Bose-Einstein condensation in finite noninteracting systems: A relativistic gas with pair production. II

Surjit Singh and R. K. Pathria
Guelph-Waterloo Program for Graduate Work in Physics, Waterloo Campus, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

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An asymptotic evaluation of the specific heat of an ideal relativistic Bose gas confined to a cuboidal enclosure ($L_1 \times L_2 \times L_3$) is carried out, under periodic boundary conditions, taking into account the possibility of particle-antiparticle pair production in the system. Finite-size corrections to the standard bulk behavior are calculated explicitly in the regions $t>0$ and $t<0$, where $t=(T-T_c)/T_c$, such that $|t| \ll 1$ and $|L_i t| \gg 1$. While for $t>0$ finite-size corrections turn out to be exponential for all geometries, for $t<0$ this is true only in the case of a film; for other geometries, such as a cuboid or a rectangular channel, these corrections conform to a power law instead. Finally, we consider the situation in the core region, where $|L_i t| = O(1)$, and examine the location t^* and the height c_ρ^* of the specific-heat maximum; finite-size corrections in this region turn out to be $O(L_<^{-1})$, where $L_<$ denotes the shortest side of the enclosure.

I. INTRODUCTION

In a recent paper,[1] hereafter referred to as I, we presented a theoretical analysis of the onset of Bose-Einstein condensation in a relativistic Bose gas confined to a cuboidal enclosure of *finite* physical dimensions ($L_1 \times L_2 \times L_3$), taking into account the possibility of particle-antiparticle pair production in the system. Through an extensive use of the Poisson summation formula, we carried out an explicit evaluation of the summations over states appearing in the problem, which enabled us to make a rigorous study of the temperature dependence of the thermogeometric parameter y of the system in the case of a cubical enclosure under periodic boundary conditions. This, in turn, led us to determine the growth of the condensate fraction ρ_0/ρ as a *smooth* function of temperature from $T \gtrsim T_c$ down to $T=0$ K. Finite-size corrections to the standard bulk results were obtained in explicit terms and were shown to be consistent with the Fisher-Barber scaling theory for such effects.[2-4] Finally, the situation encountered in the case of special geometries, such as narrow channels and thin films, was also examined at some length.

In the present paper we extend the aforementioned analysis to a similar study of the specific heat of the system at constant volume. Once again it turns out that the inclusion of antiparticles into the problem renders the analysis of the relativistic problem far more tractable than it otherwise is, with the result that finite-size corrections to the bulk behavior of the system can be derived in a closed form. An important finding of the present investigation is that the algebraic nature of these corrections depends rather crucially on the geometry of the enclosure. While for $T>T_c$, the dependence on $L_<$, where $L_<$ denotes the shortest side of the container, is quite generally exponential, the situation for $T<T_c$ is not so straightforward. Here, the asymptotic correction varies exponentially *only* in the case of a film (for which $L_{1,2} \to \infty$ while L_3 is large but finite); for other geometries, it varies as a power law—$L_<^{-4}$ in the case of a rectangular channel (for which $L_1 \to \infty$ while $L_{2,3}$ are large but finite) and $L_<^{-3}$ in the case of a cuboid (for which $L_{1,2,3}$ are all large and finite). These differences of behavior are clearly significant in the context of the finite-size scaling formulation for Bose systems.[5-8]

Finally we examine the situation in the core region ($T \simeq T_c$) and determine precisely the location and the height of the specific-heat maximum. We find that, apart from the expected effect of changing the cusplike singularity characteristic of the bulk system into a smooth maximum, the finiteness of the enclosure results in displacing the maximum towards higher temperatures and reducing its height—both corrections being of order $L_<^{-1}$. Remarkably enough, our theoretical findings are qualitatively the same, irrespective of the severity of the relativistic effects; the influence of the latter on the various physical results turns out to be only quantitative. For continuity with our previous work we shall follow the notation of I throughout this paper; in particular, we shall use units such that $\hbar = c = k_B = 1$.

II. FORMULATION OF THE PROBLEM

We consider an ideal Bose gas composed of N_1 particles and N_2 antiparticles, each of mass m, confined to a cuboidal enclosure of sides L_1, L_2, and L_3. Since particles and antiparticles are supposed to be created in pairs, the system is governed by the conservation of the number Q ($=N_1-N_2$), rather than of the numbers N_1 and N_2 separately; the conserved quantity Q may be looked upon as a kind of generalized "charge." In equilibrium, we have[1]

$$N_1 = \sum_\epsilon (e^{\beta(\epsilon-\mu)}-1)^{-1} ,$$
$$N_2 = \sum_\epsilon (e^{\beta(\epsilon+\mu)}-1)^{-1} , \quad (1)$$

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where $\beta=1/T$, $\epsilon=(\vec{k}^2+m^2)^{1/2}$, while $|\mu|\leq m$. Assuming that, to begin with, $\mu>0$, it readily follows that $N_1>N_2$ and hence $Q>0$. In view of the conservation of Q, μ then stays positive under all circumstances. Without loss of generality, we shall assume that this indeed is the case.

Under periodic boundary conditions, the eigenvalues k_i $(i=1,2,3)$ of the wave vector $\vec{k}$ are given by

$$k_i=(2\pi/L_i)n_i \quad (n_i=0,\pm1,\pm2,\dots)\ . \tag{2}$$

The expression for the energy density u of the system may, therefore, be written as

$$u=\frac{1}{V}\sum_{\vec{n}}\epsilon_n[(e^{\beta(\epsilon_n-\mu)}-1)^{-1}+(e^{\beta(\epsilon_n+\mu)}-1)^{-1}] \tag{3}$$

$$=\frac{2m}{V}\sum_{j=1}^{\infty}\cosh(j\beta\mu)\sum_{n_1,n_2,n_3=-\infty}^{\infty}\left[1+\frac{4\pi^2}{m^2}\sum_{i=1}^{3}\left[\frac{n_i}{L_i}\right]^2\right]^{1/2}\exp\left\{-j\beta m\left[1+\frac{4\pi^2}{m^2}\sum_{i=1}^{3}\left[\frac{n_i}{L_i}\right]^2\right]^{1/2}\right\}. \tag{4}$$

Using the Poisson summation formula (PSF)

$$\sum_{n_1,n_2n_3=-\infty}^{\infty}f(n_1,n_2,n_3)=\sum_{q_1,q_2,q_3=-\infty}^{\infty}\mathcal{F}(q_1,q_2,q_3)\ ,$$

where

$$\mathcal{F}(\vec{q})=\int_{-\infty}^{\infty}f(\vec{n})e^{2\pi i(\vec{q}\cdot\vec{n})}d^3n\ , \tag{5}$$

Eq. (4) takes the form

$$u=\frac{m^2}{\pi^2\beta^2}\sum_{j=1}^{\infty}\cosh(j\beta\mu)\sum_{q_1,q_2,q_3=-\infty}^{\infty}\left[\beta m j^2\frac{K_3(\beta mz)}{z^3}-\frac{K_2(\beta mz)}{z^2}\right], \tag{6}$$

where $K_\nu(\beta mz)$ are the modified Bessel functions while

$$z=(j^2+q'^2)^{1/2}\ ,\quad q'=\beta^{-1}\gamma(\vec{q})\ ,\quad \gamma(\vec{q})=(q_1^2L_1^2+q_2^2L_2^2+q_3^2L_3^2)^{1/2}\ . \tag{7}$$

The $\vec{q}=0$ term in (6) yields the bulk result, viz.,

$$u_B(\beta,\mu)=\frac{m^2}{\pi^2\beta^2}\sum_{j=1}^{\infty}\cosh(j\beta\mu)\left[\beta m\frac{K_3(\beta mj)}{j}-\frac{K_2(\beta mj)}{j^2}\right], \tag{8}$$

while the remaining terms—those with $\vec{q}\neq0$—can be simplified by replacing the summation over j by an integration. We hasten to add that, as demonstrated in I, this corresponds to, first, replacing the summation over j by a summation over l (with the help of PSF) and, then, retaining the term with $l=0$ only. This introduces errors $O(e^{-L_i/\lambda_T})$ or $O(e^{-L_i/\lambda_C})$, where λ_T $(=\sqrt{2\pi\beta/m}\,)$ and λ_C $(=1/m)$ denote, respectively, the mean thermal wavelength and the Compton wavelength of the particles. Assuming that, for all i, $L_i\gg\lambda_T$ and λ_C, these errors can be ignored with impunity. The important thing to note is that no errors of order $(\lambda_T/L_i)^n$ or $(\lambda_C/L_i)^n$ are committed if, for $\gamma(\vec{q})\neq0$, the summation over j is replaced by an integration.

Now, the integration encountered here is somewhat involved but can be done exactly; see the Appendix. Using Eq. (A6) and remembering that

$$K_{1/2}(z)=\left[\frac{\pi}{2z}\right]^{1/2}e^{-z}\ , \tag{9}$$

we obtain

$$u=u_B(\beta,\mu)+\frac{\mu^2}{2\pi\beta}H_1(\mu)\ , \tag{10}$$

where

$$H_n(\mu)=\sum_{\vec{q}}{}'[\gamma(\vec{q})]^{-n}e^{-(m^2-\mu^2)^{1/2}\gamma(\vec{q})}\ . \tag{11}$$

The primed summation here implies that the term with $\vec{q}=0$ is excluded; accordingly, $\gamma(\vec{q})>0$. The corresponding expression for the "charge density" ρ $[\equiv Q/V=(N_1-N_2)/V]$ is given by Eq. (29) of I, viz.,

$$\rho=\rho_B(\beta,\mu)+\frac{\mu}{2\pi\beta}H_1(\mu)\ , \tag{12}$$

where

$$\rho_B(\beta,\mu)=\frac{m^3}{2\pi^2}W(\beta,\mu)\ , \tag{13}$$

with

$$W(\beta,\mu)=2\sum_{j=1}^{\infty}(j\beta m)^{-1}\sinh(j\beta\mu)K_2(j\beta m)\ . \tag{14}$$

Combining (10) with (12), we obtain for the *thermal* energy density $\bar{u}$ of the system

$$\bar{u}\equiv(u-m\rho)=\bar{u}_B(\beta,\mu)-\frac{\mu(m-\mu)}{2\pi\beta}H_1(\mu)\ . \tag{15}$$

where

$$\bar{u}_B(\beta,\mu)\equiv[u_B(\beta,\mu)-m\rho_B(\beta,\mu)]=\frac{m^4}{2\pi^2}Z(\beta,\mu)\ , \quad (16)$$

with

$$Z(\beta,\mu)=2\sum_{j=1}^{\infty}\left[\cosh(j\beta\mu)\left[\frac{K_3(j\beta m)}{j\beta m}-\frac{K_2(j\beta m)}{(j\beta m)^2}\right]-\sinh(j\beta\mu)\frac{K_2(j\beta m)}{j\beta m}\right]. \quad (17)$$

For $\mu\to m$, the function $Z(\beta,\mu)$ takes the form

$$Z(\beta,\mu)=Z(\beta,m)-\left[\frac{\partial Z}{\partial\mu}\right]_{\mu=m}(m-\mu)+\frac{2^{1/2}\pi}{\beta m^{5/2}}(m-\mu)^{3/2}+O((m-\mu)^2)\ . \quad (18)$$

In view of the fact that

$$\left[\frac{\partial Z(\beta,\mu)}{\partial\mu}\right]_{\mu=m}=-\frac{\beta}{m}\frac{dW}{d\beta}\ ,$$

where $W\ [\equiv W(\beta,m)]$ is given by (14), Eqs. (15)–(18) yield the desired expression

$$\bar{u}=\frac{m^4}{2\pi^2}\left[Z(\beta,m)+\frac{\beta(m-\mu)}{m}\left[\frac{dW}{d\beta}\right]\right]+\frac{(m^2-\mu^2)^{3/2}}{4\pi\beta}\left[1-\frac{H_1(\mu)}{(m^2-\mu^2)^{1/2}}\right]+O((m^2-\mu^2)^2)\ . \quad (19)$$

In the same vein, Eqs. (12)–(14) give [see also Eq. (31) of I]

$$\rho=\frac{m^3}{2\pi^2}W(\beta,m)-\frac{m(m^2-\mu^2)^{1/2}}{2\pi\beta}\left[1-\frac{H_1(\mu)}{(m^2-\mu^2)^{1/2}}\right]+O(m^2-\mu^2)\ . \quad (20)$$

It can be shown that despite their apparent structure, which involves the quantity $(m^2-\mu^2)^{1/2}$, the functions $\bar{u}$ and ρ are smooth, analytic functions of the variable μ; see, for instance, Chaba and Pathria,[9] especially the identity embodied in Eq. (63) of their paper.

To determine the specific heat of the system at constant volume, we need to know the quantity $(\partial\mu/\partial\beta)_\rho$, which can be obtained from Eq. (20). To leading order in $m^2-\mu^2$, we have

$$\left[\frac{\partial\mu}{\partial\beta}\right]_\rho=\frac{\beta m}{\pi}\left|\frac{dW}{d\beta}\right|\frac{(m^2-\mu^2)^{1/2}}{1+H_0(\mu)}\ , \quad (21)$$

whence

$$c_\rho\equiv-\beta^2\left[\frac{\partial\bar{u}}{\partial\beta}\right]_\rho=\frac{m^4\beta^2}{2\pi^2}\left|\frac{dZ}{d\beta}\right|-\frac{(\beta m)^4}{2\pi^3}\left[\frac{dW}{d\beta}\right]^2\frac{(m^2-\mu^2)^{1/2}}{1+H_0(\mu)}\ , \quad (22)$$

where $Z\equiv Z(\beta,m)$. Equation (22) should enable us to study the leading finite-size corrections to the bulk specific heat of the system in different geometries, including the rounding off of the bulk singularity into a smooth maximum.

III. ASYMPTOTIC ANALYSIS OF THE SPECIFIC HEAT

We shall now examine the asymptotic behavior of c_ρ in the vicinity of the bulk critical point $\beta=\beta_c$ which is determined by the condition

$$W(\beta_c,m)=\frac{2\pi^2\rho}{m^3} \quad (23)$$

[see Eq. (13)]. For β close to β_c, Eq. (20) reduces to

$$[(m^2-\mu^2)^{1/2}-H_1(\mu)]\simeq\frac{(\beta_c m)^2}{\pi}\left|\frac{dW}{d\beta}\right|_c t\ , \quad (24)$$

where

$$t=\frac{T-T_c}{T_c}\simeq\frac{\beta_c-\beta}{\beta_c}\quad(|t|\ll1)\ .$$

At the same time, Eq. (22) takes the form

$$c_\rho=\frac{m^4\beta_c^2}{2\pi^2}\left\{\left|\frac{dZ}{d\beta}\right|_c+\left[\beta_c\left[\frac{d^2Z}{d\beta^2}\right]_c-2\left|\frac{dZ}{d\beta}\right|_c\right]t\right\}-\frac{(\beta_c m)^4}{2\pi^3}\left[\frac{dW}{d\beta}\right]_c^2\frac{(m^2-\mu^2)^{1/2}}{1+H_0(\mu)}\ . \quad (25)$$

We shall now consider the regions $t>0$ and $t<0$ separately.

(a) $t>0$, $L_i\to\infty$. In this case, the quantity $m^2-\mu^2$ is of order unity and the functions $H_n(\mu)$ tend to zero exponentially; see Eq. (11). Equation (24) then gives

$$(m^2-\mu^2)^{1/2}\simeq\frac{(\beta_c m)^2}{\pi}\left|\frac{dW}{d\beta}\right|_c t\ , \quad (26)$$

whence

$$c_\rho\simeq\frac{m^4\beta_c^2}{2\pi^2}\left\{\left|\frac{dZ}{d\beta}\right|_c+\left[\beta_c\left[\frac{d^2Z}{d\beta^2}\right]_c-2\left|\frac{dZ}{d\beta}\right|_c\right]t\right\}-\frac{(\beta_c m)^6}{2\pi^4}\left|\frac{dW}{d\beta}\right|_c^3 t+O(te^{-L_<t})\ , \quad (27)$$

where $L_<$ denotes the shortest side of the container.

(b) $t<0$, $L_i\to\infty$. In this case, the quantity $m^2-\mu^2$ tends to zero, and the functions $H_n(\mu)$ diverge, in a manner which depends crucially on the geometry of the container. The various possibilities of interest are covered adequately by considering the following three cases.

(i) A *cuboid*, for which L_1, L_2, and L_3 are all large and comparable to one another. Here

$$H_1(\mu)\simeq\int_{-\infty}^{\infty}\frac{e^{-(m^2-\mu^2)^{1/2}\gamma(\vec{q})}}{\gamma(\vec{q})}d^3q$$

$$=\frac{4\pi}{L_1L_2L_3(m^2-\mu^2)} \tag{28}$$

and

$$H_0(\mu)\simeq\int_{-\infty}^{\infty}e^{-(m^2-\mu^2)^{1/2}\gamma(\vec{q})}d^3q$$

$$=\frac{8\pi}{L_1L_2L_3(m^2-\mu^2)^{3/2}}\ . \tag{29}$$

Equations (24) and (25) then give

$$m^2-\mu^2\simeq\frac{4\pi^2}{(\beta_c m)^2\,|\,dW/d\beta\,|_c L_1L_2L_3\,|\,t\,|} \tag{30}$$

and, hence,

$$c_\rho\simeq\frac{m^4\beta_c^2}{2\pi^2}\left\{\left|\frac{dZ}{d\beta}\right|_c+\left[\beta_c\left[\frac{d^2Z}{d\beta^2}\right]_c-2\left|\frac{dZ}{d\beta}\right|_c\right]t\right\}-\frac{1}{L_1L_2L_3t^2}\ . \tag{31}$$

(ii) A *rectangular channel,* for which $L_1=\infty$ while L_2 and L_3 are both large and comparable to each other. The functions $H_n(\mu)$ now involve summations over q_2 and q_3 only, so that asymptotically

$$H_1(\mu)\simeq\int_{-\infty}^{\infty}\frac{e^{-(m^2-\mu^2)^{1/2}\gamma(\vec{q})}}{\gamma(\vec{q})}d^2q$$

$$=\frac{2\pi}{L_2L_3(m^2-\mu^2)^{1/2}} \tag{32}$$

and

$$H_0(\mu)\simeq\int_{-\infty}^{\infty}e^{-(m^2-\mu^2)^{1/2}\gamma(\vec{q})}d^2q$$

$$=\frac{2\pi}{L_2L_3(m^2-\mu^2)}\ , \tag{33}$$

with the result that

$$m^2-\mu^2\simeq\frac{4\pi^4}{(\beta_c m)^4(dW/d\beta)_c^2(L_2L_3)^2t^2} \tag{34}$$

and

$$c_\rho\simeq\frac{m^4\beta_c^2}{2\pi^2}\left\{\left|\frac{dZ}{d\beta}\right|_c+\left[\beta_c\left[\frac{d^2Z}{d\beta^2}\right]_c-2\left|\frac{dZ}{d\beta}\right|_c\right]t\right\}-\frac{2\pi^2}{(\beta_c m)^2\,|\,dW/d\beta\,|_c(L_2L_3)^2\,|\,t\,|^3}\ . \tag{35}$$

(iii) A *film,* for which $L_{1,2}=\infty$ while L_3 is large. The functions $H_n(\mu)$ now involve summations over q_3 only and can be expressed in a closed form:

$$H_1(\mu)=2\sum_{q_3=1}^{\infty}\frac{e^{-(m^2-\mu^2)^{1/2}q_3L_3}}{q_3L_3}$$

$$=-\frac{2}{L_3}\ln(1-e^{-(m^2-\mu^2)^{1/2}L_3}) \tag{36a}$$

$$\simeq-\frac{2}{L_3}\ln[(m^2-\mu^2)^{1/2}L_3] \tag{36b}$$

and

$$H_0(\mu)=2\sum_{q_3=1}^{\infty}e^{-(m^2-\mu^2)^{1/2}q_3L_3}$$

$$=2(e^{(m^2-\mu^2)^{1/2}L_3}-1)^{-1} \tag{37a}$$

$$\simeq\frac{2}{(m^2-\mu^2)^{1/2}L_3}\ . \tag{37b}$$

Using the approximations (36b) and (37b), we obtain from Eqs. (24) and (25)

$$m^2-\mu^2\simeq\frac{1}{L_3^2}\exp\left[-\frac{(\beta_c m)^2}{\pi}\left|\frac{dW}{d\beta}\right|_c L_3\,|\,t\,|\right] \tag{38}$$

and

$$c_\rho\simeq\frac{m^4\beta_c^2}{2\pi^2}\left\{\left|\frac{dZ}{d\beta}\right|_c+\left[\beta_c\left[\frac{d^2Z}{d\beta^2}\right]_c-2\left|\frac{dZ}{d\beta}\right|_c\right]t\right\}-\frac{(\beta_c m)^4}{4\pi^3}\left[\frac{dW}{d\beta}\right]_c^2\frac{1}{L_3}\times\exp\left[-\frac{(\beta_c m)^2}{\pi}\left|\frac{dW}{d\beta}\right|_c L_3\,|\,t\,|\right]\ . \tag{39}$$

We thus observe that while, for $t>0$, the finite-size correction to the bulk result,

$$(c_\rho)_B=\frac{m^4\beta_c^2}{2\pi^2}\left\{\left|\frac{dZ}{d\beta}\right|_c+\left[\beta_c\left[\frac{d^2Z}{d\beta^2}\right]_c-2\left|\frac{dZ}{d\beta}\right|_c\right]t\right\}-\frac{(\beta_c m)^6}{2\pi^4}\left|\frac{dW}{d\beta}\right|_c^3 t\quad(t>0) \tag{40}$$

is exponential for *all* geometries, this is generally not true for $t<0$. Here, the corresponding correction to the bulk result,

$$(c_\rho)_B=\frac{m^4\beta_c^2}{2\pi^2}\left\{\left|\frac{dZ}{d\beta}\right|_c+\left[\beta_c\left[\frac{d^{2Z}}{d\beta^2}\right]_c-2\left|\frac{dZ}{d\beta}\right|_c\right]t\right\}\quad(t<0) \tag{41}$$

depends crucially on the geometry of the enclosure. It is only in the case of a film that the relevant correction turns out to be exponential in nature, as was indeed found previously by Barber[5] and by Barber and Fisher[6] in their analysis of the thermodynamic properties of an ideal, nonrelativistic Bose film. In other geometries, such as the cuboid or the rectangular channel, the relevant corrections are found to depend on L_i through a power law instead. This, in turn, affects their t dependence as well for, in the spirit of the scaling theory for finite-size effects,[7,8] one expects these corrections to be of the form

$$L_<^{-1} f(L_< t) \; . \tag{42}$$

It is gratifying to note that all our results, viz., Eqs. (27), (31), (35), and (39), are consistent with this expectation.

At this stage we would like to point out that, in the special case of a film geometry, Eqs. (36a) and (37a) enable us to derive our final results in a closed form which holds over a considerable range of temperatures—in fact, from $T \gtrsim T_c$ down to $T=0$ K. Using (36a) in conjunction with (20), we obtain

$$\rho = \frac{m^3}{2\pi^2} W(\beta,m) - \frac{m}{\pi\beta L_3} \ln(2\sinh y) \; , \tag{43}$$

where y is the thermogeometric parameter appropriate to the system,[1] viz.,

$$y \equiv \tfrac{1}{2}(m^2-\mu^2)^{1/2} L_3 \; . \tag{44}$$

It follows that

$$y(\beta) = \sinh^{-1}\left[\tfrac{1}{2}\exp\left[\frac{m^2\beta L_3}{2\pi}[W(\beta,m)-W(\beta_c,m)]\right]\right] . \tag{45}$$

Equations (22) and (37a) then give

$$c_\rho = \frac{m^4\beta^2}{2\pi^2}\left|\frac{dZ}{d\beta}\right| - \frac{(\beta m)^4}{\pi^3}\left[\frac{dW}{d\beta}\right]^2 \frac{1}{L_3} y \tanh y \; . \tag{46}$$

As a check we observe that in the nonrelativistic limit, where

$$Z(\beta,m) = \frac{3}{2}\left[\frac{\pi}{2m^5}\right]^{1/2} \zeta\left[\frac{5}{2}\right]\beta^{-5/2} ,$$
$$W(\beta,m) = \left[\frac{\pi}{2m^3}\right]^{1/2} \zeta\left[\frac{3}{2}\right]\beta^{-3/2} , \tag{47}$$
$$\beta_c m = \frac{1}{2\pi}\left[\zeta\left[\frac{3}{2}\right]\frac{m^3}{\rho}\right]^{2/3} ,$$

Eqs. (45) and (46) reduce to

$$y(\beta) = \sinh^{-1}\left[\tfrac{1}{2}\exp\left\{\frac{\pi\beta\rho L_3}{m}\left[1-\left[\frac{\beta_c}{\beta}\right]^{3/2}\right]\right\}\right] \tag{48a}$$

and

$$c_\rho = \frac{15}{4}\frac{\zeta(5/2)}{\zeta(3/2)}\rho\left[\frac{\beta_c}{\beta}\right]^{3/2} - \frac{9[\zeta(3/2)]^2 m}{8\pi^2\beta L_3} y \tanh y \; , \tag{48b}$$

which agree with the corresponding results of Pathria[10] who first studied the problem of Bose-Einstein condensation in ideal Bose films using the concept of the thermogeometric parameter y. In the extreme relativistic limit, on the other hand,

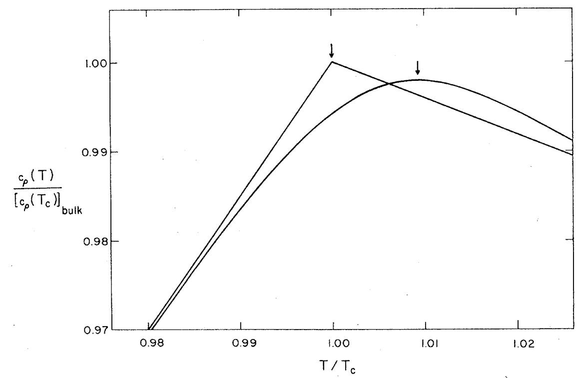

FIG. 1. The nonrelativistic variation of $c_\rho(T)/[c_\rho(T_c)]_{\text{bulk}}$ with T/T_c for a film of thickness L_3 such that $\rho^{1/3}L_3=50$; for comparison, the bulk limit $\rho^{1/3}L_3 \to \infty$ is also displayed. Reduction in the height of the maximum and its shift towards higher temperatures are clearly seen.

$$Z(\beta,m) = \frac{2\pi^4}{15m^4}\beta^{-4} ,$$
$$W(\beta,m) = \frac{2\pi^2}{3m^2}\beta^{-2} , \tag{49}$$
$$\beta_c m = (m^3/3\rho)^{1/2} ,$$

with the result that

$$y(\beta) = \sinh^{-1}\left\{\tfrac{1}{2}\exp\left\{-\frac{\pi\beta\rho L_3}{m}\left[1-\left[\frac{\beta_c}{\beta}\right]^2\right]\right\}\right\} \tag{50a}$$

and

$$c_\rho = \frac{4\pi^2}{5}\left[\frac{3\rho}{m^3}\right]^{1/2}\rho\left[\frac{\beta_c}{\beta}\right]^3 - \frac{16\pi}{9\beta^2 L_3} y \tanh y \; . \tag{50b}$$

The leading term in (50b) is in agreement with the corresponding bulk result of Haber and Weldon[11] who introduced the possibility of particle-antiparticle pair production into this problem, while the finite-size correction to the relativistic bulk result is derived here for the first time. The variation of c_ρ with T, near $T=T_c$, is shown in Figs. 1 and 2.

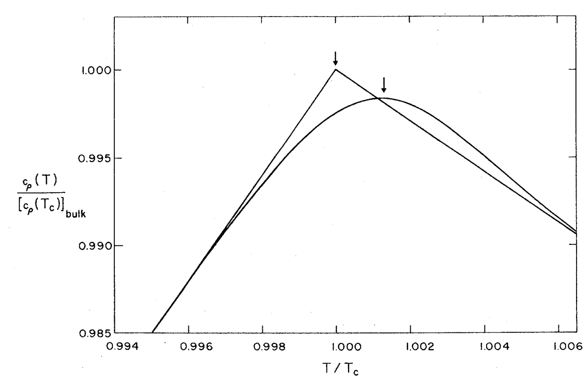

FIG. 2. Same as in Fig. 1, except that now $\rho/m^3=100$, which makes this case essentially extreme relativistic.

IV. THE SPECIFIC-HEAT MAXIMUM

The regions (a) and (b) studied in the preceding section correspond to the limiting situations $(m^2-\mu^2)^{1/2}L_< \to\infty$ and $(m^2-\mu^2)^{1/2}L_<\to 0$, respectively. In between (a) and (b) lies a "core region," characterized by the condition $(m^2-\mu^2)^{1/2}L_<=O(1)$, which marks the onset of the phase transition in the system. It is not difficult to see that the bulk critical point $(t=0)$ lies in this region for, according to Eq. (20) or (24).

$$[H_1(\mu)/(m^2-\mu^2)^{1/2}]_{t=0}=1 \tag{51}$$

and hence, by (11),

$$[(m^2-\mu^2)^{1/2}L_<]_{t=0}=O(1) . \tag{52}$$

It turns out that the specific heat of the system possesses a smooth maximum which also lies in the core region. To see this, we maximize c_ρ, as given by Eq. (25), with the help of the formula (21) for $(\partial\mu/\partial\beta)_\rho$, which leads to the condition

$$\frac{1+H_0(\mu^*)+(m^2-\mu^{*2})^{1/2}H_{-1}(\mu^*)}{[1+H_0(\mu^*)]^3} =\frac{\pi^2[\beta_c(d^2Z/d\beta^2)_c-2\,|\,dZ/d\beta\,|_c]}{\beta_c^4 m^2\,|\,dW/d\beta\,|_c^3} , \tag{53}$$

where μ^* is the value of μ at the location of the maximum, i.e., at $t=t^*$, say. Now, the right-hand side of Eq. (53) is seen to be $O(1)$, ranging from the value of 0.788. . . in the nonrelativistic limit to the value 0.675 in the extreme relativistic limit. Accordingly, the quantity $[(m^2-\mu^{*2})^{1/2}L_<]$ would also be $O(1)$, with the result that, for a precise evaluation of μ^*, the functions $H_0(\mu)$ and $H_{-1}(\mu)$, which appear on the left-hand side of (53) and depend crucially on the geometry of the enclosure, will have to be computed numerically.

In the case of a film geometry, however, the expression on the left-hand side of (53) can be written down in more familiar terms. Noting that the function $H_{-1}(\mu)$ in this case is given by

$$H_{-1}(\mu)=2\sum_{q_3=1}^{\infty} q_3L_3e^{-(m^2-\mu^2)^{1/2}q_3L_3}=\tfrac{1}{2}L_3 \operatorname{csch}^2 y , \tag{54}$$

where y is defined in (44), and recalling Eq. (37a) for $H_0(\mu)$, we obtain for this expression the functional form

$$f(y)=(\coth y+y \operatorname{csch}^2 y)/(\coth y)^3 ; \tag{55}$$

the corresponding value of y^* turns out to be 0.854. . . in the nonrelativistic case and 0.738. . . in the extreme relativistic case. In passing, we note the formal similarity between Eqs. (53) and (55) on one hand and the corresponding nonrelativistic results derived earlier by Greenspoon and Pathria[12] and by Pathria[10] on the other; cf. their Eqs. (27) and (30), respectively.

Once μ^* is determined, Eq. (24) enables us to derive t^* and Eq. (25) c_ρ^*. Quite generally, t^* turns out to be positive, which implies a shift towards higher temperatures, while c_ρ^* turns out to be smaller than the bulk $(c_\rho)_c$, which implies a reduction in the height of the maximum; it is easily seen that both these effects are $O(L_<^{-1})$. As $L_<\to\infty$, the whole of the core region gets submerged into a singularity at $t=0$ and the specific-heat maximum turns into a cusp singularity characteristic of the bulk system.

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APPENDIX

We wish to evaluate the integrals

$$J_\nu(x,y;\xi)=\int_0^\infty \cosh(jx)\frac{K_\nu[y(j^2+\xi^2)^{1/2}]}{(j^2+\xi^2)^{\nu/2}}dj \tag{A1}$$

and

$$L_\nu(x,y;\xi)=\int_0^\infty j^2\cosh(jx)\frac{K_\nu[y(j^2+\xi^2)^{1/2}]}{(j^2+\xi^2)^{\nu/2}} =\left[\frac{\partial^2 J_\nu(x,y;\xi)}{\partial x^2}\right]_{y,\xi} . \tag{A2}$$

For this we employ the integral representation[13]

$$K_\nu(\alpha z)=\frac{z^\nu}{2}\int_0^\infty \exp\left[-\tfrac{1}{2}\alpha\left[t+\frac{z^2}{t}\right]\right]t^{-\nu-1}dt , \tag{A3}$$

whereby (A1) takes the form

$$J_\nu(x,y;\xi)=\tfrac{1}{2}\int_0^\infty\int_0^\infty \cosh(jx)\exp\left[-\frac{1}{2}y\left[t+\frac{j^2+\xi^2}{t}\right]\right]t^{-\nu-1}dt\,dj .$$

The integration over j is now straightforward; using the formula[13]

$$\int_0^\infty \cosh(aj)e^{-bj^2}dj=\frac{1}{2}\left[\frac{\pi}{b}\right]^{1/2}e^{a^2/4b} ,$$

we get

$$J_\nu(x,y;\xi)=\tfrac{1}{2}\left[\frac{\pi}{2y}\right]^{1/2}\int_0^\infty \exp\left\{-\frac{1}{2}\left[\left[y-\frac{x^2}{y}\right]t+\frac{y\xi^2}{t}\right]\right\}t^{-\nu-(1/2)}dt .$$

We now use the representation (A3) in reverse and obtain the desired result

$$J_\nu(x,y;\xi)=\left[\frac{\pi\xi}{2}\right]^{1/2}\frac{(y^2-x^2)^{\nu/2-(1/4)}}{(y\xi)^\nu}K_{\nu-(1/2)}[\xi(y^2-x^2)^{1/2}]\ . \tag{A4}$$

It follows that

$$L_\nu(x,y;\xi)=\left[\frac{\pi\xi^3}{2}\right]^{1/2}\frac{1}{(y\xi)^\nu}\{(y^2-x^2)^{\nu/2-(3/4)}K_{\nu-(3/2)}[\xi(y^2-x^2)^{1/2}]$$
$$+\xi x^2(y^2-x^2)^{\nu/2-(5/4)}K_{\nu-(5/2)}[\xi(y^2-x^2)^{1/2}]\}\ , \tag{A5}$$

whence

$$yL_{\nu+1}(x,y;\xi)-J_\nu(x,y;\xi)=\left[\frac{\pi\xi^3}{2}\right]^{1/2}\frac{x^2(y^2-x^2)^{\nu/2-(3/4)}}{(y\xi)^\nu}K_{\nu-(3/2)}[\xi(y^2-x^2)^{1/2}]\ . \tag{A6}$$

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