# Activation rates for nonlinear stochastic flows driven by non-Gaussian noise

C. Van den Broeck* and P. Hänggi

*Polytechnic Institute of New York, Department of Physics, Brooklyn, New York 11201*

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Activation rates are calculated for stochastic bistable flows driven by asymmetric dichotomic Markov noise (a two-state Markov process). This noise contains as limits both a particular type of non-Gaussian white shot noise and white Gaussian noise. Apart from investigating the role of colored noise on the escape rates, one can thus also study the influence of the non-Gaussian nature of the noise on these rates. The rate for white shot noise differs in leading order (Arrhenius factor) from the corresponding rate for white Gaussian noise of equal strength. In evaluating the rates we demonstrate the advantage of using transport theory over a mean first-passage time approach for cases with generally non-white and non-Gaussian noise sources. For white shot noise with exponentially distributed weights we succeed in evaluating the mean first-passage time of the corresponding integro-differential master-equation dynamics. The rate is shown to coincide in the weak noise limit with the inverse mean first-passage time.

## I. INTRODUCTION

This paper considers dynamical aspects of nonlinear systems driven by random forces of the white-shot-noise type. Such noise is composed of a series of weighted impulses which occur at Poisson arrival times (see Fig. 1). Our interest is in nonlinear systems in which the nonlinearity leads to instabilities and bistable behavior in certain ranges of the control parameter. Typical systems would be semiconductor instabilities such as Esaki diodes, Josephson-tunneling junctions, or optical bistability, all being driven by a noisy control parameter. In the bistable region one expects that the deterministic stability of the metastable state corresponds in a stochastic description to a very slow activation or transition rate from the metastable to the globally stable state. Historically such rate processes have interested scientists and engineers over many decades, most notably in the fields of chemical kinetics, transport in semiconductors, and biological systems.

Bistable systems often resemble the model of a Brownian particle moving in a potential with two (or perhaps more) adjacent wells and a barrier in between, which prevent the particles from jumping too often. In this context, Kramers' work[1] represents a milestone in the field. Since Kramers, a number of investigators have improved or extended the theory in several points. As a sample of papers, we mention here the connection between transport theory (rate approach) and the mean first-passage time for one-dimensional Fokker-Planck processes,[2–5] the extension to multidimensional Fokker-Planck systems obeying detailed balance,[6–8] and those generally not obeying detailed balance,[9,10] the influence of a non-Gaussian, Markovian thermal heat bath,[11] and the recent work on the role of frequency-dependent damping in various viscosity regimes.[12–15]

In the study of dynamical effects, the evaluation of the mean first-passage time[16–18] represents an important concept yielding estimates for the various physical time scales in the system. Montroll and Shuler[19] were probably the first to obtain explicit results for a special unit-step Markov process modeling low-damped activated escape. It is worth pointing out that for a one-dimensional unit-step (birth and death) Markov process, the mean first-passage time can be obtained in closed form.[17] It solely can be expressed in terms of the stationary probability and a jump rate,[2,20] very much like in the case of one-dimensional Fokker-Planck processes.[16–18] Moreover, the concept of the mean first-passage time can be formulated for general non-Markov processes and exact closed-form expressions can be derived for processes with unit-step and two-step transitions.[21]

Compared to the vast number of papers published on processes driven by a Fokker-Planck dynamics, results for nonlinear systems driven by white shot noise,[22–24] or more generally, by nonwhite and non-Gaussian noise,[24] are scarce. Results on the stationary state have been published in the context of phase diagrams of noise-induced transitions.[25–27]

The purpose of this paper is precisely to discuss dynam-

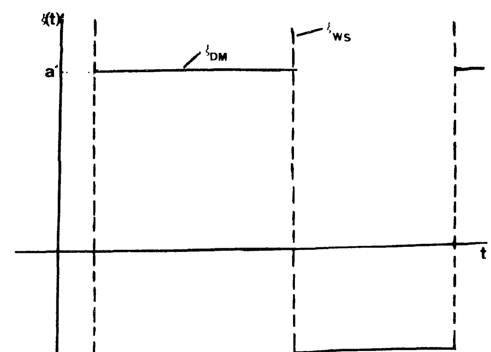


FIG. 1. Sketch of a realization of asymmetric dichotomic noise (solid line). Dashed line denotes the corresponding white-shot-noise realization.

 

ical questions such as escape rates and mean first-passage times for systems driven by an asymmetric two-state Markov process with exponentially distributed waiting times (asymmetric dichotomic noise[25(b),27]). One of the main findings of a recent communication with one of the authors[28] was a characteristic exponential decrease of the activation rate in systems driven by a symmetric dichotomic noise as compared to the corresponding Smoluchowsky rate (limit of white Gaussian noise). Because asymmetric dichotomic noise contains as limits both white shot noise and white Gaussian noise,[27] the interesting comparison between the Smoluchowsky rate and the rate obtained with white shot noise of equal strength becomes possible. Moreover, for white shot noise, we obtain the exact result for the mean first-passage time and show the equivalence with the result obtained from the rate approach in the limit of weak noise.

The paper is organized as follows. In Sec. II stochastic flows driven by the asymmetric dichotomic Markov process are introduced. In this context, we review the results of Ref. 27 because those will be most relevant for the calculations in Secs. III and IV. In Sec. III we derive the exact expression for the activation rate of a general nonlinear bistable flow driven by asymmetric dichotomic noise. The white-shot-noise limit and the white-Gaussian-noise limit are discussed. Section IV contains the exact result for the mean first-passage time of a system driven by white shot noise and the conclusions are given in Sec. V.

## II. STOCHASTIC FLOWS WITH DICHOTOMIC MARKOV NOISE

We consider a (one-dimensional) stochastic flow $\dot{x}$ defined by a stochastic differential equation characterized by the macroscopic flow $f_\alpha(x)$ being driven by generally multiplicative [coupling function $g(x)$] stationary dichotomic noise, i.e.,

$$\dot{x}=f_\alpha(x)+g(x)\xi_{\rm DM}(t)\ . \tag{2.1}$$

Here $f_\alpha$ and $g$ are nonlinear functions of $x$, and $\xi_{\rm DM}$ is the stationary (asymmetric) dichotomic Markov noise. The latter process is a discrete two-state Markov process taking the values $\xi_{\rm DM}=a$ and $\xi_{\rm DM}=a'$ (see Fig. 1). Stochastic flows of the type in (2.1) occur in driven nonequilibrium systems if the control parameter $\alpha$ is subjected to external noise $\alpha\to\alpha+\xi_{\rm DM}(t)$.[28] Examples with $f_\alpha$ being bistable are an Esaki diode with a fluctuating supply current, an optically bistable system with externally injected slightly incoherent light, a Josephson-junction circuit under a fluctuating current bias or a periodically driven Van der Pol oscillator with a fluctuating driving force.[28] Because the system is not in equilibrium and the nonthermal external noise $\xi_{\rm DM}$ can be structured and controlled by the experimenter, the fluctuation-dissipation relation between the noise correlation and the dissipative deterministic flow does not hold. Our situation is thus drastically different from an equilibrium situation with thermal shot noise[11] or thermal, generally non-white, Gaussian noise.[1,12–15]

The transition rates of $\xi_{\rm DM}(t)$ from $a$ to $a'$ and vice versa are denoted by $\mu$ and $\mu'$, respectively. The master equation for $P_t(\xi_{\rm DM})$ thus reads

$$\begin{aligned}\dot{P}_t(a)&=-\mu P_t(a)+\mu' P_t(a')\ ,\\ \dot{P}_t(a')&=\mu P_t(a)-\mu' P_t(a')\ .\end{aligned} \tag{2.2}$$

The steady-state probabilities are

$$\bar{P}(a)=\frac{\mu'}{\mu+\mu'},\quad \bar{P}(a')=\frac{\mu}{\mu+\mu'}\ . \tag{2.3}$$

Without loss of generality, we will assume that the steady-state average value of $\xi_{\rm DM}$ is zero, i.e.,

$$\frac{a}{\mu}+\frac{a'}{\mu'}=0\ , \tag{2.4}$$

and for the following we take $a'>0$ (hence $a<0$). We also define the correlation time $\tau_c$;

$$\tau_c=\frac{1}{\mu+\mu'},\quad \langle \xi_{\rm DM}(t)\xi_{\rm DM}(s)\rangle=\frac{D}{\tau_c}e^{-|t-s|/\tau_c}\ , \tag{2.5}$$

and the intensity (zero-frequency spectral density) of the noise,

$$D=\tfrac{1}{2}\int_{-\infty}^{\infty}\langle \xi_{\rm DM}(\tau)\xi_{\rm DM}(0)\rangle d\tau=a'\,|a|\,\tau_c\ , \tag{2.6}$$

*which in the sequel is always held a constant.*

Since the joint process $(x,\xi_{\rm DM})$ constitutes a Markov process, the master-equation equivalent to (2.1) reads

$$\dot{P}_t(x,a)=-\frac{\partial}{\partial x}\{[f(x)+g(x)a]P_t(x,a)\}+\mu' P_t(x,a')-\mu P_t(x,a)\ , \tag{2.7a}$$

$$\dot{P}_t(x,a')=-\frac{\partial}{\partial x}\{[f(x)+g(x)a']P_t(x,a')\}+\mu P_t(x,a)-\mu' P_t(x,a')\ . \tag{2.7b}$$

For the reduced probability density $p_t(x)$,

$$p_t(x)=P_t(x,a)+P_t(x,a')\ , \tag{2.8}$$

one obtains from (2.7a) and (2.7b) the following closed equation [taking as initial preparation at time $t_0=0$, $P_{t_0}(x,a')=0$]:[29]

$$\dot{p}_t(x)=-\frac{\partial}{\partial x}\{[f(x)+g(x)a]p_t(x)\}+\mu(a-a')\frac{\partial}{\partial x}g(x)\int_0^t \exp\left[-\left[\frac{\partial}{\partial x}[f(x)+g(x)a']+\mu+\mu'\right](t-\tau)\right]p_\tau(x)d\tau\ . \tag{2.9}$$

The steady-state solution $\bar{p}(x)$ of (2.9) is readily obtained as

$$\bar{p}(x)=Z^{-1}\frac{|g(x)|}{D^{\text{eff}}(x)}\exp\left[\int^x \frac{f(y)}{D^{\text{eff}}(y)}dy\right]\Theta(D^{\text{eff}}(x)) \ . \tag{2.10}$$

$Z$ is a normalization constant and $D^{\text{eff}}$ is an "effective diffusion coefficient" given by

$$D^{\text{eff}}(x)=D\left[g(x)-\frac{f(x)}{|a|}\right]\left[g(x)+\frac{f(x)|a|}{D}\tau_c\right] . \tag{2.11}$$

$\Theta$ is the Heaviside function, expressing that the probability is zero in the "unstable" region of negative $D^{\text{eff}}$. The extrema $\{\bar{x}_e\}$ of $\bar{p}(x)$ are the solutions of the following equation:

$$f-Dgg'+\left[\frac{D}{|a|\tau_c}-|a|\right]\tau_c f'g + \tau_c\left[2ff'-\frac{f^2g'}{g}\right]\Bigg]\Bigg|_{x=\bar{x}_e}=0 \ . \tag{2.12}$$

Next we consider the following two limits[27] (to be understood as convergence in probability to the underlying limiting characteristic functionals).

(1) *White-shot-noise limit:*

$$\tau_c \to 0, \quad a, D=a^2/\mu \text{ const.} \tag{2.13}$$

This is equivalent to the limit $a' \to +\infty$ and $\mu' \to +\infty$ with fixed ratio $a'/\mu'=D/|a|$. In this limit $\xi_{\text{DM}}$ reduces to a white non-Gaussian shot noise $\xi_{\text{WS}}$. The realizations of the latter process (see Fig. 1) consist of Dirac $\delta$ peaks at random time points. The weights $w$ of the peaks have an exponential distribution $\phi(w)$,[27]

$$\phi(w)=\frac{|a|}{D}\exp\left[-\frac{w|a|}{D}\right]\Theta(w) \ . \tag{2.14}$$

In between the Dirac $\delta$ peaks, $\xi_{\text{WS}}$ assumes the constant value $a<0$, and the average waiting time between two subsequent $\delta$ peaks is $1/\mu=D/|a|^2$ [cf. (2.4)]. In this limit, the stationary probability density has the form (2.10), but with the effective diffusion coefficient given by

$$D_{\text{WS}}^{\text{eff}}(x)=D\left[g(x)-\frac{f(x)}{|a|}\right]g(x) \ , \tag{2.15}$$

and the extrema $\{\bar{x}_e\}$ obey the equation [cf. (2.12)]

$$f-Dgg'+\frac{D}{|a|}f'g\,|_{x=\bar{x}_e}=0 \ . \tag{2.16}$$

(2) *White-Gaussian-noise limit:*[25,27]

$$\mu,\mu' \to \infty$$

i.e.,

$$\tau_c \to 0, \quad a', |a| \to +\infty \ , \quad D \text{ const.} \tag{2.17}$$

The stationary probability density is given by (2.10) with

$$D_{\text{WG}}^{\text{eff}}(x)=Dg^2(x) \tag{2.18}$$

and the extrema of $\bar{p}$ are solutions of the following equation:[25,27]

$$f-Dgg'\,|_{x=\bar{x}_e}=0 \ . \tag{2.19}$$

## III. RATES IN BISTABLE SYSTEMS

We consider a deterministic macroscopic flow

$$\dot{x}=f(x) \tag{3.1}$$

with (locally) stable steady-state solutions $x_1$ and $x_2$ and an unstable state at $x_u$ (see Fig. 2). Starting from (2.1), we now evaluate the activation rates. Moreover, we assume in the following that $D^{\text{eff}}$ is positive in the interval $[x_1,x_2]$, thereby guaranteeing a nonzero support of $\bar{p}(x)$ over the bistable region. In order to calculate the forward rate $r$ from $x_1$ to $x_2$, we inject particles at $x=\bar{x}$ [$\bar{p}(\bar{x})$ still positive; see (2.10) and (2.11)] left of the stable state $x_1$ at a rate $j_0$ and remove them the moment they reach a state around the stable state $x_2$.[28] The resulting particle density $\bar{n}(x)$ in the interval $[\bar{x},x_2]$ is a solution of (2.9) subject to the condition

$$\bar{n}(x=x_2)=0 \ . \tag{3.2}$$

At the steady state, one obtains for the particle density $\bar{n}(x)$

$$\bar{n}(x)=\beta(x)\bar{p}(x) \ , \tag{3.3}$$

where $\bar{p}(x)$ is given by (2.10) and $\beta(x)$ reads

$$\beta(x)=-j_0\int_{x_2}^{x}\frac{1+\tau_c g(y)\left[\frac{f(y)}{g(y)}\right]'}{\bar{p}(y)D^{\text{eff}}(y)}dy \ . \tag{3.4}$$

Dividing the constant flux $j_0$ by the number of particles, $\int_{\bar{x}}^{x_2}\bar{n}(x)dx$, $\bar{x}<x_1$, one obtains the rate $r$:

$$r=\left[\int_{\bar{x}}^{x_2}\bar{p}(x)dx\int_{x}^{x_2}\frac{1+\tau_c g(y)\left[\frac{f(y)}{g(y)}\right]'}{\bar{p}(y)D^{\text{eff}}(y)}dy\right]^{-1} \ . \tag{3.5}$$

In the limit of weak noise $D$ (i.e., $\ln[\bar{p}(x_1)/\bar{p}(x_u)]\gtrsim 5$)

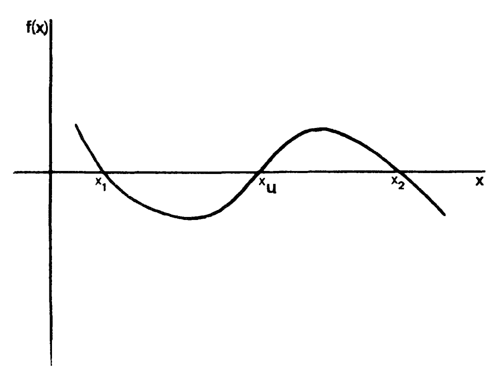


FIG. 2. Deterministic bistable flow with two locally stable states $x_1$, $x_2$ and an unstable state $x_u$.

the probability $\bar{p}(x)$ has sharp extrema. Then, the rate of escape $r$ will be of the order of inverse of the mean first-passage time from a point $x < x_u$ to $x_2$.

In the weak noise limit, we can evaluate the rate by use of the method of steepest descent, yielding for the forward rate $r$

$$r = \frac{(\lambda_1 | \lambda_u |)^{1/2}}{2\pi(1 + | \lambda_u | \tau_c)} \exp\left[-\frac{\Delta\phi}{D}\right] , \tag{3.6}$$

where $\lambda_1$ and $\lambda_u$ are the negative slope of $f$ at $x_1$ and $x_u$,

$$\lambda_1 = -f'(x_1) > 0, \quad \lambda_u = -f'(x_u) < 0 , \tag{3.7}$$

and $\exp(-\Delta\phi/D)$ is the Arrhenius factor, where

$$\Delta\phi = -\int_{x_1}^{x_u} \frac{f(y)}{\left[g(y) - \frac{f(y)}{|a|}\right]\left[g(y) + \frac{f(y)|a|\tau_c}{D}\right]} dy . \tag{3.8}$$

For the backward rate $\hat{r}$, $x_2 \to x_1$ and Eqs. (3.6)–(3.8) are subjected to the trivial replacement $x_1 \to x_2$. In the case of symmetric dichotomic noise,

$$|a| = a', \quad \mu = \mu', \quad D = a^2\tau_c , \tag{3.9}$$

and one recovers the results of Ref. 28. In this case, it is found that the rates increase exponentially with decreasing correlation time $\tau_c$, being maximal in the limit of white Gaussian noise.[28] This conclusion generally no longer holds for asymmetric dichotomic Markov noise. Therefore, an increase of the correlation time $\tau_c$ does not in general imply lower rates or longer mean first-passage times.

Let us next consider the white-shot-noise limit (2.13). One readily finds in the weak noise limit from (3.6) with $\tau_c = 0$

$$r^{\rm WS} = \frac{1}{2\pi}(\lambda_1 | \lambda_u |)^{1/2} \exp\left[-\frac{\Delta\phi^{\rm WS}}{D}\right] \tag{3.10}$$

with

$$\Delta\phi^{\rm WS} = -\int_{x_1}^{x_u} \frac{f(y)}{g(y)\left[g(y) - \frac{f(y)}{|a|}\right]} dy . \tag{3.11}$$

For white Gaussian noise (2.17), one recovers the well-known Smoluchowsky rate:[1–5]

$$r^{\rm WG} = \frac{1}{2\pi}(\lambda_1 | \lambda_u |)^{1/2} \exp\left[-\frac{\Delta\phi^{\rm WG}}{D}\right] \tag{3.12}$$

with

$$\Delta\phi^{\rm WG} = -\int_{x_1}^{x_u} \frac{f(y)}{g^2(y)} dy . \tag{3.13}$$

For white shot noise and white Gaussian noise, the pre-exponential factor remains the same. Since, with $g(x) > 0$ and with $f(x) < 0$ in $(x_1, x_u)$ (see Fig. 2),

$$\Delta\phi^{\rm WS} < \Delta\phi^{\rm WG} , \tag{3.14}$$

the forward rate $r$, (3.10), is with positive $g(x)$ exponentially larger for white shot noise $\xi_{\rm WS}(t)$, defined by (2.4) and (2.13), as compared with white Gaussian noise of equal strength $D$. This is a consequence of the asymmetry of the considered white shot noise $\xi_{\rm WS}(t)$, $\langle \xi_{\rm WS}(t) \rangle = 0$, whose Dirac $\delta$ peaks (see Fig. 1) act with $g(x) > 0$ as positive, destabilizing Dirac $\delta$-force peaks of infinite strength, thereby *shortening* the escape time (i.e., the rate is increasing) as compared to the case driven by white Gaussian noise for which the Dirac $\delta$ peaks are distributed symmetrically [see (2.13); $|a|, a' \to \infty$; $\mu, \mu' \to \infty$]. Indeed, it is easily verified that an opposite result holds for white shot noise of vanishing average with Dirac $\delta$ peaks pointing towards negative values, i.e., $g(x) < 0$. For the backward rate $\hat{r}$, $x_2 \to x_1$, where positive Dirac $\delta$ peaks have a stabilizing effect, just the opposite results are found.

As an example we mention here the case of overdamped Brownian motion in a bistable potential field $V(x) = -\frac{1}{2}cx^2 + \frac{1}{4}dx^4$, $c > 0$, $d > 0$, driven by white shot noise $\xi_{\rm WS}(t)$. The nonequilibrium bistable stochastic flow then reads

$$\dot{x} = cx - dx^3 + \xi_{\rm WS}(t), \quad c > 0, \; d > 0 . \tag{3.15}$$

In virtue of (3.11) and (3.14), this bistable flow yields with $x_{1/2} = \mp (c/d)^{1/2}$, $x_u = 0$, an exponentially enhanced forward rate as compared to the standard Smoluchowsky rate (3.12) (white Gaussian noise of equal strength), i.e., from (3.11), (3.14), and $g = 1$, we have $\Delta\phi^{\rm WS} < \Delta\phi^{\rm WG} = c^2/4d$. An application to a model of genetic selection in population dynamics, originally introduced in Ref. 25 (genetic model), will be presented elsewhere.

## IV. MEAN FIRST-PASSAGE TIME FOR BISTABLE FLOWS DRIVEN BY MULTIPLICATIVE WHITE SHOT NOISE

In contrast to the case of white Gaussian noise,[1–5] the problem of calculating the mean first-passage time of one-dimensional flows of the type in (2.1) is not straightforward. Basically this is due to the fact that with a non-Gaussian white noise the master operator becomes an integral operator or equivalently an infinite-order differential operator. In the following, we will derive the master-equation dynamics for the flow in (2.1) driven by white shot noise, (2.13)–(2.15), and derive from it an exact equation obeyed by the mean first-passage time [see (4.3) and (4.5) below]. The resulting equation is of the same structure as the mean-first-passage-time equation of a Fokker-Planck dynamics. The boundary conditions, however, differ from those for simple diffusion processes.[30] Because our prime interest is only in the (exponential) Arrhenius factor of the rate and in the leading term of the prefactor of the rate expression, an elegant alternative method, (4.14)–(4.18), is developed which bypasses the difficult problem of deriving the absorbing boundary condition for the mean first-passage time of a master-equation dynamics. This method will be based on a Fokker-Planck modeling of the long-time dynamics of the underlying master-equation dynamics.[31,32]

Performing the white-shot-noise limit (2.13), we obtain from (2.9) a Markovian master-equation dynamics, which is given explicitly by

$$\dot{p}_t(x) = -\frac{\partial}{\partial x}\{[f(x) - |a|g(x)]p_t(x)\} - |a|\frac{\partial}{\partial x}\left[g(x)\frac{1}{1+\frac{|a|}{\mu}\frac{\partial}{\partial x}g(x)}p_t(x)\right]$$

$$\equiv \Gamma_x p_t(x) \; . \tag{4.1}$$

Note that the deterministic limit (3.1) follows from (4.1) in the limit $\mu \to \infty$, $D = a^2/\mu \to 0$. The master equation (4.1) is within the Stratonovich interpretation[24,27] equivalent to a stochastic differential equation driven by white shot noise with exponentially distributed weights,

$$\dot{x} = f(x) + g(x)\xi_{\rm WS}(t) \; , \tag{4.2a}$$

where with $w_0 = D/|a|$, $|a| = \mu w_0$,

$$\xi_{\rm WS}(t) = \sum_i w_i \delta(t - t_i) - \mu w_0 \tag{4.2b}$$

and $\phi(w)$ is the distribution of weights $\{w\}$ in (2.14), i.e.,

$$\phi(w) = \frac{1}{w_0}\exp\left[-\frac{w}{w_0}\right]\Theta(w) \; . \tag{4.2c}$$

Because the dynamics (4.1) is Markovian, the mean of the first-passage time $T(x)$ of a random walker which started out at $x(0) = x$ and is moving towards an exit point $x_f = x_2$ obeys[17,18]

$$\Gamma_x^+ T(x) = -1 \; , \tag{4.3}$$

where $\Gamma_x^+$ is the adjoint master operator

$$\Gamma_x^+ = [f(x) - \mu w_0 g(x)]\frac{\partial}{\partial x} + g(x)\frac{\partial}{\partial x}\frac{\mu w_0}{1 - w_0 g(x)\frac{\partial}{\partial x}} \tag{4.4a}$$

$$= [f(x) - \mu w_0 g(x)]\frac{\partial}{\partial x} + \frac{\mu w_0}{1 - w_0 g(x)\frac{\partial}{\partial x}}g(x)\frac{\partial}{\partial x} \; . \tag{4.4b}$$

Multiplying both sides of (4.3) from the left by the operator $[1 - w_0 g(x)(\partial/\partial x)]$ one finds (prime denotes differentiation after $x$)

$$\{w_0 g(x)[\mu w_0 g(x) - f(x)]\}\frac{\partial^2}{\partial x^2}T + [f(x) - w_0 g(x) f'(x) + \mu w_0^2 g(x) g'(x)]\frac{\partial}{\partial x}T = -1 \; . \tag{4.5}$$

This equation is of first order in $\partial T/\partial x$ and thus can easily be integrated. In order to solve (4.5), two boundary conditions must be supplied which must be consistent with the Markovian dynamics (4.1).[21] For a master-equation dynamics governed by discrete finite step-size transitions, there is usually no problem in determining the corresponding boundary conditions[21] (reflecting or absorbing) for $T(x)$. Because

$$\int^t \xi_{\rm WS}(\tau)d\tau = \sum_i w_i \Theta(t - t_0) - \mu w_0 t \tag{4.6}$$

we have a continuous spectrum of jump widths given by (4.2c) and hence the determination of the corresponding boundary conditions becomes nontrivial. Considering the interval $(\bar{x}, x_2)$, $\bar{x} \ll x_u$, for the random walker, we observe that $T(x)$, $(x \to \bar{x})$, approaches asymptotically a constant value. Therefore, we can use the natural boundary condition

$$\left.\frac{dT(x)}{dx}\right|_{x\to\bar{x}} = 0 \; . \tag{4.7}$$

The solution for $T(x)$ still contains one more undetermined constant which is fixed by the value of $T(x)$ at the exit point $x = x_2$. In principle, $T(x_2)$ must be determined from the full master-equation dynamics and the physical restrictions on the transition probabilities.[21] Generally, the values $T(x_2)$ and $T'(x_2)$ are not independent of each other.[30] Clearly, different boundary conditions for $T(x)$ at $x = x_2$ will give different results for the prefactor of a rate $r = 1/T(x)$. In the following we use without further justification the simple boundary condition

$$T(x = x_2) = 0 \; . \tag{4.8}$$

In order to write down the solution $T(x)$ of (4.5), subject to the boundary conditions (4.7) and (4.8), it is convenient to introduce the following quantity $\psi(x)$:

$$\psi(x) = \frac{\exp\left[\int^x dy \frac{f(y) + \mu w_0^2 g(y) g'(y) - w_0 g(y) f'(y)}{D^{\rm eff}(y)}\right]}{D^{\rm eff}(x)} \; , \tag{4.9}$$

where $D^{\rm eff}$ is given by the expression (2.15). $\psi(x)$ has the meaning of the stationary probability of a substitutive Fokker-Planck process with diffusion $D^{\rm eff}$ [first term in the left-hand side (lhs) in (4.5)] and drift given by the second term in the lhs in (4.5). It is related to the true stationary probability $\bar{p}(x)$, (2.10), of the master equation (4.1) as follows:

$$\psi(x) = \bar{p}(x)\exp[-\phi_1(x)] \tag{4.10}$$

with

$$\phi_1(x) = \ln|g(x)| - \int^x \frac{|a|g'(y) - f'(y)}{|a|g(y) - f(y)}dy \; . \tag{4.11}$$

In terms of this substitutive stationary probability, the mean first-passage time takes the following familiar form:

$$T(x) = \int_x^{x_2} \frac{dy}{\psi(y) D^{\rm eff}(y)} \int_{\bar{x}}^y \psi(z)dz \tag{4.12}$$

which within a steepest descent approximation reduces to

$$\frac{1}{T(x)} \simeq \frac{1}{T} = \frac{1}{2\pi}(\lambda_1 |\lambda_u|)^{1/2} \times \exp\left[\phi_1(x_1) - \phi_1(x_u) - \frac{\Delta\phi^{WS}}{D}\right] = \exp[\phi_1(x_1) - \phi_1(x_u)] r^{WS} . \tag{4.13}$$

Most importantly, we note that the Arrhenius factor in (4.13) coincides with the Arrhenius factor in the escape rate (3.10). The prefactor differs by a term of order 1 which in the white Gaussian limit, i.e., $|a| \to +\infty$, equals 1. Clearly in this limit, (4.13) coincides precisely with (3.12) with $\Delta\phi = \Delta\phi^{WG}$.

The problem with the prefactor between (3.10) and (4.13) can be resolved as follows. First, note that for a master-equation dynamics the boundary condition (4.8) is not equivalent with an absorbing boundary condition.[30] In the white-Gaussian-noise limit the boundary condition in (4.8) becomes an *exact absorbing* boundary condition of the resulting Fokker-Planck dynamics and the corresponding prefactor difference between (3.12) and (4.13) vanishes. This is just the situation for which within the weak noise limit of a Fokker-Planck dynamics the equivalence between the rate obtained from transport theory and the inverse mean first-passage time with $x = x_2$ absorbing has been shown.[3–5,10] Thus, in order to correctly compare the prefactors one should look for a Fokker-Planck approximation to the long-time dynamics of (4.1). Such an equivalent Fokker-Planck modeling of the master-equation long-time dynamics has been put forward recently in Refs. 31 and 32. Then, the boundary conditions for the mean first-passage time of the resulting Fokker-Planck dynamics are well known.[2–4,16–18] Moreover, the mean first-passage time can be evaluated in this case by use of recently developed techniques.[2–10] If applied to the master equation in (4.1), we first note that the stationary probability $\bar{p}(x)$ of (4.1) can be recast into WKB form

$$\bar{p}(x) = Z^{-1} \exp\{-[\varphi_0(x) + D\varphi_1(x)]/D\} , \tag{4.14}$$

with $\varphi_0(x)$, $\varphi_1(x)$ determined via (2.10) and (2.15). The deterministic flow (3.1) can then be recast as a transport law[31,32]

$$\dot{x} = -L(x)\chi_0(x) = f(x) , \tag{4.15}$$

where $\chi_0(x) = \partial\varphi_0/\partial x$ is a generalized thermodynamic force and $L(x)$,

$$L(x) = D^{\text{eff}}(x)/D = g(x)\left[g(x) - \frac{f(x)}{|a|}\right] , \tag{4.16}$$

is the corresponding "Onsager coefficient." Following Ref. 32, the bistable master-equation long-time dynamics in (4.1) can be modeled by the Fokker-Planck dynamics $(\chi_1(x) = [\partial\varphi_1(x)/\partial x])$

$$\dot{p}_t(x) = \frac{\partial}{\partial x} L(x)\left[\chi_0(x) + D\chi_1(x) + D\frac{\partial}{\partial x}\right] p_t(x) \tag{4.17}$$

which has (4.14) as the unique stationary probability. The mean first-passage time is now readily evaluated.[2–4] Using the well-known fact that the *absorbing* boundary condition at $x = x_2$ for $T(x)$ for a *Fokker-Planck dynamics is given by (4.8),* we readily obtain from (4.17), observing also (4.7),

$$T(x) = \frac{1}{D}\int_x^{x_2} \frac{dy}{\bar{p}(y)L(y)} \int_{\bar{x}}^{y} \bar{p}(z)dz . \tag{4.18}$$

Within the transport-theory approach [(3.2)–(3.5)], one models via (3.2) an absorbing boundary condition which for the long-time Fokker-Planck dynamics (4.17) of (4.1) is given by (4.8), yielding (4.18). From (4.18), one now recovers for the activation rate $r = 1/T(x)$ within the steepest descent approximation exactly the result in (3.10). Moreover, note that the two Fokker-Planck structures in (4.5) and (4.17), which serve different purposes, have identical diffusion coefficients; the noise-induced drift terms of order $D$, however, are not identical.

## V. CONCLUSIONS

As mentioned earlier, the evaluation of activation rates and mean first-passage times is of considerable interest for a large variety of physical, chemical, and biological applications. In this paper we have obtained activation rates for bistable flows driven by non-Gaussian and generally nonwhite (colored) noise, Eqs. (3.5), (3.6), and (3.10), without having to refer to the concept of a mean first-passage time. In general, the evaluation of the mean first-passage time in flows not driven by Gaussian white noise is nontrivial. Only for the exceptional case of white shot noise with exponentially distributed weights have we been able to derive a finite-order differential equation satisfied by the mean first-passage time. We have not succeeded in deriving a similar type of equation for the case of a nonlinear non-Markovian flow driven by multiplicative dichotomic noise (2.1). This clearly demonstrates the advantage of using a transport theory approach [(3.2)–(3.5)] in cases with non-Gaussian and generally colored noise sources.

Keeping the noise strength $D$ constant (implying identical free self-diffusion coefficients $D$), we have investigated the influence of a finite correlation time of the noise and the character of the noise statistics on the activation rate. No general conclusions could be drawn in the case of asymmetric dichotomic noise. In particular, an increase in correlation time does not generally imply a decrease of the escape rate. In contrast, for symmetric dichotomic noise the forward and backward rates are enhanced exponentially with decreasing correlation time, independent of the multiplicative coupling $g(x)$ and specific form of deterministic bistable flow.[28] Interestingly, the activation rates also depend generally on the noise statistics: White shot noise and white Gaussian noise of equal strength yield exponentially different rates depending on the sign of the shot noise impulses.

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*Permanent address: Departement Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussel, Belgium.

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