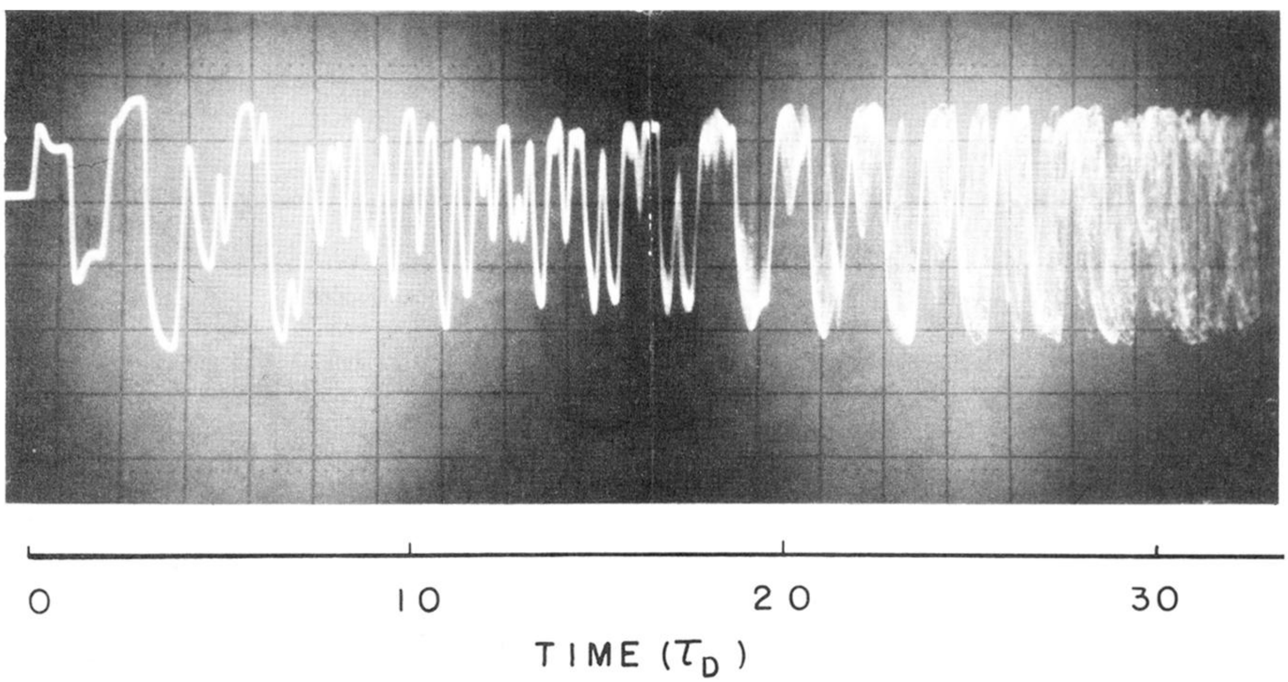

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