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# Vector electromagnetic modes of an optical resonator

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Maxwell's equations are solved for the self-reproducing TEM Gaussian-Hermite modes of a stable resonator formed by two curved mirrors. The boundary condition is satisfied that the electric field vector of a mode is everywhere perpendicular to the mirror surfaces, which are regarded as perfect conductors. The vector solution for a semiconfocal or confocal resonator geometry is compared with the standard scalar solution, and a correction to the scalar representation of the field is discussed.

## I. INTRODUCTION

Our aim is to calculate the electromagnetic modes of a stable Fabry-Perot resonator defined by two curved reflectors facing each other. The approach taken in this work is first to obtain paraxial solutions of Maxwell's equations for the electric and magnetic field configurations of Gaussian beam modes in free space, and then fit the reflectors of an open resonator to the Gaussian mode solutions so obtained.

In the microwave region of the spectrum, electromagnetic theory is nearly always based on Maxwell's equations. By contrast, optical beams—whether unguided or supported by open resonators—have mostly been treated with use of a scalar theory. A difficulty involved in characterizing an electromagnetic field by a single scalar wave function is that the boundary conditions on the surfaces of curved reflectors cannot be satisfied. To see this, let us imagine that the end reflectors of an optical cavity are perfect conductors. Then the tangential component of the electric field must everywhere vanish on the mirror surfaces. We recall that each Cartesian component of the electric and magnetic vectors obeys the scalar wave equation. For a plane-polarized beam propagating along the $z$ axis, the scalar wave function is usually taken to be the transverse component of the electric field, say $E_x$. In solving the scalar wave equation, one assumes that the wave function vanishes on the cavity boundaries. It is clear, however, that for curved reflectors the stipulation that $E_x$ vanishes on the mirror surfaces is not equivalent to the boundary condition that the electric field lines are perpendicular to the mirror surfaces.

It is generally assumed that, so long as the physical dimensions of an open resonator are large compared with the wavelength of the radiation, a scalar theory is satisfactory for describing the cavity TEM modes. In the optical region, where open resonators currently used have mirror dimensions which are many orders of magnitude larger than a wavelength, as expected the scalar theory has enjoyed remarkable success. Nevertheless, even in this short-wavelength portion of the spectrum, there is motivation for working out a vector field solution of an ideal (infinite-aperture) open resonator problem which can be compared with the conventional scalar wave solution of the same problem. By having a vector field solution at hand, precise knowledge about the interpretation and accuracy of the scalar representation is obtained. In the millimeter wave and microwave range, where cavity dimensions may be comparable to a wavelength, one does not expect the scalar theory to be satisfactory. In this spectral region, it will be shown that a vector field treatment, in which the Gaussian scalar potential includes nonparaxial correction terms, is needed in some practical situations.

Another consideration is that situations frequently arise where, because of the complexity of the problem, one must be content with a scalar representation of the electromagnetic field. For stable resonators of such configuration that a significant fraction of the energy "spills over" the outer edges of the mirrors, and also for unstable resonators, it is virtually impossible to obtain the electromagnetic field configurations of the modes. Another important example where a vector wave theory is not feasible is the case where the resonator, either stable or unstable, contains a nonuniform gain medium. In cases like these, one employs a computer to solve the scalar wave equation subject to the boundary conditions mentioned above. Therefore, far from being merely an academic question, practical considerations compel us to question when the scalar description is reasonably valid. The vector Gaussian mode solution obtained in this paper provides a basis for conjecture about how satisfactory the scalar representation is in a variety of circumstances.

## II. GENERAL FORMALISM

The free-space electromagnetic field may be expressed in terms of two partial fields which are the electromagnetic duals of each one. On the one hand, the field is derived from the electric Hertz vector $\vec{\Pi}$ according to (time factor $e^{i\omega t}$ understood)

$$\vec{H}=\vec{\nabla}\times\vec{\Pi}\ , \tag{1}$$

$$\vec{E}=\frac{1}{ik}\vec{\nabla}\times\vec{H}=\frac{1}{ik}\vec{\nabla}\times\vec{\nabla}\times\vec{\Pi}\ , \tag{2}$$

where $k=\omega/c$ is the wave number. The field satisfies Maxwell's equations provided $\vec{\Pi}$ obeys the vector

 

Helmholtz equation

$$\nabla^2\vec{\Pi}+k^2\vec{\Pi}=\vec{0} \ . \tag{3}$$

By $\nabla^2$ acting on a vector one understands

$$\nabla^2\vec{\Pi}=\vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi})-\vec{\nabla}\times\vec{\nabla}\times\vec{\Pi} \ . \tag{4}$$

Equation (3) expressed in curvilinear coordinates corresponds to three coupled scalar equations, but the solution of this system for a component of $\vec{\Pi}$ is in most cases intractable. If the vector function $\vec{\Pi}$ is resolved into its Cartesian components, however, one obtains the three independent scalar Helmholtz equations

$$\nabla^2\Pi_j+k^2\Pi_j=0, \quad j=1,2,3 \ . \tag{5}$$

As explained by Stratton,[1] for example, the operator $\nabla^2$ in Eq. (5) may then be expressed in curvilinear as well as Cartesian coordinates.

We now set $\vec{\Pi}=\hat{a}V$, in which $\hat{a}$ is a *constant* vector of unit length. The resulting field is said to be of the $E$ type. If the beam is propagating along the $z$ axis, we further distinguish between the cases where $\vec{\Pi}$ is transverse and longitudinal:

(a) $\hat{a}=\hat{e}_1$, describing what are called TEM modes in the literature,

(b) $\hat{a}=\hat{e}_3$, describing axially symmetric TM modes.

On the other hand, the free-space complementary field is derived from the equations

$$\vec{E}=\vec{\nabla}\times\vec{\Pi}^* \ , \tag{6}$$

$$\vec{H}=\frac{1}{ik}\vec{\nabla}\times\vec{E}=-\frac{1}{ik}\vec{\nabla}\times\vec{\nabla}\times\vec{\Pi}^* \ , \tag{7}$$

where $\Pi^*$ is the magnetic Hertz vector. By duality, this partial field also obeys Maxwell's equations provided $\vec{\Pi}^*$ satisfies the vector Helmholtz equation. As above, one lets $\vec{\Pi}^*=\hat{a}\psi$, with $\hat{a}$ denoting a constant vector, whereupon $\psi$ obeys the scalar Helmholtz equation. This procedure describes an $H$-type field. For the cases where $\vec{\Pi}^*$ is transverse and longitudinal we now have

(c) $\hat{a}=\hat{e}_2$, describing what also are called TEM modes in the literature,

(d) $\hat{a}=\hat{e}_3$, describing axially symmetric TE modes.

It is perhaps worth mentioning that the TEM modes, including the fundamental mode, lack axial symmetry. The scalar field has axial symmetry but the TEM vector field, being nearly plane polarized, clearly is not axially symmetric.

The axially symmetric TM and TE Gaussian-Laguerre unguided beam modes were first described by Goubau and Schwering.[2] The properties of these vector modes were later treated in more detail by Davis and Patsakos.[3] Laser oscillators employing a stable open resonator normally generate the $TEM_{00}$ mode. This mode is favored because (1) the fundamental TM and TE modes have larger diameters than the fundamental TEM mode (see Fig. 1), and therefore they may be apertured by the plasma tube; and

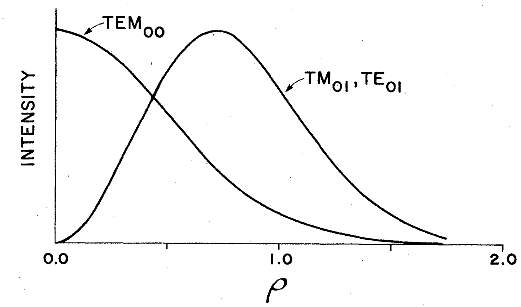


FIG. 1. Intensity profiles of the fundamental TEM mode and the lowest-order axially symmetric TM and TE modes.

(2) laser resonators invariably have some axial asymmetry arising from Brewster angle surfaces (an extreme example of symmetry-breaking elements) or reflector anisotropies. For this reason, in the remainder of this paper we will consider solutions of the scalar Helmholtz equation $\nabla^2V+k^2V=0$ in a Cartesian coordinate system, as is required to describe the usual TEM modes. Incidentally, the term TEM mode conceals the fact that neither the electric nor the magnetic fields of any bounded electromagnetic wave in free space are purely transverse. Some field lines of a $TEM_{00}$ beam mode are shown in Fig. 2.

For a TEM partial field of the type given by Eqs. (1) and (2), with $\vec{\Pi}=\hat{e}_1V$, one obtains

$$\vec{H}=\hat{e}_2\frac{\partial V}{\partial z}-\hat{e}_3\frac{\partial V}{\partial y} \ . \tag{8}$$

For this case the lines of $\vec{H}$ are confined to the $yz$ plane. The associated electric field is

$$\vec{E}=\hat{e}_1\left[\frac{\partial^2V}{\partial x^2}+k^2V\right]+\hat{e}_2\frac{\partial^2V}{\partial x\partial y}+\hat{e}_3\frac{\partial^2V}{\partial x\partial z} \ . \tag{9}$$

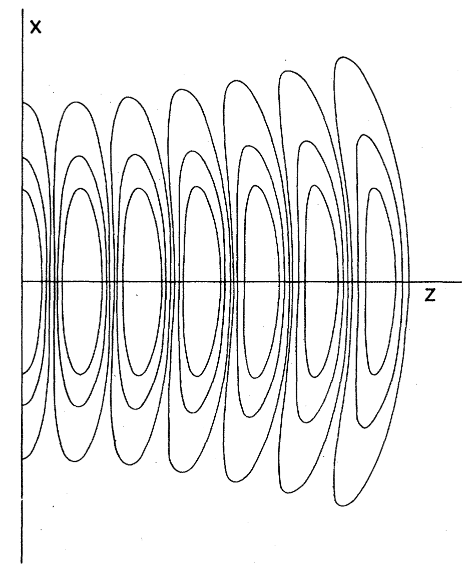


FIG. 2. Section $y=0$ showing electric field lines of a Gaussian $TEM_{00}$ mode at a fixed time. Plot is for the case $w_0=\lambda$ so that the Rayleigh length is $l=\pi\lambda$.

## III. PARAXIAL TEM GAUSSIAN MODES

Gaussian beam mode solutions of the paraxial scalar wave equation are discussed in several papers and textbooks. In a Cartesian coordinates system, one obtains the well-known Gaussian-Hermite traveling-wave solutions[4]

$$V=iQH_m\left[\sqrt{2}\frac{x}{w}\right]H_n\left[\sqrt{2}\frac{y}{w}\right]\times\exp[-iQ\rho^2+i(m+n)\phi-ikz]\ , \tag{10}$$

where $\rho=(x^2+y^2)^{1/2}/w_0$ is the dimensionless radial coordinate, $\phi=\arctan(z/l)$, $l=\frac{1}{2}kw_0^2$ is the Rayleigh length parameter, and one introduces the complex beam parameter

$$iQ=\frac{1}{1-iz/l}\ . \tag{11}$$

The quantity $w_0$ is the spot size at the beam waist (plane $z=0$), and

$$w(z)=w_0(1+z^2/l^2)^{1/2} \tag{12}$$

is the beam contour function. Finally, $H_n(x)$ is a Hermite polynomial

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\ .$$

If needed, higher-order corrections to the paraxial solution (10) could be obtained using the approach of Agrawal and Pattanayak.[5]

At this point, we restrict attention to the $\mathrm{TEM}_{00}$ mode ($m=n=0$) to simplify the discussion. Since a resonator designed to support the fundamental mode also will support the higher-order $\mathrm{TEM}_{mn}$ modes, this is not an essential limitation. From Eq. (10) the paraxial solution of the scalar wave equation for the fundamental mode is

$$V=iQe^{-i(Q\rho^2+kz)}\ , \tag{13}$$

where it is useful to write

$$iQ=\frac{w_0}{w}e^{i\phi}\ . \tag{14}$$

For the $E$-type fundamental TEM mode, the field components are[6]

$$H_x=0\ (\text{exact}),\quad H_y=V,\quad H_z=-\frac{Qy}{l}V\ ,$$
$$E_x=V,\quad E_y=(iQ)^2\frac{xy}{l^2}V,\quad E_z=-\frac{Qx}{l}V\ . \tag{15}$$

In deriving these expressions for the field components, only leading-order terms in the small expansion parameter[7] $f=w_0/2l$ have been retained. Specifically, terms of order $f^2$ are neglected in our expressions for $H_y$ and $E_x$, terms of order $f^3$ are neglected in the expressions for $H_z$ and $E_z$, and terms of order $f^4$ are ignored in the result for $E_y$.

For the $H$-type mode which is the electromagnetic dual of the $E$-type fundamental mode just treated, the six field components are

$$E_x=V,\quad E_y=0\ (\text{exact}),\quad E_z=-\frac{Qx}{l}V\ ,$$
$$H_x=(iQ)^2\frac{xy}{l^2}V,\quad H_y=V,\quad H_z=-\frac{Qy}{l}V\ , \tag{16}$$

where again these expressions for the field component are correct to leading order in the power-series expansion parameter $f$. By analogy to the $E$-type mode case, terms of order $f^2$ are neglected in these expressions for $E_x$ and $H_y$, terms of order $f^3$ are neglected in the expressions for $E_z$ and $H_z$, and terms of order $f^4$ are ignored in the expression for $H_x$.

Let us now consider the properties of a mode which is the symmetrized linear combination of the above $E$-type and $H$-type $\mathrm{TEM}_{00}$ modes. The electric field of a symmetrized mode is conveniently written

$$\vec{E}=\frac{1}{ik}\vec{\nabla}\times\vec{\nabla}\times(\hat{e}_1V)+\vec{\nabla}\times(\hat{e}_2V)\ ,$$

where the scalar potential is given by Eq. (13). When two such counterpropagating traveling Gaussian beams are superposed, this prescription leads to the standing-wave electric field ($r^2=x^2+y^2$)

$$E_x=\frac{w_0}{w}e^{-r^2/w^2}\sin[k(z+r^2/2R)-\phi]\cos(\omega t)\ , \tag{17a}$$

$$E_y=\frac{1}{2}\left[\frac{w_0}{w}\right]^3\frac{xy}{l^2}e^{-r^2/w^2}\times\sin[k(z+r^2/2R)-3\phi]\cos(\omega t)\ , \tag{17b}$$

$$E_z=-\left[\frac{w_0}{w}\right]^2\frac{x}{l}e^{-r^2/w^2}\times\cos[k(z+r^2/2R)-2\phi]\cos(\omega t)\ , \tag{17c}$$

in which

$$R=z+\frac{l^2}{z} \tag{18}$$

is commonly known as the radius of curvature of a wave front. For use below, in Eqs. (17) we have written the actual electric components of the mode rather than the phasor components. It is seen from these expressions that on the plane surface $z=0$, the transverse electric field is zero: $E_x=E_y=0$. Thus if a flat perfectly conducting mirror is placed in the plane $z=0$ (beam waist), the analytic solution given above does satisfy the boundary condition that the tangential component of $\vec{E}$ vanishes on the reflecting surface.

We consider now the modes of a Fabry-Perot resonator which consists of a concave perfectly conducting reflector facing the flat reflector. The curvature of this reflector is to be selected so as to fit the field lines described by the expressions we have developed. We wish to discover the surfaces which are perpendicular to the electric field. If

$d\vec{r}$ denotes a translation tangent to such a surface, then $\vec{E}\cdot d\vec{r}=0$. For the moment let us limit our discussion to the planes of constant $y$, whereupon $d\vec{r}=\hat{e}_1 x+\hat{e}_3 dz$. Therefore, a perfect conductor will cut the planes of constant $y$ in lines with slope

$$\frac{dz}{dx}=-\frac{E_x}{E_z}\ .$$

From Eqs. (17) one obtains

$$\frac{dz}{dx}=\frac{wl\sin[k(z+r^2/2R)-\phi]}{w_0 x\cos[k(z+r^2/2R)-2\phi]}\ . \tag{19}$$

This differential equation could be integrated numerically to obtain the required reflector curvature to any desired degree of accuracy. However, for our present purpose an approximate analytic result will suffice. Although $w$, $R$, and $\phi$ weakly depend on $z$, for a paraxial beam we can suppose these quantities are constant over the surface of a curved mirror. In this work we assume a semiconfocal resonator geometry, so that the curved mirror is located a distance $z=l$ from the flat mirror. The phase angle of the beam on the curved mirror is then $\phi=\pi/4$. On the reflector surface the relation

$$k(z+r^2/2R)-\phi=q\pi,\quad q=1,2,3,\ldots$$

is approximately obeyed, so that

$$\sin[k(z+r^2/2R)-\phi]=(-1)^q(z-z_0+x^2/2R)\ ,$$

where $kz_0=q\pi+\phi-ky^2/2R$. Furthermore

$$\cos[k(z+r^2/2R)-2\phi]=\cos(q\pi-\phi)=(-1)^q\frac{1}{\sqrt{2}}\ .$$

Hence, using $w/w_0=\sqrt{2}$ when $z=l$, one obtains

$$x\frac{dz'}{dx}=2kl(z'=x^2/2R)\ , \tag{20}$$

where we have let $z'=z-z_0$. Solving this differential equation gives the parabola

$$z=z_0-\frac{x^2}{2R'}\ , \tag{21}$$

where

$$R'=R(1-1/kl)\ . \tag{22}$$

This result may be compared with the prediction of scalar diffraction theory. The fundamental Gaussian beam mode is represented by expression (13):

$$V=\frac{w_0}{w}\exp[-r^2/w^2-i(kz+kr^2/2R-\phi)]\ .$$

Given a beam of this type, it is assumed one can form an open resonator simply by inserting two mirrors with radii of curvature which match those of the propagating beam phase fronts. It is seen that at an axial distance $z_0$ from the beam waist, the phase fronts of the Gaussian scalar disturbance are paraboloids which obey

$$z=z_0-\frac{r^2}{2R}\ .$$

Thus, according to scalar theory the radius of curvature of the spherical mirror is $R$ rather than $R'=R(1-1/kl)$ obtained in the vector theory. We can understand this discrepancy as follows. In a scalar theory, when describing a TEM mode one identifies the wave disturbance as a single transverse electric field component, say $E_x$. Referring to Fig. 3, we see that for an arbitrary electric field line $E_x=0$ at the point $S$. However, if the reflector surface is represented by the dashed line, the electric field line in question is normal to the reflector surface at point $P$. Thus one can reason that the reflector curvature $R'$ is smaller than the curvature $R$ obtained by assuming $E_x=0$ over the surface.

To this point, we have examined the reflector curvature in planes of constant $y$. We now consider lines on the reflector surface in planes of constant $x$. A small displacement along such a line is $d\vec{r}=\hat{e}_2 dy+\hat{e}_3 dz$, so that the condition $\vec{E}\cdot d\vec{r}=0$ yields

$$\frac{dz}{dy}=-\frac{E_y}{E_z}\ .$$

From Eqs. (17) one obtains

$$\frac{dz}{dy}=\frac{w_0 y\sin[k(z+r^2/2R)-3\phi]}{2wl\cos[k(z+r^2/2R)-2\phi]}\ . \tag{23}$$

In the present case, to a close approximation

$$\sin[k(z+r^2/2R)-3\phi]=(-1)^{q+1}\ ,$$

and as before,

$$\cos[k(z+r^2/2R)-2\phi]=(-1)^q\frac{1}{\sqrt{2}}\ .$$

Thus Eq. (23) becomes approximately

$$\frac{dz}{dy}=-\frac{y}{2l}\ , \tag{24}$$

which has the solution

$$z=z_0-\frac{y^2}{2R}\ . \tag{25}$$

In deriving this expression we have used $R=z+l^2/z=2l$.

These findings indicate that, under the assumed conditions, in planes of constant $y$ the lines cut by the surface perpendicular to the electric field have curvature $R'=R(1-1/kl)$, while in planes of constant $x$ the lines

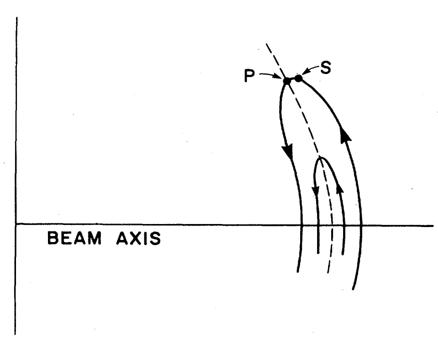


FIG. 3. Several electric field lines in plane $y=0$. Dashed line corresponds to the contour of a perfect reflector.

cut by this surface have curvature $R$, the curvature of the phase front of the scalar disturbance. A more precise calculation might reveal that surfaces perpendicular to the electric field are in fact axially symmetric for the assumed symmetric superposition of $E$-type and $H$-type Gaussian modes, a point which we hope to explore in future work. In any event, in actual practice cavity mirrors will continue to be surfaces of revolution, so the field supported by these open structures may turn out to be slightly different than the symmetric linear combination of $E$-type and $H$-type Gaussian modes proposed here.

Although we have treated a semiconfocal resonator, our solutions apply equally well to a confocal resonator. In the latter case, of course, the transverse electric field need not vanish in the plane $z=0$.

## IV. CONCLUSIONS

The development of a vector field description of the modes of a Fabry-Perot resonator serves to illuminate some of the difficulties which arise when the electromagnetic field is represented by a single scalar function. Furthermore, a vector framework yields the field configurations, which in some applications are of interest. A difference between the scalar and vector theories lies in the manner the boundary conditions are met. In this paper, the mirror surfaces are modeled as perfect conductors, and consequently we require that the electric field is everywhere perpendicular to these boundaries. To our knowledge, this is the first time such an approach has been used to solve an open resonator problem, though it is standard procedure for microwave cavity resonators.

For a semiconfocal or confocal resonator, we find that on a curved mirror the relationship between the phase front curvature $R$ of the scalar potential $V(\vec{r})$ and the mirror radius $R'$ is given by Eq. (22). One notes that $1/kl=2f^2$, where we recall that for a paraxial beam the power-series expansion parameter $f$ is much less than unity. In the visible and infrared portion of the spectrum, typically, $f$ is roughly $10^{-3}$, so our correction, being of order $f^2$, is not of practical significance. In the millimeter wave and microwave regions, however, open resonators do exist for which the modes are not paraxial. If reasonable accuracy is to be achieved in describing such cases it must be appreciated that the approximate Gaussian-Hermite scalar potentials of Eq. (10) are no longer valid and the vector representation of the field must be employed in order to closely satisfy the boundary conditions on the curved reflector surfaces.

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