PHYSICAL REVIEW A VOLUME 30, NUMBER 6 DECEMBER 1984

Sum rules and static local-field corrections of electron liquids in two and three dimensions

Naoki Iwamoto*
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106
(Received 23 July 1984)

The dielectric functions of electron liquids which take into account short-range electron-electron correlations via the static local-field corrections are examined in the light of the frequency-moment sum rules. The formulation is given for degenerate as well as classical electron liquids in arbitrary (d) spatial dimensions, which is suitable for comparison between the two- and three-dimensional cases. By using the virial equations of state it is shown that such dielectric functions cannot satisfy the compressibility sum rule and the third-frequency-moment sum rule simultaneously. In the degenerate case, the plasmon, single-pair, and multipair contributions to the sum rules are analyzed, and the reason for this incompatibility is discussed.

I. INTRODUCTION

In the study of the dielectric properties of electron liquids, several exact relations which do not require a small value of the coupling constant in atomic units are known.[1] Such relations are useful since the perturbation theory is not reliable in the strong coupling regime, where the dimensionless coupling constant is larger than unity. The frequency moment sum rules provide some of the exact boundary conditions which the dielectric response function of electron liquids must satisfy.[1–4] Not only are these sum rules used in testing the validity of the various theoretical models but are powerful as guiding principles to find satisfactory response functions in the theoretical formulation of electron liquids.[5–12] In this paper we consider the approximation scheme beyond the random-phase approximation (RPA) in which short-range electron-electron correlations are taken into account via the *static* local-field correction $G(q)$. Physically, the function $G(q)$ may be interpreted as the screening charge density (in units of the electronic charge) in the Fourier space, and expresses the degree to which the dielectric response deviates from the RPA.[12(a)] There are at least four cases in the strong coupling regime which are experimentally or observationally accessible: (1) degenerate case in three dimensions (bulk-metallic electrons),[13(a),13(b)] (2) degenerate case in two dimensions (silicon-inversion layers),[13(c)] (3) classical case in three dimensions (ions inside large planets, white dwarfs, and neutron stars; inertially confined thermonuclear fusion plasmas),[13(b),13(d)] and (4) classical case in two dimensions (electrons on the surface of liquid helium).[13(b)–13(d)] In all these cases the screened response function in such a scheme is assumed to have the form

$$\chi_d^{sc}(q,\omega)=\frac{\chi_d^0(q,\omega)}{1+v_d(q)G(q)\chi_d^0(q,\omega)} \quad \text{(model)}\,, \tag{1.1a}$$

where the spatial dimensionality is indicated by the suffix d, $v_d(q)$ is the Fourier transformed Coulomb potential e^2/r, and $\chi_d^0(q,\omega)$ is taken to be the Lindhard response function $\chi_d^L(q,\omega)$ for the degenerate case and the Vlasov response function $\chi_d^V(q,\omega)$ for the classical case. By definition, the dielectric function is given by[14]

$$\epsilon_d(q,\omega)=1-v_d(q)\chi_d^{sc}(q,\omega)\,. \tag{1.1b}$$

Various approximate schemes lead to a screened response function of the form (1.1a). For degenerate electron liquids in three dimensions, many models since the early works by Hubbard[15] and by Singwi, Tosi, Land, and Sjölander[5] (STLS) reduce to this form.[13(a),13(b)] In two dimensions, Jonson studied the STLS scheme.[16] For classical electron liquids there are also many model screened response functions of this type.[13(b),13(d)] We examine the dielectric function (1.1b) with the screened response function (1.1a) in the light of the compressibility sum rule and the third frequency moment sum rule. We find that the form (1.1) cannot satisfy both sum rules simultaneously in the long wavelength limit. Using the virial equation of state, we explicitly give two constraints on the static local-field correction from these sum rules in terms of the dimensionless coupling constant [r_s or Γ of (2.3) and (6.6), respectively] derivative of the interaction energy, and show that these two constraints are incompatible. We also consider the weak coupling limit, in particular, and express these two constraints in the form of the coupling constant (r_s or Γ) expansion, using the RPA equation of state.

For degenerate electron liquids in three dimensions, a similar conclusion has been reached by Vaishya and Gupta,[17] who show that the two requirements on $G(q)$ from these sum rules lead to the violation of Ferrell's condition on the ground-state energy.[18] However, our form is simpler and more transparent than theirs because of the use of the virial theorem. Also Kugler[6(b)] has studied the constraints from these sum rules on the *dynamic* local-field correction $G(q,\omega)$.

In the degenerate case in two and three dimensions, we further discuss the reason for the impossibility of satisfying the two constraints simultaneously by analyzing the plasmon, single pair, and multipair contributions to the sum rules.

The paper is organized as follows. We begin with degenerate electron liquids. For the sake of generality we shall formulate the problem in an arbitrary number of (d)

spatial dimensions before reducing to two and three dimensions. In Sec. II, we give definitions of various physical quantities in degenerate electron liquids in d dimensions. In Secs. III and IV, we derive the two constraints on the static local-field correction from the two sum rules in the long-wavelength limit, and give the expressions of these constraints in the high-density limit. In Sec. V, we analyze the sum rule contributions of three types of density fluctuation excitations in the long-wavelength limit, and discuss the reason why the two constraints are different. Then we consider the classical electron liquids in Sec. VI. Summary and discussions are given in Sec. VII. In the appendixes, we give the calculational details of the exchange energy, the frequency moment sum rules for the Lindhard function, together with the plasmon dispersion in the RPA and from the ω^3 sum rule in d dimensions, and discuss the plasmon dispersion for classical electron liquids.

II. DEGENERATE ELECTRON LIQUIDS IN d DIMENSIONS

Consider a system of electrons of total number N at zero temperature in a d-dimensional region $V=L^d$, interacting with the Coulomb potential e^2/r. Note that V actually stands for the area for $d=2$. A uniform neutralizing positive charge background is assumed. In this section we define fundamental physical quantities for this system.

(i) *Mean particle distance* r_0: The mean-particle distance is defined as

$$n\int_{r<r_0} d^dr = V_d r_0^d n = 1 , \tag{2.1}$$

where d^dr is the volume element in d dimensions, $V_d \equiv S_d/d = \pi^{d/2}/\Gamma(\frac{1}{2}d+1)$ is the volume of the d-dimensional sphere of unit radius, and $n=N/V$, which is the volume number density for $d=3$ and the area number density for $d=2$. Thus,[19]

$$r_0 = \pi^{-1/2}[\Gamma(\tfrac{1}{2}d+1)]^{1/d} n^{-1/d} . \tag{2.2}$$

(ii) *Dimensionless coupling constant* r_s: The ratio of the mean particle distance to the Bohr radius $a_B=\hbar^2/me^2$ gives the dimensionless coupling constant

$$r_s \equiv r_0/a_B = \pi^{-1/2}[\Gamma(\tfrac{1}{2}d+1)]^{1/d} n^{-1/d} me^2/\hbar^2 . \tag{2.3a}$$

Thus,

$$r_s = \begin{cases} (3/4\pi)^{1/3} n^{-1/3} me^2\hbar^{-2}, & d=3 \qquad (2.3b) \\ \pi^{-1/2} n^{-1/2} me^2\hbar^{-2}, & d=2 . \qquad (2.3c) \end{cases}$$

(iii) *Fermi wave number* q_F: The Fermi wave number is defined as a sum over wave vector q and spin σ of the electron states in the Fermi sphere; viz.,

$$\sum_{\substack{\vec{q},\sigma \\ |\vec{q}|<q_F}} = 2V(2\pi)^{-d} V_d q_F^d = N , \tag{2.4}$$

so that

$$q_F = 2^{1-(1/d)} \pi^{1/2} [\Gamma(\tfrac{1}{2}d+1)]^{1/d} n^{1/d} , \tag{2.5a}$$

$$q_F = \begin{cases} (3\pi^2 n)^{1/3}, & d=3 \qquad (2.5b) \\ (2\pi n)^{1/2}, & d=2 . \qquad (2.5c) \end{cases}$$

The Fermi wave number and the mean particle distance are related:

$$q_F = 1/\alpha_d r_s a_B , \tag{2.6}$$

where

$$\alpha_d = 2^{(1/d)-1}[\Gamma(\tfrac{1}{2}d+1)]^{-2/d} , \tag{2.7a}$$

$$\alpha = \begin{cases} (4/9\pi)^{1/3}, & d=3 \qquad (2.7b) \\ 2^{-1/2}, & d=2 . \qquad (2.7c) \end{cases}$$

(iv) *Fourier component of the Coulomb potential* $v_d(q)$: Using the formula,[20]

$$\int d^dr\, f(\vec{r}\cdot\vec{r}_1) = S_{d-1}\int_0^\infty dr\, r^{d-1} \int_0^\pi d\vartheta (\sin\vartheta)^{d-2} f(r^2, r_1 r\cos\vartheta) \tag{2.8}$$

one obtains

$$v_d(q) \equiv \int d^dr \frac{e^2}{r} e^{-i\vec{q}\cdot\vec{r}} = 2^{d-1}\pi^{(d-1)/2}\Gamma(\tfrac{1}{2}(d-1)) e^2 q^{1-d} \equiv Q_q e^2 q^{1-d} , \tag{2.9a}$$

$$v(q) = \begin{cases} 4\pi e^2/q^2, & d=3 \qquad (2.9b) \\ 2\pi e^2/q, & d=2 . \qquad (2.9c) \end{cases}$$

(v) *Free-particle kinetic energy per particle* $\epsilon_{\rm kin}$ $(=\langle E_{\rm kin}\rangle_0)$ is proportional to $n^{2/d}$ as

$$\epsilon_{\rm kin} \equiv \sum_{\substack{\vec{q},\sigma \\ |\vec{q}|<q_F}} \frac{\hbar^2 q^2}{2m}\frac{1}{N} = \frac{d}{d+2}\frac{\hbar^2 q_F^2}{2m} = \frac{d}{d+2}\frac{1}{\alpha^2 r_s^2}\,{\rm Ry} \tag{2.10a}$$

$$\epsilon_{\rm kin} = \begin{cases} \dfrac{3}{5}\dfrac{1}{\alpha^2 r_s^2}\,{\rm Ry}, & d=3 \qquad (2.10b) \\ \dfrac{1}{r_s^2}\,{\rm Ry}, & d=2 \qquad (2.10c) \end{cases}$$

where 1 Ry $=e^2/2a_B$.

(vi) *Hartree-Fock exchange energy per particle* $\epsilon_{\rm ex}$ is proportional to $n^{1/d}$ as

$$\epsilon_{\rm ex} \equiv -\frac{1}{VN}\sum_{\vec{q}} v_d(q) \sum_{\substack{\vec{k} \\ |\vec{k}|<q_F,\, |\vec{k}+\vec{q}|<q_F}} 1 = -\frac{4}{\pi}\frac{d}{d^2-1}\frac{1}{\alpha r_s}\,{\rm Ry} , \tag{2.11a}$$

which reduces to

$$\epsilon_{\rm ex} = \begin{cases} -(3/2\pi)/\alpha r_s \text{ Ry}, & d=3 \quad (2.11\text{b}) \\ -8\sqrt{2}/3\pi r_s \text{ Ry}, & d=2 \quad (2.11\text{c}) \end{cases}$$

where the results in Eqs. (2.11b) and (2.11c) reproduce those given, respectively, in Refs. 1 and 21. The derivation is given in Appendix A.

(vii) *Fermi-Thomas wave number* $q_{\rm FT}$:

$$q_{\rm FT} \equiv \left[\frac{dQ_d ne^2}{2E_F} \right]^{1/(d-1)} = \left[\frac{d2^{d-2}\pi^{(d-1)/2}\Gamma(\frac{1}{2}(d-1))ne^2}{E_F} \right]^{1/(d-1)}, \tag{2.12a}$$

$$q_{\rm FT} = \begin{cases} \left[\dfrac{6\pi ne^2}{E_F} \right]^{1/2}, & d=3 \quad (2.12\text{b}) \\ \dfrac{2\pi ne^2}{E_F}, & d=2 \quad (2.12\text{c}) \end{cases}$$

where $E_F = \hbar^2 q_F^2/2m$ is the Fermi energy.

The relation of the Fermi-Thomas wave number to the Fermi wave number is

$$q_{\rm FT} = \left[\frac{2\Gamma(\frac{1}{2}(d-1))}{\pi^{1/2}\Gamma(\frac{1}{2}d)} \right]^{1/(d-1)} (\alpha r_s)^{1/(d-1)} q_F , \tag{2.13a}$$

$$q_{FT}/q_F = \begin{cases} (4\alpha r_s/\pi)^{1/2}, & d=3 \quad (2.13\text{b}) \\ \sqrt{2} r_s, & d=2 . \quad (2.13\text{c}) \end{cases}$$

The meaning of this definition becomes clear later [cf. Eqs. (3.2b) and (3.3)].

(viii) *Plasma frequency* $\omega_p(q)$:

$$\omega_p^2(q) \equiv (nq^2/m) v_d(q) = 2^{(d-1)}\pi^{(d-1)/2}\Gamma(\tfrac{1}{2}(d-1)) ne^2 q^{3-d}/m , \tag{2.14a}$$

$$\omega_p^2(q) = \begin{cases} 4\pi ne^2/m, & d=3 \quad (2.14\text{b}) \\ 2\pi ne^2 q/m, & d=2 . \quad (2.14\text{c}) \end{cases}$$

(ix) *Isothermal compressibility* κ_T: The isothermal compressibility is defined as

$$\frac{1}{\kappa_T} = -V \left[\frac{\partial p}{\partial V} \right]_{T,N} = n \left[\frac{\partial p}{\partial n} \right]_{T,N} , \tag{2.15a}$$

where the pressure may be obtained from the ground-state energy E ($\equiv \epsilon N$),

$$p = -\left[\frac{\partial E}{\partial V} \right]_N = n^2 \left[\frac{\partial \epsilon}{\partial n} \right]_N . \tag{2.16a}$$

Changing the variable from n to $r_s \propto n^{-1/d}$ [Eq. (2.3a)], one obtains

$$p = -\frac{1}{d} n r_s \frac{\partial \epsilon}{\partial r_s} \tag{2.16b}$$

and

$$\frac{1}{\kappa_T} = -\frac{1}{d^2} n r_s \left[(d-1)\frac{\partial \epsilon}{\partial r_s} - r_s \frac{\partial^2 \epsilon}{\partial r_s^2} \right] . \tag{2.15b}$$

Writing the ground-state energy per particle as

$$\epsilon = \epsilon_{\rm kin} + \epsilon_{\rm ex} + \epsilon_c ,$$

and using Eqs. (2.10a) and (2.11a), one obtains

$$\frac{1}{\kappa_T} = \frac{1}{d^2} n r_s \left[\frac{2d}{\alpha^2 r_s^3} - \frac{4d}{\pi(d-1)} \frac{1}{\alpha r_s^2} - (d-1)\frac{\partial \epsilon_c}{\partial r_s} + r_s \frac{\partial^2 \epsilon_c}{\partial r_s^2} \right] , \tag{2.15c}$$

where ϵ_c is the correlation energy per particle in units of rydbergs. The isothermal compressibility of a free Fermi gas κ_T^0 is

$$\frac{1}{\kappa_T^0} = \frac{2}{d} n \frac{1}{\alpha^2 r_s^2} . \tag{2.17}$$

From Eqs. (2.15c) and (2.17) one obtains

$$1 - \frac{\kappa_T^0}{\kappa_T} = \frac{2}{(d-1)\pi} \alpha r_s + \frac{d-1}{2d} \alpha^2 r_s^3 \frac{\partial \epsilon_c}{\partial r_s} - \frac{1}{2d} \alpha^2 r_s^4 \frac{\partial^2 \epsilon_c}{\partial r_s^2} . \tag{2.18}$$

The isothermal compressibility may be expressed in terms of the isothermal sound velocity,

$$s = \left[\frac{1}{m} \left[\frac{\partial p}{\partial n} \right]_{T,N} \right]^{1/2} , \tag{2.19}$$

as

$$1/\kappa_T = mns^2 . \tag{2.20a}$$

In particular, for a free Fermi gas,

$$1/\kappa_T^0 = mns_0^2 , \tag{2.20b}$$

where $s_0 = v_F/\sqrt{d}$ with $v_F = \hbar q_F/m$ the Fermi velocity.

(x) *Lindhard function* $\chi_d^L(q,\omega)$: The Lindhard function is defined as[22]

$$\chi_d^L(q,\omega) \equiv -\frac{1}{\hbar V} \sum_{\vec{p},\sigma} n_{\vec{p}\sigma}(1 - n_{\vec{p}+\vec{q},\sigma}) \left[\frac{1}{\omega + \omega_{\vec{p}\vec{q}} + i0} - \frac{1}{\omega - \omega_{\vec{p}\vec{q}} + i0} \right] , \tag{2.21}$$

where $V = L^d$ is the normalization volume, $n_{\vec{p}\sigma}$ is the Fermi distribution function, and $\omega_{\vec{p}\vec{q}} = \hbar\vec{q}\cdot\vec{p}/m + \hbar q^2/m$. Its static long-wavelength behavior is

$$\lim_{q\to 0}\chi_d^L(q,0) = -n^2\kappa_T^0 = -\frac{n}{ms_0^2}\,, \tag{2.22}$$

where κ_T^0 is given by Eqs. (2.17) and (2.20b). At high frequencies, it has the asymptotic form

$$\lim_{\omega\to\infty}\chi_d^L(q,\omega) = \sum_{j=1}^{\infty}\frac{L_{2j-1}(q)}{\omega^{2j}}\,, \tag{2.23}$$

where

$$L_{2j-1}(q) = \frac{nq^2}{m}\sum_{l=0}^{j-1}\frac{2^{2l}}{\pi^{1/2}}\frac{(2j-1)!}{(2j-1-2l)!(2l)!}\frac{\Gamma(\frac{1}{2}d+1)\Gamma(l+\frac{1}{2})}{\Gamma(\frac{1}{2}d+l+1)}[\omega_0(q)]^{2j-l-2}(E_F/\hbar)^l\,, \tag{2.24}$$

and $\omega_0(q) = \hbar q^2/2m$. The derivation of Eq. (2.24) is given in Appendix B. The first few moments are

$$L_1(q) = \frac{nq^2}{m}\,, \tag{2.25a}$$

$$L_3(q) = \frac{nq^2}{m}[\omega_0^2(q) + (12/d)\langle E_{\rm kin}\rangle_0\,\omega_0(q)/\hbar]\,, \tag{2.25b}$$

where $\langle E_{\rm kin}\rangle_0 = \epsilon_{\rm kin} = [d/(d+2)]E_F$ is the average kinetic energy per particle of a noninteracting system. In the RPA the dielectric function is given by

$$\epsilon_d^{\rm RPA}(q,\omega) = 1 - v_d(q)\chi_d^L(q,\omega) \quad \text{(model)}\,. \tag{2.26}$$

The RPA plasmon dispersion relation is given in Appendix C.

III. CONSTRAINT FROM THE COMPRESSIBILITY SUM RULE

The requirement that the response of the system to a static long-wavelength perturbation (a uniform compression) must give the compressibility which is obtained thermodynamically from the ground-state energy provides a constraint on the long-wavelength behavior of the static screened response function and hence on $G(q)$. The exact form of the static screened response function in the long-wavelength limit may be obtained by generalizing the argument in the three-dimensional case[1] and is given by

$$\lim_{q\to 0}\chi_d^{sc}(q,0) = -n^2\kappa_T\,, \tag{3.1}$$

where κ_T is the true isothermal compressibility. From Eqs. (1.1b) and (3.1), the exact form of the static dielectric function in the long-wavelength limit is

$$\lim_{q\to 0}\epsilon_d(q,0) = 1 + \frac{\omega_p^2(q)}{q^2s^2} \quad \text{(exact)} \tag{3.2a}$$

$$= 1 + \left[\frac{q_{\rm FT}}{q}\right]^{d-1}\frac{\kappa_T}{\kappa_T^0} \quad \text{(exact)}\,, \tag{3.2b}$$

where we have used Eq. (2.20a) and the definitions (2.12a) and (2.14a). In particular, the RPA dielectric function (2.26) in the long-wavelength limit has the form

$$\lim_{q\to 0}\epsilon_d^{\rm RPA}(q,0) = 1 + \left[\frac{q_{\rm FT}}{q}\right]^{d-1}\,, \tag{3.3}$$

which naturally defines the Fermi-Thomas wave number (2.12a). If one adopts a model for the screened response function [Eq. (1.1a)] one obtains a constraint on the static local-field correction from Eq. (3.1). Since from Eqs. (1.1a) and (2.22)

$$\lim_{q\to 0}\chi_d^{sc}(q,0) = -\frac{n^2\kappa_T^0}{1-(q_{\rm FT}/q)^{d-1}G(q)} \quad \text{(model)}\,, \tag{3.4}$$

the static local-field correction that satisfies the compressibility sum rule [which we shall denote by $G_{-1}(q)$][23] has the following long-wavelength behavior:

$$\lim_{q\to 0}G_{-1}(q) = \left[1-\frac{\kappa_T^0}{\kappa_T}\right]\left[\frac{q}{q_{FT}}\right]^{d-1} \tag{3.5a}$$

$$= \frac{\pi^{1/2}\Gamma(\frac{1}{2}d)}{2\alpha r_s\Gamma(\frac{1}{2}(d-1))}\left[1-\frac{\kappa_T^0}{\kappa_T}\right]\left[\frac{q}{q_F}\right]^{d-1}\,, \tag{3.5b}$$

where Eq. (2.13a) has been used. Using Eq. (2.18) one finally obtains

$$\lim_{q\to 0}G_{-1}(q) = \frac{\Gamma(\frac{1}{2}d)}{2\pi^{1/2}\Gamma(\frac{1}{2}(d+1))}\left[1+\frac{(d-1)^2}{4d}\pi\alpha r_s^2\frac{\partial\epsilon_c}{\partial r_s} - \frac{d-1}{4d}\pi\alpha r_s^3\frac{\partial^2\epsilon_c}{\partial r_s^2}\right]\left[\frac{q}{q_F}\right]^{d-1}\,. \tag{3.6a}$$

In particular,

$$\lim_{q\to 0}G_{-1}(q) = \begin{cases} \left[\frac{1}{4}+\frac{\pi}{12}\alpha r_s^2\frac{\partial\epsilon_c}{\partial r_s} - \frac{\pi}{24}\alpha r_s^3\frac{\partial^2\epsilon_c}{\partial r_s^2}\right]\left[\frac{q}{q_F}\right]^2, & d=3 \quad (3.6\text{b}) \\ \left[\frac{1}{\pi}+\frac{1}{8}\alpha r_s^2\frac{\partial\epsilon_c}{\partial r_s} - \frac{1}{8}\alpha r_s^3\frac{\partial^2\epsilon_c}{\partial r_s^2}\right]\left[\frac{q}{q_F}\right], & d=2\,. \quad (3.6\text{c}) \end{cases}$$

In Eqs. (3.6a)–(3.6c) ϵ_c is in rydberg units.

Finally we give the expressions for Eqs. (3.6b) and (3.6c) in the high-density limit. The correlation energy per particle in the high-density limit ($r_s \ll 1$) is

$$\epsilon_c = \begin{cases} \frac{2}{\pi^2}(1-\ln 2)\ln r_s - 0.094 + 0.018 r_s \ln r_s + O(r_s)\ \text{Ry},\ d=3 & (3.7a) \\ -0.38 - \frac{2\sqrt{2}}{3\pi}(10-3\pi) r_s \ln r_s + O(r_s)\ \text{Ry},\ d=2 & (3.7b) \end{cases}$$

where Eqs. (3.7a) and (3.7b) are the results given, respectively, in Refs. 24 and 25. Thus, at high densities, the long-wavelength behavior is the following:

$$\lim_{q\to 0} G_{-1}(q) = \begin{cases} \left[\frac{1}{4} + \frac{1}{4\pi}(1-\ln 2)\alpha r_s + O(r_s^2, r_s^2 \ln r_s)\right]\left[\frac{q}{q_F}\right]^2,\ d=3 & (3.8a) \\ \left[\frac{1}{\pi} - \frac{1}{12\pi}(10-3\pi) r_s^2 \ln r_s + O(r_s^2)\right]\left[\frac{q}{q_F}\right],\ d=2\ . & (3.8b) \end{cases}$$

We note that these local-field corrections remain finite even in the weak-coupling limit $r_s \to 0$. The Hartree-Fock exchange energy is responsible for these finite terms.

IV. CONSTRAINT FROM THE THIRD-FREQUENCY-MOMENT SUM RULE

The linear density-density response function,[1] generalized to d dimensions, $\chi_d(q,\omega)$, is related to the dielectric function as

$$\frac{1}{\epsilon_d(q,\omega)} = 1 + v_d(q)\chi_d(q,\omega)\ . \quad (4.1)$$

It may be expanded in the high-frequency limit as

$$\lim_{\omega\to\infty} \chi_d(q,\omega) = \sum_{l=1}^{\infty} \frac{\langle \omega^{2l-1}\rangle}{\omega^{2l}}\ . \quad (4.2)$$

The frequency moments

$$\langle \omega^{2l-1}\rangle \equiv -\int_{-\infty}^{\infty} \frac{d\omega}{\pi}\omega^{2l-1}\text{Im}\chi_d(q,\omega)\ , \quad (4.3)$$

may be calculated exactly by generalizing the procedure in three dimensions from the commutator algebra of the Hamiltonian and the density operator in the Heisenberg representation.[2] The fluctuation-dissipation theorem at zero temperature in d dimensions has the same form as in the three-dimensional case

$$S(q,\omega) = -\frac{2\hbar}{n}\text{Im}\chi_d(q,\omega),\quad \omega>0 \quad (4.4)$$

where $S(q,\omega)$ is the dynamic form factor.[26] Equation (4.1) or (4.4) may be used to obtain the alternative forms of Eq. (4.3)

$$\langle \omega^{2l-1}\rangle = \frac{2n}{\hbar}\int_0^{\infty} \frac{d\omega}{2\pi}\omega^{2l-1} S(q,\omega)\ , \quad (4.5)$$

$$\langle \omega^{2l-1}\rangle = -\frac{2}{v_d(q)}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\omega^{2l-1}\,\text{Im}\frac{1}{\epsilon_d(q,\omega)}\ . \quad (4.6)$$

The first moment is the f-sum rule

$$\langle \omega^1\rangle = \frac{nq^2}{m}\ , \quad (4.7)$$

and the third moment may be written in the form

$$\langle \omega^3\rangle = \frac{nq^2}{m}\{\omega_0^2(q) + (12/d)\langle E_{\text{kin}}\rangle\omega_0(q)/\hbar + \omega_p^2(q)[1-I_d(q)]\}\ , \quad (4.8)$$

where $\omega_0(q) = \hbar q^2/2m$, $\langle E_{\text{kin}}\rangle$ is the average kinetic energy per particle of an interacting system, and

$$I_d(q) \equiv -\frac{1}{N}\sum_{\vec{k}\,(\neq \vec{q},\vec{0})} \frac{\vec{k}\cdot\vec{q}}{q^2} K_d(\vec{q},\vec{k})[S(|\vec{q}-\vec{k}|)-1] \quad (4.9)$$

with

$$K_d(\vec{q},\vec{k}) \equiv \frac{\vec{k}\cdot\vec{q}}{q^2}(q/k)^{d-1} + \frac{(\vec{q}-\vec{k})\cdot\vec{q}}{q^2}\left[\frac{q}{|\vec{q}-\vec{k}|}\right]^{d-1}\ , \quad (4.10)$$

and $S(|\vec{q}-\vec{k}|)$ the static form factor. From Eqs. (4.1)–(4.3), (4.7), and (4.8) one obtains the exact high-frequency behavior of the dielectric function:

$$\lim_{\omega\to\infty} \epsilon_d(q,\omega) = 1 - \frac{\omega_p^2(q)}{\omega^2} - \frac{\omega_p^4(q)}{\omega^4}\left[\frac{\omega_0^2(q)}{\omega_p^2(q)} + \frac{12\langle E_{\text{kin}}\rangle\omega_0(q)}{\hbar\omega_p^2(q) d} - I_d(q)\right]\ . \quad (4.11)$$

The plasmon dispersion obtained from the ω^3 sum rule is given in Appendix E. On the other hand, the model [Eqs. (1.1a) and (1.1b)] has the high-frequency behavior

$$\lim_{\omega\to\infty} \epsilon_d(q,\omega)=1-\frac{\omega_p^2(q)}{\omega^4}-\frac{\omega_p^4(q)}{\omega^2}\left[\frac{\omega_0^2(q)}{\omega_p^2(q)}+\frac{12\langle E_{\text{kin}}\rangle_0\omega_0(q)}{\hbar\omega_p^2(q)d}-G(q)\right], \text{ (model)} \tag{4.12}$$

where Eqs. (2.23) and (2.25) have been used. One sees that the model Eq. (4.12) satisfies the f-sum rule. Let us denote the static local-field correction which satisfies the ω^3 sum rule by $G_3(q)$. Then, from Eqs. (4.11) and (4.12), one obtains

$$G_3(q)=I_d(q)-\frac{12\omega_0(q)}{\hbar\omega_p^2(q)d}(\langle E_{\text{kin}}\rangle-\langle E_{\text{kin}}\rangle_0)\ . \tag{4.13}$$

The long-wavelength limit of $I_d(q)$ is calculated in Appendix D and has the form

$$\lim_{q\to 0} I_d(q)=-\frac{(7-d)\pi^{1/2}\Gamma(\frac{1}{2}d)}{4(d+2)\Gamma(\frac{1}{2}(d-1))}\alpha r_s(q/q_F)^{d-1}\frac{\langle\mathscr{V}\rangle}{me^4/2\hbar^2}\ , \tag{4.14}$$

where

$$\langle\mathscr{V}\rangle=\frac{1}{2V}\sum_{\vec{q}} v_d(q)[S(q)-1]\ \left[=\frac{U_{\text{ex}}}{N}\right] \tag{4.15}$$

is the average potential energy per particle (which is also referred to as the excess internal energy per particle, U_{ex}/N, in the classical case) of an interacting system.

The kinetic and potential energies per particle which appear in Eqs. (4.13) and (4.14) may be expressed in terms of the correlation energy per particle ϵ_c by applying the virial theorem,[27] generalized to d dimensions:[28]

$$\langle E_{\text{kin}}\rangle=\langle E_{\text{kin}}\rangle_0-\frac{\partial}{\partial r_s}(r_s\epsilon_c)\ , \tag{4.16}$$

$$\langle\mathscr{V}\rangle=\epsilon_{\text{ex}}+\frac{1}{r_s}\frac{\partial}{\partial r_s}(r_s^2\epsilon_c)\ , \tag{4.17}$$

where ϵ_{ex} is the Hartree-Fock exchange energy per particle given in Eq. (2.11a). From Eqs. (2.11a), (4.13), (4.14), (4.16), and (4.17) one obtains

$$\lim_{q\to 0} G_3(q)=\frac{d(7-d)\Gamma(\frac{1}{2}d)}{4\pi^{1/2}(d+2)\Gamma(\frac{1}{2}(d+3))}\left[1+\frac{4(d-1)^2(d+1)}{2d(7-d)}\pi\alpha r_s\epsilon_c+\frac{(7d+5)(d^2-1)}{4d(7-d)}\pi\alpha r_s^2\frac{\partial\epsilon_c}{\partial r_s}\right](q/q_F)^{d-1}\ . \tag{4.18a}$$

In particular,

$$\lim_{q\to 0} G_3(q)=\begin{cases}\left[\dfrac{3}{20}+\dfrac{11}{20}\pi\alpha r_s\epsilon_c+\dfrac{13}{20}\pi\alpha r_s^2\dfrac{\partial\epsilon_c}{\partial r_s}\right](q/q_F)^2,\ d=3 & (4.18\text{b})\\[2ex] \left[\dfrac{5}{6\pi}+\dfrac{7}{8}\alpha r_s\epsilon_c+\dfrac{19}{16}\alpha r_s^2\dfrac{\partial\epsilon_c}{\partial r_s}\right](q/q_F),\ d=2\ . & (4.18\text{c})\end{cases}$$

In the high-density limit, one may use Eqs. (3.7a) and (3.7b) for ϵ_c to obtain

$$\lim_{q\to 0} G_3(q)=\begin{cases}\left[\dfrac{3}{20}+\dfrac{11}{10\pi}(1-\ln 2)\alpha r_s\ln r_s+O(r_s)\right](q/q_F)^2,\ d=3 & (4.19\text{a})\\[2ex] \left[\dfrac{5}{6\pi}-0.24r_s-\dfrac{33}{24\pi}(10-3\pi)r_s^2\ln r_s+O(r_s)\right](q/q_F),\ d=2\ . & (4.19\text{b})\end{cases}$$

Upon comparing Eqs. (3.6a) and (4.18a) [Eqs. (3.6b) and (4.18b) for $d=3$, and Eqs. (3.6c) and (4.18c) for $d=2$] one finds that the static local-field correction which satisfies the compressibility sum rule [Eqs. (3.6)] cannot satisfy the constraint from the ω^3 sum rule [Eqs. (4.18)]. In other words, the constraints which these sum rules impose are incompatible in uniquely determining a static local-field correction. This may be seen more clearly when one compares two constraints in the high-density limit [Eqs. (3.8a) and (4.19a) for $d=3$, and Eqs. (3.8b) and (4.19b) for $d=2$].

V. SUM RULE CONTRIBUTIONS OF DENSITY FLUCTUATION EXCITATIONS IN THE LONG-WAVELENGTH LIMIT

In this section we discuss the reason for $G_{-1}(q) \neq G_3(q)$ by analyzing the plasmon, single-pair, and multipair contributions to the sum rules in the long-wavelength limit. For this purpose, it is convenient to express Eq. (4.5) in terms of the matrix elements for the density operator:[1]

$$\langle \omega^{2l-1} \rangle = \frac{2}{\hbar V} \sum_n |(\rho_{\mathbf{q}}^{+})_{n0}|^2 (\omega_{n0})^{2l-1} , \tag{5.1}$$

where ω_{n0} is the excitation frequency. The plasmon, single-pair, and multipair contributions to the various physical quantities and sum rules are listed in Tables I–III for d, three, and two dimensions, respectively. The notations are the same as those in Ref. 1. The quantity $\langle \omega^{-1} \rangle'$ in these tables is defined as[29]

$$\langle \omega^{-1} \rangle' = \frac{2n}{\hbar} \int_0^\infty \frac{d\omega}{2\pi} \omega^{-1} S(q,\omega) \, |\epsilon_d(q,\omega)|^2 \tag{5.2a}$$

$$\equiv \frac{2}{\hbar V} \sum_n |(\rho_{\mathbf{q}}^{+})_{n0}|^2 (\omega_{n0})^{-1} |\epsilon_d(q,\omega_{n0})|^2 \tag{5.2b}$$

and the compressibility sum rule [Eq. (3.1)] yields

$$\langle \omega^{-1} \rangle' = \frac{n}{ms^2} = n^2 \kappa_T \; . \tag{5.3}$$

From these tables one sees the following similarities between the two- and three-dimensional cases.

(i) The plasmon contribution determines the static form factor in the long-wavelength limit.

(ii) The single-pair contribution exhausts the compressibility sum rule. The matrix element of the single-pair excitations reproduces Eq. (5.3) exactly.

(iii) The plasmon contribution exhausts the f-sum rule.

(iv) The plasmon and multipair excitations give major contributions to the ω^3 sum rule. One can identify the term $(nq^2/m)\omega_p^2(q)$ in Eq. (4.8) as coming from the lowest-order plasmon contribution with the excitation frequency $\omega_{n0} = \omega_p(q)$. One can further identify the terms proportional to $\langle E_{\text{kin}} \rangle$ and $I_d(q)$ in Eq. (4.8) as the higher-order plasmon contributions due to plasmon dispersion ($\propto q^4$) and the multipair contributions ($\propto q^4$).

On the other hand, the plasmon excitation frequency is wave number dependent, in general, and the dielectric screening property at the single-pair excitation frequency depends on the dimensionality, both of which lead to the change in the relative importance of the contributions from these three types of density fluctuation excitations to the sum rules.

(v) In three dimensions, multipair excitations contribute to the static form factor to order q^4 next to the plasmon contribution ($\propto q^2$), while in two dimensions single-pair excitations ($\propto q^3$) follow the plasmon contribution ($\propto q^{2/3}$).

(vi) In three dimensions, the single-pair contribution to the f-sum rule ($\propto q^6$) is negligible, while in two dimensions single-pair excitations contribute to the same order ($\propto q^4$) as multipair excitations.

(vii) In the ω^3 sum rule in two dimensions the multipair contribution is by only one power of q higher than the plasmon contribution.

Therefore, compared with the three-dimensional case, one sees somewhat different interplay among the three types of density fluctuation excitations in two dimensions. It is possible to identify further the origin of the terms in the ω^3 sum rule [Eq. (4.8)] by using a more accurate plasmon excitation frequency. First, let us use the RPA plasmon dispersion relation, given in Appendix C [Eq. (C2a)]. Then we obtain the plasmon contribution to the ω^3 sum rule

$$\langle \omega^3 \rangle_{\text{pl}}^{\text{RPA}} = \frac{nq^2}{m} \omega_p^2(q) \left[1 + \frac{12}{d+2} \left[\frac{E_F}{\hbar\omega_p(q)} \right]^2 (q/q_F)^2 \right] . \tag{5.4}$$

TABLE I. Matrix elements, excitation frequencies, and sum rule contributions of density fluctuation excitations in the long-wavelength limit in d dimensions.

		Plasmon	Single pair	Multipair
Matrix element	$(\rho_{\mathbf{q}}^{+})_{n0}$	$q[\hbar N/2m\omega_p(q)]^{1/2}$	$q^2/\omega_p^2(q)$	q^2
Excitation frequency	ω_{n0}	$\omega_p(q) \propto q^{(3-d)/2}$	qv_F	$\bar{\omega}$
Pauli-principle restriction		none	q/q_F	none
Dielectric function	$\epsilon(q,\omega_{n0})$	$\simeq 0$	$\omega_p^2(q)/s^2q^2 \propto q^{1-d}$	1
Static form factor	$S(q) = \sum_n \|(\rho_{\mathbf{q}}^{+})_{n0}\|^2 / N$	$\hbar q^2/2m\omega_p(q) \propto q^{(1+d)/2}$	$q^5/\omega_p^4(q) \propto q^{2d-1}$	q^4
Compressibility sum rule	$\langle \omega^{-1} \rangle'$	$\simeq 0$	$\frac{n}{ms^2} = n^2\kappa_T$	q^4
f-sum rule	$\langle \omega \rangle$	$\frac{nq^2}{m}$	$q^6/\omega_p^4(q) \propto q^{2d}$	q^4
ω^3 sum rule	$\langle \omega^3 \rangle$	$\frac{nq^2}{m}\omega_p^2(q) \propto q^{5-d}$	$q^8/\omega_p^4(q) \propto q^{2d+2}$	q^4

TABLE II. Same as Table I in three dimensions.

	Plasmon	Single pair	Multipair
Matrix element	$q(\hbar N/2m\omega_p)^{1/2}$	q^2	q^2
Excitation frequency	ω_p	qv_F	$\bar{\omega}$
Pauli-principle restriction	none	q/q_F	none
Dielectric function	$\simeq 0$	ω_p^2/s^2q^2	1
Static form factor	$\hbar q^2/2m\omega_p$	q^5	q^4
Compressibility sum rule	$\simeq 0$	$\frac{n}{ms^2}=n^2\kappa_T$	q^4
f-sum rule	$\frac{nq^2}{m}$	q^6	q^4
ω^3 sum rule	$\frac{nq^2}{m}\omega_p^2$	q^8	q^4

Since

$$\frac{12}{d+2}\left[\frac{E_F}{\hbar\omega_p(q)}\right]^2(q/q_F)^2=\frac{12}{d}\langle E_{\rm kin}\rangle_0\,\omega_0(q)/\hbar \tag{5.5}$$

the RPA plasmon dispersion accounts for a part of the term proportional to $\langle E_{\rm kin}\rangle$ in Eq. (4.8).

Second, we may use the plasmon dispersion from Eq. (4.11) to obtain

$$\langle\omega^3\rangle_{\rm pl}=\frac{nq^2}{m}\omega_p^2(q)\left[1+\frac{12\langle E_{\rm kin}\rangle\omega_0(q)}{\hbar\omega_p^2(q)d}-I_{d1}(q)\right]+\cdots, \tag{5.6}$$

where $I_{d1}(q)$ is the part of $I_d(q)$ which contributes to order q^{d-1} in the long-wavelength limit. One thus sees the plasmon contribution due to dispersion more clearly. One can further take the plasmon dispersion relation up to $O(q^4)$ in order to reproduce the term proportional to $\omega_0^2(q)$ in Eq. (4.8). However, it is not possible to distinguish the plasmon contribution from that of multipair excitations to order q^4 in the ω^3 sum rule. This is because $\langle E_{\rm kin}\rangle$ and $I_d(q)$ include both contributions. In other words, the plasmon dispersion relation which gives Eq. (5.6) already includes the contribution from multipair excitations through $\langle E_{\rm kin}\rangle$ and $I_d(q)$.

From the above analysis it is now clear why $G_{-1}(q)$ differs from $G_3(q)$: The single-pair excitations exhaust the compressibility sum rule, while the plasmon and multipair excitations contribute to the ω^3 sum rule. Therefore, $G_{-1}(q)$ incorporates the modification due to single-pair excitations in order to satisfy the compressibility sum rule. On the other hand, $G_3(q)$ incorporates the plasmon and multipair excitations in order to satisfy the ω^3 sum rule. Since $G_{-1}(q)$ and $G_3(q)$ take into account different types of density fluctuation excitations in this way, they are intrinsically different, and thus it is natural that they do not agree.

VI. CLASSICAL ELECTRON LIQUIDS

We now turn to classical electron liquids. The analysis proceeds in a manner similar to the degenerate case. We consider the dielectric function (1.1b) with a model screened response function (1.1a) in which $\chi_d^0(q,\omega)$ is taken to be the Vlasov response function,

$$\chi_d^V(q,\omega)=-\int d^dp\frac{1}{\omega-\vec{q}\cdot\vec{v}+i0}\vec{q}\cdot\frac{\partial f(\vec{p})}{\partial\vec{p}}, \tag{6.1a}$$

TABLE III. Same as Table I in two dimensions.

	Plasmon	Single pair	Multipair
Matrix element	$q[\hbar N/2m\omega_p(q)]^{1/2}$	$q^2/\omega_p^2(q)$	q^2
Excitation frequency	$\omega_p(q)\propto q^{1/2}$	qv_F	$\bar{\omega}$
Pauli-principle restriction	none	q/q_F	none
Dielectric function	$\simeq 0$	$\omega_p^2(q)/s^2q^2\propto q^{-1}$	1
Static form factor	$\hbar q^2/2m\omega_p(q)\propto q^{3/2}$	$q^5/\omega_p^4(q)\propto q^3$	q^4
Compressibility sum rule	$\simeq 0$	$\frac{n}{ms^2}=n^2\kappa_T$	q^4
f-sum rule	$\frac{nq^2}{m}$	$q^6/\omega_p^4(q)\propto q^4$	q^4
ω^3 sum rule	$\frac{nq^2}{m}\omega_p^2(q)\propto q^3$	$q^8/\omega_p^4(q)\propto q^6$	q^4

where

$$f(\vec{p})=n(2\pi m k_B T)^{-d/2}e^{-p^2/2mk_BT} \quad (6.2)$$

is the Maxwellian momentum distribution function with a temperature T, k_B the Boltzmann constant, and $\vec{v}=\vec{p}/m$. Introducing the plasma dispersion function[30,31]

$$W(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx\frac{xe^{-x^2/2}}{x-z-i0}\ , \quad (6.3)$$

one can rewrite Eq. (6.1a) as

$$\chi_d^V(q,\omega)=-\frac{n}{k_BT}W\left[\frac{\omega}{q(k_BT/m)^{1/2}}\right]. \quad (6.1b)$$

The function $W(z)$ has the following behavior:[30,31]

$$W(z)=\begin{cases} i(\pi/2)^{1/2}ze^{-z^2/2}+1-z^2+\frac{z^4}{3}-\cdots\ , & z\ll 1 \quad (6.4a)\\ i(\pi/2)^{1/2}ze^{-z^2/2}-\frac{1}{z^2}-\frac{3}{z^4}-\cdots\ , & z\gg 1\ . \quad (6.4b)\end{cases}$$

The RPA corresponds to setting $G(q)=0$ in Eq. (1.1a), so that

$$\epsilon_d^{\rm RPA}(q,\omega)=1+\frac{nv_d(q)}{k_BT}W\left[\frac{\omega}{q(k_BT/m)^{1/2}}\right]. \quad (6.5)$$

The RPA plasmon dispersion relation and the damping rate are given in Appendix C.

In classical electron liquids, the dimensionless coupling constant is defined in terms of the mean particle distance (2.2) as

$$\Gamma=e^2/r_0k_BT$$
$$=\pi^{1/2}[\Gamma(\tfrac{1}{2}d+1)]^{-1/d}n^{1/d}e^2(k_BT)^{-1}\ . \quad (6.6a)$$

Thus,

$$\Gamma=\begin{cases}(4\pi/3)^{1/3}n^{1/3}e^2(k_BT)^{-1}, & d=3 \quad (6.6b)\\ \pi^{1/2}n^{1/2}e^2(k_BT)^{-1}, & d=2\ . \quad (6.6c)\end{cases}$$

A. Compressibility sum rule

The pressure and isothermal compressibility (2.15a) may be expressed in terms of the coupling constant (Γ) derivative of the excess internal energy via the thermodynamic relation:[28]

$$p=nk_BT\left[1+\frac{1}{d}\frac{U_{\rm ex}}{Nk_BT}\right], \quad (6.7)$$

$$\frac{1}{\kappa_T}=nk_BT\left[1+\frac{1}{d}\frac{U_{\rm ex}}{Nk_BT}+\frac{1}{d^2}\Gamma\frac{d}{d\Gamma}\left[\frac{U_{\rm ex}}{Nk_BT}\right]\right], \quad (6.8a)$$

where the excess internal energy $U_{\rm ex}$ is defined as the difference between the internal energy and the kinetic energy $(\frac{1}{2}d)Nk_BT$ [cf. Eq. (4.15)]. In this equation $d/d\Gamma$ means differentiation with respect to the parameter Γ. The isothermal compressibility of an ideal gas is

$$\kappa_T^0=1/nk_BT\ , \quad (6.9)$$

so that

$$1-\frac{\kappa_T^0}{\kappa_T}=-\frac{1}{d}\frac{U_{\rm ex}}{Nk_BT}-\frac{1}{d^2}\Gamma\frac{d}{d\Gamma}\left[\frac{U_{\rm ex}}{Nk_BT}\right]. \quad (6.8b)$$

The exact form of the static screened response function is given by Eq. (3.1). The static dielectric function is then

$$\epsilon_d(q,0)=1+\frac{\omega_p^2(q)}{q^2s^2}\ \text{(exact)} \quad (6.10a)$$

$$=1+(q_D/q)^{d-1}\frac{\kappa_T}{\kappa_T^0}\ \text{(exact)}\ , \quad (6.10b)$$

where $\omega_p(q)$ is the plasma frequency [Eqs. (2.14)], s is the isothermal sound velocity [Eq. (2.19) whose ideal gas value is now $s_0=(k_BT/m)^{1/2}$], and q_D the Debye wave number defined by

$$q_D\equiv 2\pi^{1/2}[\Gamma(\tfrac{1}{2}(d-1))ne^2/k_BT]^{1/(d-1)}\ . \quad (6.11)$$

The Debye wave number is related to the plasma frequency as $\omega_p^2(q)=s_0^2q^2(q_D/q)^{d-1}$, and has the values $q_D=(4\pi ne^2/k_BT)^{1/2}$ for $d=3$, and $q_D=2\pi ne^2/k_BT$ for $d=2$. In particular, the static RPA dielectric function (6.5) has the form

$$\epsilon_d^{\rm RPA}(q,0)=1+(q_D/q)^{d-1}\ , \quad (6.12)$$

where we have used the Vlasov response function (6.1b) in the static limit [cf. (6.4a)],

$$\chi_d^V(q,0)=-\frac{n}{k_BT}\left[=-\frac{n}{ms_0^2}=-n^2\kappa_T^0\right]. \quad (6.13)$$

On the other hand, the model screened response function [(1.1a) and (6.1b)] in the static limit is

$$\chi_d^{sc}(q,0)=-\frac{n^2\kappa_T^0}{1-(q_D/q)^{d-1}G(q)}\ . \quad (6.14)$$

Let us denote the static local-field correction that satisfies the compressibility sum rule by $G_{-1}^T(q)$. Comparing Eqs. (3.1) and (6.14), one obtains the constraint

$$G_{-1}^{T}(q)=\left[1-\frac{\kappa_T^0}{\kappa_T}\right](q/q_D)^{d-1} \tag{6.15a}$$

$$=-\left[\frac{1}{d}\frac{U_{\text{ex}}}{Nk_BT}+\frac{1}{d^2}\Gamma\frac{d}{d\Gamma}\left[\frac{U_{\text{ex}}}{Nk_BT}\right]\right](q/q_D)^{d-1} . \tag{6.15b}$$

In the weak coupling (low-density, $\Gamma \ll 1$) limit, the excess internal energy is calculated to be

$$\frac{U_{\text{ex}}}{Nk_BT}=\begin{cases}-\frac{\sqrt{3}}{2}\Gamma^{3/2}-3\Gamma^3\left[\frac{3}{8}\ln(3\Gamma)+\frac{1}{2}\gamma-\frac{1}{3}\right]+\cdots, & d=3 \quad (6.16a)\\ 2\Gamma^2\left[\ln(2\Gamma)-\frac{1}{2}+\gamma\right]+\cdots, & d=2 \quad (6.16b)\end{cases}$$

where $\gamma=0.5772\ldots$ is the Euler constant; and Eqs. (6.16a) and (6.16b) are the results given, respectively, in Refs. 32 and 33. Using Eqs. (6.16a) and (6.16b), one obtains in the low-density limit

$$G_{-1}^{T}(q)=\begin{cases}\left[\left[\frac{\sqrt{3}}{4}\Gamma^{3/2}+\Gamma^3\left[\frac{3}{4}\ln(3\Gamma)+\gamma-\frac{13}{24}\right]\right]\right](q/q_D)^2, & d=3 \quad (6.17a)\\ -2\Gamma^2\left[\ln(2\Gamma)-\frac{1}{4}+\gamma\right](q/q_D), & d=2 . \quad (6.17b)\end{cases}$$

Contrary to the degenerate case, these local-field corrections vanish in the weak-coupling limit $\Gamma\to 0$.

B. Third-frequency-moment sum rule

In the classical case, the frequency moments may also be defined by Eqs. (4.2) and (4.3). The fluctuation-dissipation theorem now reads

$$S(q,\omega)=-\frac{2k_BT}{n\omega}\text{Im}\chi_d(q,\omega) , \tag{6.18}$$

for all ω, so that Eq. (4.5) is replaced by

$$\langle\omega^{2l-1}\rangle=\frac{n}{k_BT}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^{2l}S(q,\omega) , \tag{6.19}$$

while Eq. (4.6) still holds. The first two moments in the classical case have the same form (cf. Ref. 4 for $d=3$) as in the degenerate case [Eqs. (4.7) and (4.8)] except that the term proportional to $\omega_0^2(q)$ does not appear[34] in $\langle\omega^3\rangle$ and $\langle E_{\text{kin}}\rangle=\langle E_{\text{kin}}\rangle_0=(d/2)k_BT$. The long-wavelength limit of the function $I_d(q)$ [(4.9)] in Eq. (4.8) may be expressed as [cf. Eq. (D10)]

$$\lim_{q\to 0} I_d(q)=-\frac{7-d}{d(d+2)}\frac{U_{\text{ex}}}{Nk_BT}(q/q_D)^{d-1} , \tag{6.20}$$

where U_{ex}/N is identical to $\langle\mathscr{V}\rangle$ [cf. Eq. (4.15)]. The exact high-frequency behavior of the dielectric function is again given by Eq. (4.11) except that the term proportional to $\omega_0^2(q)$ is absent and $\langle E_{\text{kin}}\rangle=(\frac{1}{2}d)k_BT$. On the other hand, from Eqs. (1.1a), (1.1b), (6.1b), and (6.4b), the model dielectric function has the asymptotic behavior,[35]

$$\lim_{\omega\to\infty}\epsilon_d(q,\omega)=1-\frac{\omega_p^2(q)}{\omega^2}-\frac{\omega_p^4(q)}{\omega^4}\left[\frac{12\langle E_{\text{kin}}\rangle_0\,\omega_0(q)}{\hbar\omega_p^2(q)d}-G(q)\right] \quad (\text{model}) . \tag{6.21}$$

One sees that the model dielectric function satisfies the f-sum rule. Let us denote the static local-field correction that satisfies the ω^3 sum rule by $G_3^T(q)$. Then, comparing Eqs. (4.11) [with the term proportional to $\omega_0^2(q)$ omitted] and (6.21), one obtains the constraint from the ω^3 sum rule

$$G_3^T(q)=I_d(q) . \tag{6.22}$$

Thus, from Eq. (6.20) the constraint in the long-wavelength limit becomes

$$\lim_{q\to 0} G_3^T(q)=-\frac{7-d}{d(d+2)}\frac{U_{\text{ex}}}{Nk_BT}(q/q_D)^{d-1} . \tag{6.23}$$

In the low-density limit one may again use Eqs. (6.16a) and (6.16b) in Eq. (6.23) to obtain

$$\lim_{q\to 0} G_3^T(q) = \begin{cases} \left[\frac{2\sqrt{3}}{15}\Gamma^{3/2} + \frac{4}{5}\Gamma^3 \left[\frac{3}{8}\ln(3\Gamma) + \frac{1}{2}\gamma - \frac{1}{3} \right] + \cdots \right] (q/q_D)^2, \; d=3 & \text{(6.24a)} \\ -\frac{5}{4}\Gamma^2 \left[\ln(2\Gamma) - \frac{1}{2} + \gamma + \cdots \right] (q/q_D), \; d=2 . & \text{(6.24b)} \end{cases}$$

Upon comparing Eqs. (6.15b) and (6.23) [or (6.17a) and (6.24a) for $d=3$; (6.17b) and (6.24b) for $d=2$ in the low-density limit], one finds that

$$\lim_{q\to 0} G_{-1}^T(q) \neq \lim_{q\to 0} G_3^T(q) . \tag{6.25}$$

Therefore, for classical electron liquids the dielectric function which takes into account short-range electron-electron interactions via the static local-field correction in the form Eqs. (1.1b) with the Vlasov response function (6.1b) for $\chi_d^0(q,\omega)$ in Eq. (1.1a) cannot satisfy the compressibility sum rule and the third-frequency-moment sum rule simultaneously, as in the case of degenerate electron liquids.

VII. SUMMARY AND DISCUSSIONS

In summary, we have examined the dielectric functions which take into account short-range electron-electron correlations in terms of the static local-field corrections in the light of the compressibility sum rule and the third-frequency-moment sum rule for degenerate and classical electron liquids in d dimensions with special emphasis on $d=2$ and 3. We have shown, in a transparent fashion, that the requirements from these sum rules cannot be satisfied simultaneously in the long-wavelength limit by such dielectric functions. We, in particular, analyzed the sum rule contributions from three types of density fluctuation excitations in degenerate liquids and explained the incompatibility of those two requirements by noting that single-pair excitations contribute to the compressibility sum rule and plasmon and multipair excitations contribute to the ω^3 sum rule in the long-wavelength limit.

Here, it is worth mentioning the study of the two-dimensional degenerate electron liquids by Jonson,[16] who finds that the short-range correlation effects in the correlation energy and the pair correlation function are more pronounced in two dimensions than in three dimensions. As has been stressed in Sec. V, the change in spatial dimensionality (from $d=3$ to $d=2$, for example) affects the relative importance of the plasmon, single-pair, and multipair excitation contributions to various physical quantities. Therefore, the comparison of the systems in different dimensions provides, in principle, the possibility of allowing one to sort out the effects of each density fluctuation excitation, to a certain degree, separately.

ACKNOWLEDGMENTS

The author would like to thank Professor David Pines for helpful comments and encouragement, and the Aspen Center for Physics for hospitality. This work is supported in part by the National Science Foundation under Grants Nos. PHY-77-27084 (supplemented by funds from the U.S. National Aeronautics and Space Administration), PHY-80-25605, DMR-82-15128, and DMR-83-04213.

APPENDIX A: EXCHANGE ENERGY IN d DIMENSIONS

In this appendix we calculate the Hartree-Fock exchange energy in d dimensions. Using Eq. (2.9a) and the formula (2.8) in Eq. (2.11a), one obtains

$$\epsilon_{\rm ex} = -\frac{1}{\pi}\frac{d}{d-1} e^2 q_F J , \tag{A1}$$

where

$$J \equiv \int_0^2 dx \int_{x/2}^1 dy (1-y^2)^{(d-1)/2} , \tag{A2}$$

with $x=q/q_F$, and $y=\cos\vartheta$. Changing the order of integration and the variables to $z=x/2$ and $t=y^2$, one obtains

$$\begin{aligned} J &= 2\int_0^1 dz \int_z^1 dy (1-y^2)^{(d-1)/2} \\ &= 2\int_0^1 dy \int_0^y dz (1-y^2)^{(d-1)/2} \\ &= \int_0^1 dt (1-t)^{(d-1)/2} \\ &= B(1,\tfrac{1}{2}(d+1)) = \frac{\Gamma(\frac{1}{2}(d+1))}{\Gamma(1+\frac{1}{2}(d+1))} = \frac{2}{d+1} , \end{aligned} \tag{A3}$$

where $B(1,\frac{1}{2}(d+1))$ is the beta function. Since $e^2 q_F = (2/\alpha r_s)$ Ry the use of Eq. (A3) in Eq. (A1) gives

$$\epsilon_{\rm ex} = -\frac{4}{\pi}\frac{d}{d^2-1}\frac{1}{\alpha r_s}\ \text{Ry} . \tag{A4}$$

APPENDIX B: FREQUENCY-MOMENT SUM RULES FOR THE LINDHARD FUNCTION

In this appendix we derive the asymptotic form of the Lindhard function in terms of the frequency-moment sum rules. One can write the Lindhard function (2.21) as

$$\chi_d^L(q,\omega) = \frac{4}{\hbar}\int \frac{d^dp}{(2\pi)^d}[1-\Theta(q_F - |\vec{p}+\vec{q}|)]\Theta(q_F - |\vec{p}|) \times \frac{\omega_{\vec{p}\vec{q}}}{\omega^2 - \omega_{\vec{p}\vec{q}}^2} , \tag{B1}$$

where

$$\omega_{\vec{p}\vec{q}}=\hbar\vec{q}\cdot\vec{p}/m+\hbar q^2/2m\ .$$

Since $\Theta(q_F-|\vec{p}+\vec{q}|)\Theta(q_F-|\vec{p}|)$ is even under the interchange $\vec{p}\leftrightarrow\vec{p}+\vec{q}$, while $\omega_{\vec{p}\vec{q}}$ is odd under the same interchange, one obtains

$$\lim_{\omega\to\infty}\chi_d^L(q,\omega)=\frac{4}{\hbar}\int\frac{d^dp}{(2\pi)^d}\Theta(q_F-|\vec{p}|)\frac{\omega_{\vec{p}\vec{q}}}{\omega^2-\omega_{\vec{p}\vec{q}}^2} \tag{B2}$$

$$\equiv\sum_{j=1}^{\infty}\frac{L_{2j-1}(q)}{\omega^{2j}}\ , \tag{B3}$$

where

$$L_{2j-1}(q)\equiv\frac{4}{\hbar}\int\frac{d^dq}{(2\pi)^d}\Theta(q_F-|\vec{p}|)(\omega_{\vec{p}\vec{q}})^{2j-1}\ . \tag{B4}$$

One first expands the power of $\omega_{\vec{p}\vec{q}}$ as

$$(\omega_{\vec{p}\vec{q}})^{2j-1}=\left[\frac{\hbar qq_F}{m}k\cos\vartheta+\frac{\hbar q^2}{2m}\right]^{2j-1} \tag{B5}$$

$$=\sum_{l=0}^{2j-2}{}_{2j-1}C_l\left[\frac{\hbar q^2}{2m}\right]^{2j-1-l}\times\left[\frac{\hbar qq_F}{m}\right]^l k^l(\cos\vartheta)^l\ , \tag{B6}$$

where $\cos\vartheta\equiv\vec{q}\cdot\vec{p}/|\vec{q}||\vec{p}|$, and $k\equiv|\vec{p}|/q_F$, then uses the formula (2.8) to obtain

$$L_{2j-1}(q)=\frac{4}{\hbar}\frac{S_{d-1}}{(2\pi)^d}q_F^d\int_0^1dk\int_0^{\pi}d\vartheta\, k^{d-1}(\sin\vartheta)^{d-2}\times\sum_{l=0}^{2j-2}{}_{2j-1}C_l\left[\frac{\hbar q^2}{2m}\right]^{2j-1-l}\left[\frac{\hbar qq_F}{m}\right]^l\times k^l(\cos\vartheta)^l\ . \tag{B7}$$

In Eq. (B7), the terms with odd powers in $\cos\vartheta$ vanish, and

$$\int_0^{\pi}d\vartheta(\sin\vartheta)^{d-2}(\cos\vartheta)^{2l}=\frac{\Gamma(l+\frac{1}{2})\Gamma(\frac{1}{2}(d-1))}{\Gamma(l+\frac{1}{2}d)}\ , \tag{B8}$$

(where $\mu=\cos\vartheta$ and $t=\mu^2$). Then Eq. (B7) becomes

$$L_{2j-1}(q)=\sum_{l=0}^{j-1}\frac{2^{2l+1}\Gamma(\frac{1}{2}d+1)\Gamma(l+\frac{1}{2})}{\pi^{1/2}\Gamma(\frac{1}{2}d+l+1)}\times{}_{2j-1}C_{2l}\frac{n}{\hbar}[\omega_0(q)]^{2j-1-l}(E_F/\hbar)^l\ . \tag{B9}$$

Since ${}_{2j-1}C_{2l}=(2j-1)!/(2j-1-2l)!(2l)!$, Eq. (2.24) results.

APPENDIX C: PLASMON DISPERSION IN THE RPA IN d DIMENSIONS

In this appendix we derive the plasmon dispersion relation within the RPA for degenerate and classical electron liquids in d dimensions. In the degenerate case, the asymptotic form of the RPA dielectric function (2.26) is given by Eq. (4.12) with $G(q)$ set equal to zero. The plasmon dispersion relation, $\omega=\omega(q)$, is obtained by solving

$$\epsilon_d^{\mathrm{RPA}}(q,\omega(q))=0\ . \tag{C1}$$

The result is

$$\omega^2(q)=\omega_p^2(q)+(12/d)\langle E_{\mathrm{kin}}\rangle_0\omega_0(q)/\hbar+\cdots, \tag{C2a}$$

which justifies the definition of the plasma frequency (2.14). Equation (C2a) reproduces the well-known results

$$\omega^2(q)=\begin{cases}4\pi ne^2/m+\frac{3}{5}(qv_F)^2+\cdots,\ d=3 & \text{(C2b)}\\ 2\pi ne^2q/m+\frac{3}{4}(qv_F)^2+\cdots,\ d=2 & \text{(C2c)}\end{cases}$$

given, respectively, in Refs. 1 and 36. It is of interest to note that for $d=4$, the plasmon dispersion becomes

$$\omega^2(q)=4\pi^2ne^2/mq+\tfrac{1}{2}(qv_F)^2+\cdots,\ d=4 \tag{C2d}$$

which has a minimum at $q_c\simeq(\alpha r_s/4)^{1/3}q_F$.

In the classical case, using Eq. (6.4b) one obtains the asymptotic form of the RPA dielectric function (6.5),

$$\epsilon_d^{\mathrm{RPA}}(q,\omega)=1-\frac{\omega_p^2(q)}{\omega^2}-\frac{3\omega_p^2(q)}{\omega^4}(k_BT/m)q^2+i(\pi/2)^{1/2}\frac{\omega_p^2(q)\omega}{(k_BT/m)^{3/2}q^3}\times\exp\left[-\frac{\omega^2}{2(k_BT/m)q^2}\right]+\cdots\ . \tag{C3}$$

Solving Eq. (C1) with $\omega(q)=\omega_R(q)+i\gamma(q)$, one obtains

$$\omega_R^2(q)=\omega_p^2(q)[1+3(q/q_D)^{d-1}+\cdots]\ , \tag{C4a}$$

$$\frac{\gamma(q)}{\omega_R(q)}=-(\pi/8)^{1/2}(q_D/q)^{3(d-1)/2}\times\exp[-\tfrac{1}{2}(q_D/q)^{d-1}-\tfrac{3}{2}]\ . \tag{C5a}$$

Equations (C4a) and (C5a) reproduce the well-known results

$$\omega_R^2(q)=\begin{cases}4\pi ne^2/m+3(k_BT/m)q^2+\cdots, & d=3 & \text{(C4b)}\\ 2\pi ne^2q/m+3(k_BT/m)q^2, & d=2 & \text{(C4c)}\end{cases}$$

given, respectively, in Ref. 31 and in Refs. 37, 38, and 33(c), and

$$\gamma(q)/\omega_R(q)=\begin{cases}-(\pi/8)^{1/2}(q_D/q)^3\exp[-\tfrac{1}{2}(q_D/q)^2-\tfrac{3}{2}], & d=3 & \text{(C5b)}\\ -(\pi/8)^{1/2}(q_D/q)^{3/2}\exp[-\tfrac{1}{2}(q_D/q)-\tfrac{3}{2}], & d=2 & \text{(C5c)}\end{cases}$$

given also, respectively, in Ref. 31 and in Refs. 37, 38, and 33(c).

APPENDIX D: LONG-WAVELENGTH LIMIT OF $I_d(q)$

In this appendix we derive the long-wavelength limit of $I_d(q)$, defined by (4.9). First, change the variable $\vec{k}\rightarrow\vec{q}-\vec{k}$ in Eq. (4.9) to write

$$I_d(q)=-\frac{1}{N}\sum_{\vec{k}\,(\neq\vec{q},\vec{0})}\frac{(\vec{q}-\vec{k})\cdot\vec{q}}{q^2}\left[\frac{(\vec{q}-\vec{k})\cdot\vec{q}}{q^2}\left[\frac{q}{|\vec{q}-\vec{k}|}\right]^{d-1}+\frac{\vec{k}\cdot\vec{q}}{q^2}(q/k)^{d-1}\right][S(k)-1]\,. \tag{D1}$$

Second, expand the term $|\vec{q}-\vec{k}|^{1-d}$ for small q as

$$\frac{1}{|\vec{q}-\vec{k}|^{d-1}}=\frac{1}{k^{d-1}}\left[1+(d-1)\mu x+\left[\frac{d^2-1}{2}\mu^2x^2-\frac{d-1}{2}x^2\right]+O(x^3)\right], \tag{D2}$$

where $\mu\equiv\vec{q}\cdot\vec{k}/|\vec{q}|\,|\vec{k}|$ and $x=q/k$, to obtain

$$\lim_{q\rightarrow 0}I_d(q)=-\frac{1}{N}\sum_{\vec{k}\,(\neq\vec{q},\vec{0})}x^{d-1}\{[-1+(d-1)\mu^2]\mu/x+[1-\tfrac{5}{2}(d-1)\mu^2+\tfrac{1}{2}(d^2-1)\mu^4]+O(x)\}[S(k)-1]\,. \tag{D3}$$

In Eq. (D3) the angular integral of the term odd in μ vanishes. The use of the the formula (2.8) in (D3) then gives

$$\lim_{q\rightarrow 0}I_d(q)=-\frac{1}{n}\frac{q^{d-1}}{2^{d-1}\pi^{(d+1)/2}\Gamma(\frac{1}{2}(d-1))}A\int_0^\infty dk[S(k)-1]\,, \tag{D4}$$

where

$$A\equiv\int_{-1}^{1}d\mu(1-\mu^2)^{(d-3)/2}[1-\tfrac{5}{2}(d-1)\mu^2+\tfrac{1}{2}(d^2-1)\mu^4] \tag{D5}$$

$$=\frac{\pi^{1/2}(7-d)}{2d(d+2)}\frac{\Gamma(\frac{1}{2}(d-1))}{\Gamma(\frac{1}{2}d)}\,. \tag{D6}$$

The definition of the potential energy per particle (4.15) may be written as

$$\langle\mathscr{V}\rangle=\frac{\Gamma(\frac{1}{2}(d-1))e^2}{2\pi^{1/2}\Gamma(\frac{1}{2}d)}\int_0^\infty dk[S(k)-1]\,. \tag{D7}$$

Putting Eqs. (D6) and (D7) in (D4), one obtains

$$\lim_{q\rightarrow 0}I_d(q)=-\frac{7-d}{d(d+2)}\frac{1}{2^{d-1}\pi^{(d-1)/2}\Gamma(\frac{1}{2}(d-1))}\frac{q^{d-1}}{ne^2}\langle\mathscr{V}\rangle\,. \tag{D8}$$

In the degenerate case, the definition (2.5), and the relation, $e^2q_F=me^4/\hbar^2\alpha r_s$, may be used to write (D8) in the form

$$\lim_{q\rightarrow 0}I_d(q)=-\frac{(7-d)\pi^{1/2}\Gamma(\frac{1}{2}d)}{4(d+2)\Gamma(\frac{1}{2}(d-1))}\alpha r_s(q/q_F)^{d-1}\frac{\langle\mathscr{V}\rangle}{me^4/2\hbar^2}\,. \tag{D9a}$$

In particular,

$$\lim_{q\to 0} I(q)=\begin{cases} -\dfrac{q^2}{15\pi n e^2}\langle \mathscr{V}\rangle = -\dfrac{\pi}{10}\alpha r_s (q/q_F)^2 \dfrac{\langle \mathscr{V}\rangle}{me^4/2\hbar^2} = \left[\dfrac{3}{20}-\dfrac{\pi}{10}\left[2\alpha r_s \epsilon_c + \alpha r_s^2 \dfrac{\partial \epsilon_c}{\partial r_s}\right]\right](q/q_F)^2, \quad d=3 & \text{(D9b)} \\ -\dfrac{5}{16\pi}\dfrac{q}{ne^2}\langle \mathscr{V}\rangle = -\dfrac{5}{16}\alpha r_s (q/q_F)\dfrac{\langle \mathscr{V}\rangle}{me^4/2\hbar^2} = \left[\dfrac{5}{6\pi}-\dfrac{5}{16}\left[2\alpha r_s \epsilon_c + \alpha r_s^2 \dfrac{\partial \epsilon_c}{\partial r_s}\right]\right](q/q_F), \quad d=2 & \text{(D9c)} \end{cases}$$

where Eq. (D9b) reproduces the result given in Ref. 13(b). To obtain the last expressions in Eqs. (D9b) and (D9c), Eqs. (2.11b), (2.11c), and (4.17) have been used and the correlation energy per particle ϵ_c is in rydberg units. In the classical case one may use Eqs. (4.15) and (6.11) to write (D8) in the form

$$\lim_{q\to 0} I_d(q) = -\frac{7-d}{d(d+2)}\frac{U_{\text{ex}}}{Nk_B T}(q/q_D)^{d-1} . \tag{D10}$$

APPENDIX E: PLASMON DISPERSION FROM THE ω^3 SUM RULE

In this appendix, we derive the plasmon dispersion in the long-wavelength limit by using the ω^3 sum rule in the degenerate case in d dimensions. We also give expressions in the form of the coupling constant (r_s) expansion in the weak-coupling limit.

From Eq. (4.11) one obtains the plasmon dispersion in the long-wavelength limit

$$\omega^2(q)=\omega_p^2(q)+\frac{12}{d}\langle E_{\text{kin}}\rangle \omega_0(q)/\hbar-\omega_p^2(q)I_{d1}(q)+\cdots , \tag{E1}$$

where $I_{d1}(q)$ has been defined in Eq. (5.6). The last two terms on the right-hand side of Eq. (E1) contribute to order q^2. One may use the virial theorem (4.16) to express $\langle E_{\text{kin}}\rangle$ in terms of the correlation energy per particle ϵ_c

$$\frac{12}{d}\langle E_{\text{kin}}\rangle \omega_0(q)/\hbar=(qv_F)^2\left[\frac{3}{d+2}-\frac{3}{d}\alpha^2 r_s^2\left[\epsilon_c+r_s\frac{\partial \epsilon_c}{\partial r_s}\right]\right] . \tag{E2}$$

In addition, from Eqs. (4.17) and (D9a) one obtains

$$\omega_p^2(q)I_{d1}(q)=-\frac{7-d}{2d(d+2)}\alpha^2 r_s^2 (qv_F)^2\left[\epsilon_{\text{ex}}+2\epsilon_c+r_s\frac{\partial \epsilon_c}{\partial r_s}\right] , \tag{E3}$$

where the relation

$$\omega_p^2(q)=(s_0 q)^2 (q/q_{\text{FT}})^{1-d} , \tag{E4}$$

or, equivalently [cf. Eq. (2.13a)],

$$\omega_p^2(q)=\frac{\Gamma(\frac{1}{2}(d-1))}{\pi^{1/2}\Gamma(\frac{1}{2}d+1)}\alpha r_s (qv_F)^2 (q/q_F)^{1-d} \tag{E5}$$

has been used. In Eq. (E4) $s_0=v_F/\sqrt{d}$. Equations (E2) and (E3) together with Eq. (2.11a) yield

$$\omega^2(q)=\omega_p^2(q)+(qv_F)^2\left[\frac{3}{d+2}-\frac{2(7-d)}{\pi(d^2-1)(d+2)}\alpha r_s-\frac{1}{2d(d+2)}\alpha^2 r_s^2\left[2(4d-1)\epsilon_c+(7d+5)r_s\frac{\partial \epsilon_c}{\partial r_s}\right]\right]+\cdots , \tag{E6a}$$

where ϵ_c is in rydberg units. In particular,

$$\omega^2(q)=\begin{cases} \dfrac{4\pi n e^2}{m}+(qv_F)^2\left[\dfrac{3}{5}-\dfrac{3}{20\pi}\alpha r_s-\dfrac{1}{15}\alpha^2 r_s^2\left[11\epsilon_c+13 r_s\dfrac{\partial \epsilon_c}{\partial r_s}\right]\right]+\cdots, \quad d=3 & \text{(E6b)} \\ \dfrac{2\pi n e^2}{m}q+(qv_F)^2\left[\dfrac{3}{4}-\dfrac{5}{6\pi}\alpha r_s-\dfrac{1}{16}\alpha^2 r_s^2\left[14\epsilon_c+19 r_s\dfrac{\partial \epsilon_c}{\partial r_s}\right]\right]+\cdots, \quad d=2 . & \text{(E6c)} \end{cases}$$

In the high-density limit, one may use Eqs. (3.7a) and (3.7b) to obtain

$$\omega^2(q)=\begin{cases}\dfrac{4\pi ne^2}{m}+(qv_F)^2\left[\dfrac{3}{5}-\dfrac{3}{20\pi}\alpha r_s-\dfrac{22}{15\pi^2}(1-\ln 2)\alpha^2 r_s^2\ln r_s+\dfrac{26}{15\pi^2}(1-\ln 2)\alpha^2 r_s^2+O(r_s^3,r_s^3\ln r_s)\right]+\cdots\,, & d=3 \quad \text{(E7a)}\\ \dfrac{2\pi ne^2}{m}q+(qv_F)^2\left[\dfrac{3}{4}-\dfrac{5\sqrt{2}}{12\pi}r_s+r_s^2\left[0.17+\dfrac{11\sqrt{2}}{16\pi}(10-3\pi)r_s\ln r_s\right]+\cdots\right]+\cdots\,, & d=2\,. \quad \text{(E7b)}\end{cases}$$

In three dimensions, an expression corresponding to Eq. (E6b) has been obtained by Kugler.[6(b)] In two dimensions, the first two terms which are proportional to q^2 on the right-hand side of Eq. (E6c) agree with the results by Beck and Kumar[39] and by Rajagopal,[40] who took into account the exchange effects. The term with the coefficient 3/4 comes from the free-particle kinetic energy while the term proportional to r_s comes from the exchange energy. Equations (E6c) and (E7b) include the correlation effects up to third order in the frequency moments.

Experimentally, two dimensional plasmons have been observed in silicon inversion layers.[41–44] The observed density and wave-number dependence of the plasma frequency $[\omega_p(q)\propto n^{1/2}q^{1/2}]$, with the geometrical factor which comes from the finite thickness of the oxide layer included, agree well with theory. On the other hand, the characteristic wave numbers in these experiments $(q/q_F<10^{-2})$[41–43] and a recent one $(q/q_F\sim 0.14)$[44] are still too small to test the dispersion relation (E6c).

APPENDIX F: COMMENTS ON PLASMON DISPERSION IN CLASSICAL ELECTRON LIQUIDS

Let us briefly mention the plasmon dispersion in the classical case. Using the same procedure as in the degenerate case (cf. Appendix E) one might obtain from Eq. (4.11) and (D10)

$$\omega^2(q)=\omega_p^2(q)+(s_0q)^2\left[3-\frac{7-d}{d(d+2)}\frac{U_{\rm ex}}{Nk_BT}\right]+\cdots\,, \tag{F1}$$

where $s_0^2=k_BT/m$. However, this is *not correct.* In three dimensions, the ω^3 sum rule, which neglects the collisional effects, is known to be inappropriate to obtaining the plasmon dispersion correctly.[45,46] In the long-wavelength limit, the plasmon dispersion has been obtained by Baus.[46] In the weak coupling limit, it is

$$\omega^2(q)=\frac{4\pi ne^2}{m}+(3+0.61\Gamma^{3/2}+\cdots)\left[\frac{k_BT}{m}\right]q^2+\cdots\,, \tag{F2}$$

while in the strong coupling limit

$$\omega^2(q)=\frac{4\pi ne^2}{m}+(C_P/C_V)(\kappa_T^0/\kappa_T)\left[\frac{k_BT}{m}\right]q^2+\cdots\,, \tag{F3}$$

where C_P/C_V is the specific-heat ratio with

$$\frac{C_P}{Nk_B}=\frac{1}{Nk_B}\frac{\partial W}{\partial T}\bigg|_{P,N}=\frac{d+2}{2}-\frac{d+1}{d}\Gamma^2\frac{d}{d\Gamma}\left[\frac{1}{\Gamma}\frac{U_{\rm ex}}{Nk_BT}\right]\,, \tag{F4}$$

$$\frac{C_V}{Nk_B}=\frac{1}{Nk_B}\frac{\partial E}{\partial T}\bigg|_{V,N}=\frac{d}{2}-\Gamma^2\frac{d}{d\Gamma}\left[\frac{1}{\Gamma}\frac{U_{\rm ex}}{Nk_BT}\right]\,, \tag{F5}$$

and W the enthalpy, and κ_T^0/κ_T is given by Eq. (6.8b). The coefficient in Eq. (F3) may be expressed in terms of the adiabatic compressibility κ_S as

$$(C_P/C_V)(\kappa_T^0/\kappa_T)=\kappa_T^0/\kappa_S\,. \tag{F6}$$

Equation (F2) reduces to the expression obtained in the collisionless approximation (C4b), since the collision frequency,

$$\omega_c\sim\Gamma^{3/2}\omega_p\,, \tag{F7}$$

vanishes as $\Gamma\to 0$. In the strong-coupling limit, since the collision frequency increases as implied from the weak-coupling expression (F7), the correct plasmon dispersion is given by hydrodynamic description. Baus estimates the region of validity of hydrodynamic description as

$$\Gamma^{-3/2}\ll q/q_D\ll 1 \tag{F8}$$

for $\Gamma\gg 1$.

The situation is different in two dimensions.[47] First, the collision frequency is independent of the dimensionless coupling constant (Γ) $[\omega_c\sim\omega_p(q)]$.[33(c)] Thus, the collisionless approximation[33(c),37,38] is not valid even in the weak-coupling limit, in contrast to the three-dimensional case. Second, the collective mode has a low frequency in the long-wavelength limit $[\omega(q)\propto\omega_p(q)\propto q^{1/2}]$. Thus, the plasmon dispersion[47] is identical to that obtained from hydrodynamics:[48]

$$\omega^2(q)=\frac{2\pi ne^2}{m}q+(C_P/C_V)(\kappa_T^0/\kappa_T)(k_BT/m)q^2+\cdots\,. \tag{F9}$$

In the weak-coupling limit, Eq. (F9) reduces to[47]

$$\omega^2(q)=\frac{2\pi ne^2}{m}q+\frac{2k_BT}{m}q^2+\cdots\,, \tag{F10}$$

which is, in fact, different from the collisionless result (C4c).[33(c),37,38]

Molecular dynamic calculations have been carried out in three[4] and two dimensions,[49] and the plasmon dispersion in both cases agree well with the theoretical result of Bauss [Eqs. (F3) and (F9)], which predict negative dispersion for large values of Γ.

Experimentally, plasmons in the surface layer of electrons on liquid helium have been observed.[50] However, the characteristic wave number in the experiment is not yet large enough ($q/q_D < 10^{-4}$) to test the correlation effects in the dispersion relation.

---

*Present address: Department of Physics, Washington University, St. Louis, MO 63130.

[1]D. Pines and P. Nozières, *The Theory of Quantum Liquids* (Benjamin, New York, 1966), Vol. I.

[2]R. D. Puff, Phys. Rev. **137**, A406 (1965); N. Mihara and R. D. Puff, *ibid.* **174**, 221 (1968).

[3]D. Forster, P. C. Martin, and S. Yip, Phys. Rev. **170**, 155 (1968).

[4]J. P. Hansen, E. L. Pollock, and I. R. McDonald, Phys. Rev. Lett. **32**, 277 (1974); J. P. Hansen, I. R. McDonald, and E. L. Pollock, Phys. Rev. A **11**, 1025 (1975).

[5]K. S. Singwi, M. P. Tosi, R. H. Land, and A. Sjölander, Phys. Rev. **176**, 589 (1968).

[6](a) A. A. Kugler, Phys. Rev. A **1**, 1688 (1970); (b) J. Stat. Phys. **12**, 35 (1975).

[7]B. Goodman and A. Sjölander, Phys. Rev. B **8**, 200 (1973); G. Mukhopadyay and A. Sjölander, *ibid.* **17**, 3589 (1978).

[8]K. N. Pathak and P. Vashishta, Phys. Rev. B **7**, 3649 (1973); V. K. Jindal, H. B. Singh, and K. N. Pathak, *ibid.* **15**, 252 (1977).

[9]A. Holas, P. K. Aravind, and K. S. Singwi, Phys. Rev. B **20**, 4912 (1979).

[10]K. Utsumi and S. Ichimaru, Phys. Rev. B **22**, 1522 (1980).

[11]K. Awa, H. Yasuhara, and T. Asahi, Phys. Rev. B **25**, 3670 (1982).

[12](a) N. Iwamoto and D. Pines, Phys. Rev. B **29**, 3924 (1984); (b) N. Iwamoto, E. Krotscheck, and D. Pines, *ibid.* **29**, 3936 (1984).

[13]See, e.g., (a) K. S. Singwi and M. P. Tosi, in *Solid State Physics,* edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic, New York, 1981), Vol. 36, p. 177; (b) S. Ichimaru, Rev. Mod. Phys. **54**, 1017 (1982); (c) T. Ando, A. B. Fowler, and F. Stern, *ibid.* **54**, 437 (1982); (d) M. Baus and J. P. Hansen, Phys. Rep. **59**, 1 (1980).

[14]Note that the relation (1.1b) is exact as far as one uses the true screened response function, while (1.1a) is a model.

[15]J. Hubbard, Proc. R. Soc. London, Ser. A **243**, 336 (1957).

[16]M. Jonson, J. Phys. C **9**, 3055 (1976).

[17]J. S. Vaishya and A. K. Gupta, Phys. Rev. B **7**, 4300 (1973).

[18]R. A. Ferrell, Phys. Rev. Lett. **1**, 443 (1958).

[19]Here, the gamma function with a parameter x, $\Gamma(x)$ is not to be confused with the dimensionless coupling constant Γ defined in Eq. (6.6). In the following, the letter d always denotes the spacial dimensionality except where noted.

[20]K. G. Wilson, Phys. Rev. Lett. **28**, 548 (1972).

[21]F. Stern, Phys. Rev. Lett. **30**, 278 (1973).

[22]J. Lindhard, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **28**, 8 (1954).

[23]The subscript "-1" reflects the fact that the compressibility sum rule may be expressed as a ω^{-1} moment of the dynamic form factor (Ref. 1)

$$\lim_{q\to 0}\int_0^\infty \frac{d\omega}{2\pi}\frac{S(q,\omega)}{\omega}\,|\epsilon_d(q,\omega)|^2=\frac{\hbar}{2ms^2}\ .$$

[24]W. Macke, Z. Naturforsch. **5a**, 192 (1950); M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106**, 364 (1957); D. F. DuBois, Ann. Phys. (N.Y.) **7**, 174 (1959); W. J. Carr, Jr. and A. A. Maradudin, Phys. Rev. **133**, A371 (1964).

[25]A. K. Rajagopal and J. C. Kimball, Phys. Rev. B **15**, 2819 (1977); A. Isihara and T. Toyoda, Ann. Phys. (N.Y.) **106**, 394 (1977); **114**, 497(E) (1978); Z. Phys. B **26**, 216(A) (1977); **29**, 70(E) (1978).

[26]We define the dynamic form factor $S(q,\omega)$ such that the frequency integral of it gives the static form factor $S(q)$: $\int_{-\infty}^{\infty}(d\omega/2\pi)S(q,\omega)=S(q)$. The relation to the notation in Ref. 1 is $S(q,\omega)=(2\pi/n)S^{PN}(q,\omega)$.

[27]N. H. March, Phys. Rev. **110**, 604 (1958); P. N. Argyres, *ibid.* **154**, 410 (1967).

[28]N. Iwamoto, Phys. Rev. A **30**, 2597 (1984).

[29]Equation (5.2a) may also be written as

$$\langle\omega^{-1}\rangle'=-\int_{-\infty}^{\infty}(d\omega/\pi)\omega^{-1}[\mathrm{Im}\chi_d(q,\omega)]\,|\epsilon_d(q,\omega)|^2 =[2/v_d(q)]\int_{-\infty}^{\infty}(d\omega/2\pi)\omega^{-1}[\mathrm{Im}\epsilon_d(q,\omega)]\ .$$

[30]B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic, New York, 1961).

[31]S. Ichimaru, *Basic Principles of Plasma Physics* (Benjamin, Reading, Mass., 1973).

[32]R. Abe, Prog. Theor. Phys. **23**, 213 (1959).

[33](a) J. Chalupa, Phys. Rev. B **12**, 4 (1975); **13**, 2243(E) (1976); (b) H. Totsuji, J. Phys. Soc. Jpn. **39**, 253 (1975); (c) **40**, 857 (1976); (d) Phys. Rev. A **19**, 889 (1979).

[34]In Eq. (4.8) the term proportional to $\omega_0^2(q)$ contains a factor of $\hbar^2$ which should vanish in the classical limit, while the term proportional to $\omega_0(q)/\hbar$ is $\hbar$ independent.

[35]Unlike Eq. (4.12), Eq. (6.21) does not contain the term proportional to $\omega_0^2(q)$. This is because the recoil energy is neglected in the propagator of the Vlasov response function (6.1a) (cf. Ref. 34).

[36]F. Stern, Phys. Rev. Lett. **18**, 546 (1967).

[37]P. M. Platzman and N. Tzoar, Phys. Rev. B **13**, 3197 (1976).

[38]A. L. Fetter, Phys. Rev. B **10**, 3739 (1974).

[39]D. E. Beck and P. Kumar, Phys. Rev. B **13**, 2859 (1976); **14**, 5127(E) (1976).

[40]A. K. Rajagopal, Phys. Rev. B **15**, 4264 (1977).

[41]S. J. Allen, Jr., D. C. Tsui, and R. A. Logan, Phys. Rev. Lett. **38**, 980 (1977).

[42]T. N. Theis, J. P. Kotthaus, and P. J. Stiles, Surf. Sci. **73**, 434 (1978); Solid State Commun. **26**, 603 (1978).

[43]D. C. Tsui, E. Gornik, and R. A. Logan, Solid State Commun. **35**, 875 (1980).

[44]For a review , see T. N. Theis, Surf. Sci. **98**, 515 (1980).

[45]S. Ichimaru, H. Totsuji, T. Tange, and D. Pines, Prog. Theor. Phys. **54**, 1077 (1975).

[46]M. Baus, Phys. Rev. A **15**, 790 (1977).

[47]M. Baus, J. Stat. Phys. **19**, 163 (1978).

[48]A. Onuki, J. Phys. Soc. Jpn. **43**, 396 (1977).

[49]H. Totsuji and H. Kakeya, Phys. Rev. A **22**, 1220 (1980).

[50]C. C. Grimes and G. Adams, Phys. Rev. Lett. **36**, 145 (1976).