Ground-state energy of the one-dimensional gravitational gas

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We derive inequalities for the ground-state energy of N bosons or fermions in the two-body confining potential $V(x,y)=\gamma|x-y|$. We comment on why the results on this problem, derived previously, are incorrect.

In a paper,[1] Muriel claimed that the quantum problem of N bosons in the confining potential $V(x,y)=\gamma|x-y|$ was exactly solvable. Its solution is relevant to studying stellar dynamics of halo stars and also to the large-N limit of one-dimensional models of baryons with N quarks. This result was further quoted in a review article by Perelomov[2] as one of a few exactly solvable N-particle problems.

In this Comment we derive inequalities for the ground-state energy which disagree with the value obtained by Muriel.[1] The Schrödinger equation considered in Ref. 1 is

$$\left[-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{d^2}{dx_i^2}+\gamma\sum_{i<j}|x_i-x_j|\right]\psi=E(N)\psi \ . \tag{1}$$

We shall set a lower bound on the eigenvalues of this equation. Let us first find a solution for $N=2$ in terms of the Airy function Ai. By introducing $z=(m^{1/3}\gamma^{1/3}/\hbar^{2/3})\times(x_2-x_1)$, Eq. (1) in the c.m. system is reduced to

$$\left[-\frac{d^2}{dz^2}+|z|-\frac{m^{1/3}}{\hbar^{2/3}\gamma^{2/3}}E(2)\right]\psi=0 \ . \tag{2}$$

Equation (2) represents two equations, the one for $z>0$ and the other for $z<0$. Matching the derivative of the solutions, we obtain for the lowest eigenvalue

$$E_{\min}(2)=\frac{\hbar^{2/3}\gamma^{2/3}}{m^{1/3}}\mu_0 \ , \tag{3}$$

where μ_0 is the lowest zero defined by $\mathrm{Ai}'(-\mu)=0$. The value (3) is different from Muriel's value for $N=2$. Now we can proceed to obtain the Dyson-Lenard[3] inequality for the ground-state energy of the N-particle system. Rewriting the Hamiltonian of the system as

$$\begin{aligned} H&=\sum_{i=1}^{N}\frac{p_i^2}{2m}+\gamma\sum_{i<j}^{N}|x_i-x_j| \ ,\\ &=\sum_{i<j}^{N}\left[\frac{p_i^2}{2m(N-1)}+\frac{p_j^2}{2m(N-1)}+\gamma|x_i-x_j|\right]\\ &\equiv\sum_{i<j}^{N}H_{ij} \ , \end{aligned} \tag{4}$$

we obtain a lower bound in the form

$$E_{\min}(N)=\inf\langle H\rangle \geqslant \sum_{i<j}^{N}\inf\langle H_{ij}\rangle \ . \tag{5}$$

By using Eq. (3) we find that

$$\inf\langle H_{ij}\rangle=\mu_0\frac{\hbar^{2/3}\gamma^{2/3}}{m^{1/3}(N-1)^{1/3}} \tag{6}$$

and therefore the lower bound is

$$E_{\min}(N)\geqslant\frac{N(N-1)}{2}E_{\min}(2)=\frac{N(N-1)^{2/3}}{2}\mu_0\frac{\hbar^{2/3}\gamma^{2/3}}{m^{1/3}} \ . \tag{7}$$

For large N, the leading term behaves as

$$E_{\min}(N)\geqslant N^{5/3}\frac{\mu_0}{2}\left[\frac{\gamma^2\hbar^2}{m}\right]^{1/3}=0.51N^{5/3}\left[\frac{\gamma^2\hbar^2}{m}\right]^{1/3} \ . \tag{8}$$

This result clearly disagrees with the value obtained by Muriel[1] for the bosonic case, $E_{\mathrm{Muriel}}=0.49N^{5/3}(\gamma^2\hbar^2/m)^{1/3}$.

Let us comment on why the method used in Ref. 1 for solving Eq. (1) is not correct. If we assume the Baxter formula

$$\sum_{i<j}^{N}|x_i-x_j|=-\sum_{j=1}^{N}(N-2j+1)x_j \ , \tag{9}$$

which is valid only in the region $x_1<x_2<x_3<\cdots<x_N$, we have to solve the Schrödinger equation with an appropriate boundary condition. If we use the factorization assumption

$$\psi(x_1,\ldots,x_N)=\psi_1(x_1)\cdots\psi_N(x_N) \ ,$$

we have to solve the Schrödinger equation for $\psi(x_i)$ with the condition $x_{i-1}<x_i<x_{i+1}$. From here it follows that ϵ_i will be a function of x_{i-1} and x_{i+1}, $\epsilon_i=\epsilon_i(x_{i-1},x_{i+1})$ and $\psi_i=\psi_i(x_{i-1},x_i,x_{i+1})$. This contradicts the factorization assumption. The solutions of Eq. (1) are not factorizable. This is also clear from the (c.m.) solution for $N=2$:

$$\psi(x_1,x_2)\sim\mathrm{Ai}((m\gamma/\hbar^2)^{1/3}|x_2-x_1|-\mu_0) \ . \tag{10}$$

Although the model is not exactly solvable, in the large-N limit we can find an approximate value for the ground-state energy. We shall use the Bohm-Pines[4] collective field as elaborated by Jevicki and Sakita.[5] The N-particle system in the large-N limit is described by the particle density $\rho(x)$. The energy functional is given by

$$E(\rho)=\frac{\hbar^2}{8m}\int\frac{\rho'(x)^2}{\rho(x)}dx+\frac{\gamma}{2}\int\rho(x)|x-y|\rho(y)dx\,dy \ , \tag{11}$$

where the first term is a Weizsäcker term[6] due to the kinetic energy of the particles. The ground-state energy of the system is obtained by minimizing the functional (11) with the condition

$$\int\rho(x)dx=N \ . \tag{12}$$

It is not possible to obtain the solution of $\delta E(\rho)/\delta\rho=\lambda$ analytically (λ is a Lagrange multiplier) and therefore we investigate this problem numerically.

We have found that the numerical bounds on the ground-state energy are

$$0.5095N^{5/3}\left(\frac{\gamma^2\hbar^2}{m}\right)^{1/3} < E_{\min}(N) < 0.5239N^{5/3}\left(\frac{\gamma^2\hbar^2}{m}\right)^{1/3} . \tag{13}$$

It is also clear from our numerical investigation that the value from Ref. 1 is incorrect because it lies below the lower bound.

A similar analysis can be extended to fermions interacting via the gravitational potential. Rearranging the Hamiltonian[7]

$$H=\sum_{i=1}^{N}\frac{p_i^2}{2m}+\frac{\gamma}{2}\sum_{i\neq j}^{N}|x_i-x_j| ,$$
$$=\sum_{i=1}^{N}\left[\sum_{j\neq i}^{N}\left(\frac{p_j^2}{2(N-1)m}+\frac{\gamma}{2}|x_i-x_j|\right)\right]=\sum_{i=1}^{N}H_i , \tag{14}$$

we have written H as a sum of N similar Hamiltonians, each representing $(N-1)$ independent particles $j\neq i$ in the field of a fixed particle (namely, the ith particle). The ground-state energy of H_i is given by distributing $N-1$ fermions over the lowest $N-1$ levels of the Hamiltonian

$$H=\frac{p^2}{2m(N-1)}+\frac{\gamma}{2}|x| . \tag{15}$$

The eigenfunctions are Airy functions and the spectrum is given by

$$E_{2n}=\left[\frac{\hbar^2}{2m(N-1)}\right]^{1/3}\left(\frac{\gamma}{2}\right)^{2/3}\mu_n , \tag{16}$$

$$\mathrm{Ai}'(-\mu_n)=0, \quad \text{for } n=0,1,2,\ldots , \tag{17}$$

for even levels, and

$$E_{2n+1}=\left[\frac{\hbar^2}{2m(N-1)}\right]^{1/3}\left(\frac{\gamma}{2}\right)^{2/3}\tilde{\mu}_n , \tag{18}$$

$$\mathrm{Ai}(-\tilde{\mu}_n)=0, \quad n=0,1,2,\ldots , \tag{19}$$

for odd levels. Note the difference between Eqs. (17) and (19), which is due to matching the solutions for even and odd levels. Now we can write the bounds

$$E_{\min}(N)\geqslant\frac{N}{2}\left[\frac{\hbar^2}{m(N-1)}\right]^{1/3}\gamma^{2/3}\sum_{n=0}^{(N-3)/2}(\tilde{\mu}_n+\mu_n) \tag{20}$$

for N odd, and

$$E_{\min}(N)\geqslant\frac{N}{2}\left[\frac{\hbar^2}{m(N-1)}\right]^{1/3}\gamma^{2/3}\left[\sum_{n=0}^{(N-2)/2}\mu_n+\sum_{n=0}^{(N-4)/2}\tilde{\mu}_n\right] \tag{21}$$

for N even. These two formulas have the same large-N behavior. We can perform an approximate evaluation of the sums in Eqs. (20) and (21) by using an asymptotic expansion of Airy functions. The zeros of these functions are

$$\tilde{\mu}_n=\left(\frac{3\pi}{2}\right)^{2/3}(n+\tfrac{3}{4})^{2/3} , \tag{22}$$

$$\mu_n=\left(\frac{3\pi}{2}\right)^{2/3}(n+\tfrac{1}{4})^{2/3} . \tag{23}$$

Equations (22) and (23) are approximately valid for large n, but these are also a good approximation for small n (the order of a few percent).

Using the Euler-MacLauren summation formula, we obtain the lower bound for the system of N fermions:

$$E_{\min}(N)\geqslant N^{7/3}\frac{\pi^{2/3}3^{5/3}}{2^{7/3}\times 5}\left(\frac{\hbar^2\gamma^2}{m}\right)^{1/3} ,$$
$$=0.531N^{7/3}\left(\frac{\hbar^2\gamma^2}{m}\right)^{1/3} . \tag{24}$$

Let us mention that the problem of the two fermions in the confining potential can be solved exactly, and the result for $N=2$ is

$$E_{\min}(N=2)=\left(\frac{\hbar^2\gamma^2}{m}\right)^{1/3}\tilde{\mu}_0 . \tag{25}$$

Equation (25) also disagrees with Muriel's value for fermions.

In spite of the fact that the problem is probably not exactly solvable, the leading term in the large-N expansion for the ground-state energy of confined fermions can be determined exactly by the collective-field method.[8] By minimizing the functional

$$E(\rho)=\frac{\pi^2\hbar^2}{6m}\int\rho^3(x)dx+\frac{\gamma}{2}\int\rho(x)|x-y|\rho(y)dx\,dy , \tag{26}$$

we obtain

$$E_0(N)=\frac{9}{28}\left(\frac{\sqrt{3}}{\pi}\right)^{1/3}[\Gamma(\tfrac{2}{3})]^3N^{7/3}\left(\frac{\hbar^2\gamma^2}{m}\right)^{1/3} ,$$
$$=0.654N^{7/3}\left(\frac{\hbar^2\gamma^2}{m}\right)^{1/3} , \tag{27}$$

in agreement with the bound given by Eq. (24).

From the results given by Eqs. (8) and (24) for both systems, bosonic and fermionic, we may conclude that these systems are not extensive and therefore cannot be treated thermodynamically.

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[1]A. Muriel, Phys. Rev. A **15**, 341 (1977).
[2]A. M. Perelomov, Fiz. Elem. Chastits At. Yadra **10**, 850 (1979) [Sov. J. Part. Nucl. **10**, 336 (1979)].
[3]F. J. Dyson and A. Lenard, J. Math. Phys. **8**, 423 (1967).
[4]D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953).
[5]A. Jevicki and B. Sakita, Nucl. Phys. **B165**, 511 (1980).
[6]C. F. von Weizsäcker, Z. Phys. **96**, 431 (1935).
[7]J.-M. Lévy-Leblond, J. Math. Phys. **10**, 806 (1969).
[8]I. Andrić, V. Bardek, and A. Mikelić, Phys. Lett. **101A**, 376 (1984).