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Manifestations of Bose and Fermi statistics on the quantum distribution function for systems of spin-0 and spin-$\frac{1}{2}$ particles

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We consider the manifestations of Bose and Fermi statistics on the original quantum distribution function. Initially, we consider some general symmetry properties of the density matrix. Next, we convert these results into distribution-function language for the case of a system of spin-0 particles. Finally, we consider a system of spin-$\frac{1}{2}$ particles, for which we treat the combined effects of both spin and statistics.

I. INTRODUCTION

Our purpose here is to consider some manifestations of Bose and Fermi statistics on the original quantum distribution function.[1] In our earlier work we also used this distribution function to calculate quantum corrections to the various properties of a Boltzmann gas. The first consideration to the effects of Bose or Fermi statistics was given by Uhlenbeck and Gropper,[2] who calculated the equation of state of both a Bose and a Fermi nonideal gas. Explicit spin effects were ignored in these considerations, as they were also in the later work of Kirkwood.[3]

After a lapse of nearly 20 years, the subject was considered anew by Green,[4] who wrote a relation between the density matrix determined on the basis of classical statistics and the corresponding density matrices for particles satisfying Bose and Fermi statistics. This relation made use of symmetrization operators, which in turn were expressed as the product of a number of cyclic operators. However, we wish to establish more explicit expressions than are given by Green's equation.

In their consideration of the quantum theory of transport in gases, Ross and Kirkwood[5] wrote down the symmetrized pair distribution function in terms of singlet distribution functions by making use of symmetry operators. The latter were discussed in detail in Appendix B of their paper and they concluded that the symmetry operation on the original quantum distribution function is represented by an integral operator. A different approach to the subject was introduced by Schram and Nijboer,[6] who introduced the symmetry requirements by means of a restricted summation of states in Hilbert space. Finally, these authors were able to express the symmetrized distribution functions in terms of an integral involving the original unsymmetrized distribution function, permutation operators, and various δ functions. They then went on to express the partition function as a summation involving permutation operators—but did not obtain a very explicit result.

A particularly important contribution to the subject was the article of Stratonovich.[7] He recorded the distribution function for particles of arbitrary spin and introduced the concept of second quantization in the phase space by essentially replacing the wave functions appearing in the distribution function by operators. The second-quantization approach has been considered further by other authors and used as a method for the incorporation of Fermi and Bose statistics.[8–10]

In a recent series of papers dealing with a one-component plasma (also called "jellium"—a system of identical particles embedded in a uniform neutralizing background of opposite charge), Jancovici[10] and Alastuey and Jancovici[11,12] calculated exchange quantum corrections in the *near-classical* limit. They found, both for three-dimensional[10,11] and so-called two-dimensional systems,[11] that the exchange free energy is negligibly small—a conclusion not changed by the presence of a strong magnetic field.[12]

In this paper we wish to consider anew the general subject of the manifestations of Bose and Fermi statistics on the quantum distribution function. In Sec. II we consider some general properties of the density matrix describing a mixture of several states. Then, in Sec. III we apply these results to a calculation of the effect of statistics on a system of spin-0 particles. Finally, in Sec. IV we extend the description to a system of spin-$\frac{1}{2}$ particles, for which we treat the combined effects of both spin and statistics. Apart from the obvious potential application of our results to a calculation of the equation of state of Bose and Fermi gases, we also consider that they should prove useful in the analysis of the spin-spin correlation experiments[13] which are of interest in connection with tests of the Bell inequalities.[14]

II. DENSITY MATRIX FOR A MIXTURE OF STATES OF PARTICLES SUBJECT TO BOSE OR FERMI STATISTICS

Consider a system of n identical particles $(1,2,\ldots,i,\ldots,n)$ and let x_i and x_i' denote all the coordinates of the ith particle, including its spin. Our starting point is the density matrix ρ for a system of identical particles. This represents, by definition, a system in which a variety of wave functions ψ_μ is present with probabilities w_μ. We have, therefore, by the definition of ρ,

$$\rho=\sum_\mu w_\mu \psi_\mu^*(x_1,x_2,\ldots,x_i,\ldots)\psi_\mu(x'_1,x'_2,\ldots,x'_i,\ldots)$$
$$\equiv\rho(x'_1,x'_2,\ldots,x'_i,\ldots;x_1,x_2,\ldots,x_i,\ldots)\,. \qquad (1)$$

No matter to which permutation we subject *both* the x' and the x, the value of ρ will remain unchanged. This is evident if the particles are of Bose type—in this case both ψ factors remain unchanged. It is also true if the particles are of Fermi type since in this case either both ψ factors remain unchanged (if the permutation is "even") or both ψ factors are multiplied by -1 (if the permutation is "odd"), so that ρ remains the same in this case also. It will be assumed, therefore, that all ρ, representing only identical particles, are unchanged if both sets of variables—the x' and the x of Eq. (1)—are subject to the same permutations. Naturally, if the ρ refers to a system of several types of particles, this remark is valid only for permutations which interchange only variables referring to identical particles.

If only the so-called "column variables"—those after the semicolon (;) in the argument of ρ—are interchanged, ρ will remain unchanged if the particles are of Bose type or if the permutation is even. In these cases all the ψ_μ of Eq. (1) remain unchanged. If the particles are of Fermi type and if the permutation is odd, the sign of ρ will be changed. The same applies, of course, for the so-called "row variables" of ρ—those before the semicolon (;) in the argument of ρ.

We will demonstrate now that if ρ is symmetric in the sense previously specified, i.e., invariant under the simultaneous and identical interchange of both row and column variables, and if it has the right symmetry property with respect to the interchange of one single pair of two column variables (or any pair of two row variables), it then has the right symmetry property with respect to any interchange of variables.

What will be proved actually is that if ρ is invariant with respect to any simultaneous and identical permutation of both row and column variables (which was demonstrated for Bose- and Fermi-particle density matrices) and if, in addition, it is symmetric or antisymmetric with respect to the interchange of the first two column variables,

$$\rho(x'_1,x'_2,\ldots,x'_i,\ldots,x'_j,\ldots;x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots)$$
$$=\pm\rho(x'_1,x'_2,\ldots,x'_i,\ldots,x'_j,\ldots;x_2,x_1,\ldots,x_i,\ldots,x_j,\ldots)\,, \qquad (2a)$$

then it has the same symmetry or antisymmetry with respect to the interchange of any pair of column variables, in particular the interchange of the ith and jth column variables, so that

$$\rho(x'_1,x'_2,\ldots,x'_i,\ldots,x'_j,\ldots;x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots)$$
$$=\pm\rho(x'_1,x'_2,\ldots,x'_i,\ldots,x'_j,\ldots;x_1,x_2,\ldots,x_j,\ldots,x_i,\ldots)\,. \qquad (2b)$$

It follows then from the possibility of arranging any permutation of the x's by a succession of the interchange of two of them that any even permutation of the column indices will leave the density matrix unchanged and any odd permutation will do so also for the case of Bose particles and change the sign for fermions. It further follows from the self-adjoint nature of ρ, that is, $\rho(x';x)=[\rho(x;x')]^*$, that the same is true for the row variables. Hence the symmetry of ρ with respect to any identical interchange of row and column variables, plus the validity of Eq. (2a), establishes the fact that ρ represents a set of identical bosons or fermions, depending on whether Eq. (2a) is valid with the $+$ or $-$ sign. This theorem facilitates the establishment of Bose or Fermi statistics in density matrices and hence also in distribution functions derived from them.

In order to derive Eq. (2b) from Eq. (2a) we first notice that the interchange of both the x and the x' sets of variables at positions 1 and i and at positions 2 and j leaves ρ unchanged—as a result of the symmetry requirement. Hence

$$\rho(x'_1,x'_2,\ldots,x'_i,\ldots,x'_j,\ldots;x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots)$$
$$=\rho(x'_i,x'_j,\ldots,x'_1,\ldots,x'_2,\ldots;x_i,x_j,\ldots,x_1,\ldots,x_2,\ldots)\,. \qquad (3)$$

We now interchange the first two column variables, which gives, according to the assumptions made,

$$\rho(x'_1,x'_2,\ldots,x'_i,\ldots,x'_j,\ldots;x_1,x_2,\ldots,x_i,\ldots,x_j,\ldots)$$
$$=\pm\rho(x'_i,x'_j,\ldots,x'_1,\ldots,x'_2,\ldots;x_j,x_i,\ldots,x_1,\ldots,x_2,\ldots)\,. \qquad (4)$$

If we now interchange again, in both row and column, the first and ith variables and also the second and jth variables, the desired equation (2b) is obtained. This shows that if Eq. (2a) is valid—or a similar equation for another pair—and if ρ is invariant under the identical permutation of both row and column indices, then the density matrix takes care of the Bose or Fermi requirements of the state represented by it—Bose or Fermi depending on whether Eq. (2a) with a $+$ or $-$ is valid. This will facilitate the derivation of the proper conditions for the distribution functions. Finally, we repeat that if there are several systems of identical particles present, then Eq. (2b) is valid for each set of coordinates referring to the same type of particles.

III. SPIN-0 PARTICLES

It should be observed that in the preceding argument *the x' and the x can stand for all coordinates of a particle, including its spin.* However, in the translation of Eq. (2a) into the language of the distribution function, which follows, we distinguish between position and spin coordinates. The effect of the spin can be treated separately.

We now consider the corresponding distribution functions $P(q_1,p_1,\ldots,q_i,p_i,\ldots,q_n,p_n)$, which are functions of position and momentum coordinates $q_1,\ldots,q_i,\ldots,q_n$ and $p_1,\ldots,p_i,\ldots,p_n$, respectively. In the classical limit, $P(q,p)$ is the phase-space distribution function which gives the probability that the coordinates and momenta have the values q and p. Specifically,[1]

$$\begin{aligned}P(q_1,p_1,\ldots,q_i,p_i,\ldots,q_n,p_n)\\ =(\pi\hbar)^{-3n}\int\cdots\int dy_1\cdots dy_i\cdots dy_n\\ \times\rho(q_1-y_1,\ldots,q_i-y_i,\ldots,q_n-y_n;q_1+y_1,\ldots,q_i+y_i,\ldots,q_n+y_n)\\ \times\exp[2i(p_1y_1+\cdots+p_iy_i+\cdots+p_ny_n)/\hbar]\ .\end{aligned} \tag{5}$$

In Eq. (5), q_i, p_i, and y_i are considered to be three-dimensional vectors, py is the scalar product of p and y, and dy_i means integration over all three vector components. All integrations in this paper are from $-\infty$ to ∞.

From Eq. (5) it follows that

$$\begin{aligned}\int\cdots\int dp_1\cdots dp_i\cdots dp_n\,P(q_1,p_1,\ldots,q_i,p_i,\ldots,q_n,p_n)\exp[-2i(p_1y_1+\cdots+p_iy_i+\cdots+p_ny_n)/\hbar]\\ =\rho(q_1-y_1,\ldots,q_i-y_i,\ldots,q_n-y_n;q_1+y_1,\ldots,q_i+y_i,\ldots,q_n+y_n)\ .\end{aligned} \tag{6}$$

Hence, changing variables to u_i $(=q_i-y_i)$ and v_i $(=q_i+y_i)$ $i=1,2,\ldots,n$, we obtain

$$\begin{aligned}\rho(u_1,\ldots,u_i,\ldots,u_n;v_1,\ldots,v_j,\ldots,v_n)\\ =\int\cdots\int dp_1\cdots dp_i\cdots dp_n\,P(\tfrac{1}{2}(u_1+v_1),p_1,\ldots,\tfrac{1}{2}(u_i+v_i),p_i,\ldots,\tfrac{1}{2}(u_n+v_n),p_n)\\ \times\exp\{+i[p_1(u_1-v_1)+\cdots+p_i(u_i-v_i)+\cdots+p_n(u_n-v_n)]/\hbar\}\ .\end{aligned} \tag{7}$$

At this stage we note that since Eq. (3) refers only to two of the particles, we number these 1 and 2 and, in fact, omit the coordinates of the other particles from the equations in order to make them more simple. We will therefore express the equation [see Eqs. (2a) and (2b); we omit the $\pm$ sign since we are dealing here with spin-0 particles]

$$\rho(q_1',q_2';q_1,q_2)=\rho(q_1',q_2';q_2,q_1)\ , \tag{8}$$

in which the q stands for all three space coordinates, in terms of the corresponding distribution functions. We then have [see Eq. (5)]

$$P(q_1,p_1,q_2,p_2)=(\pi\hbar)^{-6}\int\int\rho(q_1-y_1,q_2-y_2;q_1+y_1,q_2+y_2)\exp[2i(p_1y_1+p_2y_2)/\hbar]dy_1\,dy_2 \tag{9}$$

and also [see Eq. (7)]

$$\rho(x_1',x_2';x_1,x_2)=\int\int dp_1\,dp_2\,P(\tfrac{1}{2}(x_1+x_1'),p_1,\tfrac{1}{2}(x_2+x_2'),p_2)\exp[ip_1(x_1'-x_1)/\hbar+ip_2(x_2'-x_2)/\hbar]\ . \tag{10}$$

The condition (8) can therefore be written by equating the right side of (10) with the same expression in which, however, x_1 and x_2 are interchanged.

In order to simplify the resulting equation, we introduce new variables instead of the x and the p:

$$\begin{aligned}\tfrac{1}{4}(x_1+x_1'+x_2+x_2')=q,\quad \tfrac{1}{4}(x_1-x_1'+x_2-x_2')=Y\ ,\\ \tfrac{1}{4}(x_1+x_1'-x_2-x_2')=q_1,\quad \tfrac{1}{4}(-x_1+x_1'+x_2-x_2')=q_2\end{aligned} \tag{11}$$

and

$$\tfrac{1}{2}(p_1+p_2)=p,\quad \tfrac{1}{2}(p_1-p_2)=p'\ . \tag{12}$$

Since the interchange of x_1 and x_2 is represented by the interchange of q_1 and q_2, this leads to the equation, instead of $(2a)(i=1,\ j=2)$,

$$\int\int dp\,dp'\,P(q+q_1,p+p',q-q_1,p-p')e^{4i(p'q_2-pY)/\hbar}=\int\int dp\,dp'\,P(q+q_2,p+p',q-q_2,p-p')e^{4i(p'q_1-pY)/\hbar}\ . \tag{13}$$

Since Y appears only in the exponent, the integration with respect to p can be eliminated, leading to (after replacing the dummy variable p' by p_1)

$$\int dp_1\, P(q+q_1,p+p_1,q-q_1,p-p_1)e^{4ip_1q_2/\hbar} = \int dp_1\, P(q+q_2,p+p_1,q-q_2,p-p_1)e^{4ip_1q_1/\hbar} \,. \tag{14a}$$

It must be admitted that this equation, for the distribution function, postulating the Bose statistics for a system of spin-0 particles, is much more complicated than the corresponding equation (2a) for the density matrix.

The essential equivalence of the position and momentum variables in this equation can be demonstrated by multiplying it with $\exp[-4i(p'q_1+p''q_2)/\hbar]$ and integrating over q_1 and q_2. The resulting equation is

$$\int dq_1\, P(q+q_1,p+p'',q-q_1,p-p'')e^{-4ip'q_1/\hbar} = \int dq_2\, P(q+q_2,p+p',q-q_2,p-p')e^{-4ip''q_2/\hbar} \,, \tag{14b}$$

in the right-hand side of which, naturally, q_2 can be replaced by q_1. The equation then becomes the analog of Eq. (14a) with the roles of p and q interchanged, but the signs in the exponentials reversed. Equally easy an equation can be obtained in which q and p play essentially the same roles, but the two sides of the equation are quite different.

This is achieved by multiplying Eq. (14a) with $e^{-4ip_2q_1/\hbar}$ and integrating over q_1. On the right side this gives a factor $(\pi\hbar/2)^3\delta(p_1-p_2)$ (we must not forget that the q_1p_1 in the exponent is a three-dimensional scalar product), hence one obtains

$$P(q+q_2,p+p_2,q-q_2,p-p_2) = \left[\frac{2}{\hbar\pi}\right]^3 \int\int P(q+q_1,p+p_1,q-q_1,p-p_1)e^{4i(p_1q_2-q_1p_2)/\hbar}dp_1\,dq_1 \,, \tag{15}$$

which is essentially symmetric in position and momentum coordinates. Naturally, by another Fourier transformation, this can be transformed into the analog of Eq. (14a), with the position- and momentum-coordinate roles interchanged.

As a check, we use Eq. (15) to replace the function P on the right side of Eq. (15), obtaining

$$\begin{aligned} P(q+q_2,p+p_2,q-q_2,p-p_2) &= (2/\pi\hbar)^6 \int\int\int\int P(q+q_2',p+p_2';q-q_2',p-p_2') \\ &\quad \times \exp[4i(q_2p_1-q_1p_2)/\hbar + 4i(q_1p_2'-q_2'p_1)/\hbar]dq_1\,dp_1\,dq_2\,dp_2 \,. \end{aligned} \tag{16}$$

Carrying out the integrations over q_1 and p_1 we obtain factors $(\pi\hbar/2)^3\delta(p_2-p_2')$ and $(\pi\hbar/2)^3\delta(q_2-q_2')$, as a result of which we verify that the right side of Eq. (16) reduces to the left side, proving the self-consistency of Eq. (15).

IV. SYSTEM OF SPIN-$\frac{1}{2}$ PARTICLES

In the case of a pure state $\psi(q)$ we extend our previous definition of the single-particle quantum distribution function as follows:

$$P(q,p,\kappa) = (\pi\hbar)^{-3} \sum_{m,m'=1,-1} \sigma^{\kappa}_{mm'} \int [\psi(q+y,m)]^*\psi(q-y,m')e^{2ipy/\hbar}dy \tag{17a}$$

in the case of a pure state and

$$P(q,p,\kappa) = (\pi\hbar)^{-3} \sum_{m,m'=1,-1} \sigma^{\kappa}_{mm'} \int \rho(q-y,m';q+y,m)e^{2ipy/\hbar}dy \tag{17b}$$

in the case of a mixture of states, where κ takes on the values 0, x, y, and z; the σ^0 is the unit matrix and the others are the Pauli matrices for the spin, the rows and columns being labeled 1 and -1. Specifically,

$$\begin{aligned} \sigma^0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \\ \sigma^x &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned} \tag{18}$$

If Eq. (17a) is integrated over momentum p, one obtains

$$\int P(q,p,0)dp = |\psi(q,1)|^2 + |\psi(q,-1)|^2 = \rho(q,1;q,1) + \rho(q,-1;q,-1) \,, \tag{19a}$$

$$\int P(q,p,x)dp = \{[\psi(q,1)]^*\psi(q,-1) + [\psi(q,-1)]^*\psi(q,1)\} \,, \tag{19b}$$

$$\int P(q,p,y)dp = -i\{[\psi(q,1)^*\psi(q,-1) - [\psi(q,-1)]^*\psi(q,1)\} \,, \tag{19c}$$

$$\int P(q,p,z)dp = |\psi(q,1)|^2 - |\psi(q,-1)|^2 \,. \tag{19d}$$

Thus Eq. (19a) gives the total probability of finding the particle at position q, irrespective of its spin state, whereas Eqs. (19b)—(19d) give the difference in the probabilities of finding the particle with spin up and spin down, referred to the x, y, and z directions, respectively—always at q. Hence, the normalization of P, though it involves integration over q and p, restricts the value of κ, actually to zero.

In the case of a two-particle system, Eq. (17) generalizes to

$$P(q_1,p_1,\kappa_1;q_2,p_2,\kappa_2)=(\pi\hbar)^{-6}\sum_{m_1m_1'}\sum_{m_2m_2'}\sigma^{\kappa_1}_{m_1,m_1'}\sigma^{\kappa_2}_{m_2,m_2'}\int\int[\psi(q_1+y_1,m_1;q_2+y_2,m_2)]^* \times\psi(q_1-y_1,m_1';q_2-y_2,m_2')\exp[2i(p_1y_1+p_2y_2)/\hbar]dy_1\,dy_2 \tag{20}$$

in the case of a pure state and to

$$P(q_1,p_1,\kappa_1;q_2,p_2,\kappa_2) =(\pi\hbar)^{-6}\sum_{m_1,m_1'}\sum_{m_2,m_2'}\sigma^{\kappa_1}_{m_1m_1'}\sigma^{\kappa_2}_{m_2m_2'}\int\int dy_1\,dy_2\exp[2i(p_1y_1+p_2y_2)/\hbar] \times\rho(q_1-y_1,m_1',q_2-y_2,m_2';q_1+y_1,m_1,q_2+y_2,m_2)\ , \tag{21}$$

in the case of a mixture of states. They follow from the fact that the 4×4 matrix $M_{\kappa;mm'}\equiv2^{-1/2}\sigma^{\kappa}_{mm'}$ is unitary. One has, therefore,

$$\sum_{m,m'}\sigma^{\kappa}_{mm'}(\sigma^{\lambda}_{mm'})^*=\sum_{m,m'}\sigma^{\kappa}_{mm'}\sigma^{\lambda}_{m'm}=2\delta_{\kappa\lambda}\ . \tag{22}$$

The second expression is equal to the first because all σ^λ are self-adjoint. Similarly, we have

$$\sum_{\kappa}\sigma^{\kappa}_{mm'}(\sigma^{\kappa}_{nn'})^*=\sum_{\kappa}\sigma^{\kappa}_{mm'}\sigma^{\kappa}_{n'n}=2\delta_{mn}\delta_{m'n'}\ . \tag{23}$$

It then follows from Eq. (17b) that

$$\rho(x',n';x,n)=\tfrac{1}{2}\sum_{\kappa}\sigma^{\kappa}_{n'n}\int dp\,P(\tfrac{1}{2}(x+x'),p,\kappa)e^{ip(x'-x)/\hbar} \tag{24a}$$

or, with Eq. (10), for the two-particle situation,

$$\rho(x_1',m_1',x_2',m_2';x_1,m_1,x_2,m_2)=\tfrac{1}{4}\sum_{\kappa_1,\kappa_2}\sigma^{\kappa}_{m_1'm_1}\sigma^{\kappa}_{m_2'm_2}\int\int dp_1\,dp_2\,P(\tfrac{1}{2}(x_1+x_1'),p_1,\kappa_1;\tfrac{1}{2}(x_2+x_2'),p_2,\kappa_2) \times\exp[ip_1(x_1'-x_1)/\hbar+ip_2(x_2'-x_2)/\hbar]\ , \tag{24b}$$

from whence it follows that the generalization of Eq. (14a), to the case of spin-$\frac{1}{2}$ particles, is

$$\sum_{\kappa_1}\sum_{\kappa_2}\sigma^{\kappa_1}_{m_1'm_1}\sigma^{\kappa_2}_{m_2'm_2}\int P(q+q_1,p+p_1,\kappa_1;q-q_1,p-p_1,\kappa_2)e^{4ip_1q_2/\hbar}dp_1 =-\sum_{\kappa_1}\sum_{\kappa_2}\sigma^{\kappa_1}_{m_1'm_2}\sigma^{\kappa_2}_{m_2'm_1}\int P(q+q_2,p+p_1,\kappa_1;q-q_2,p-p_1,\kappa_2)e^{4ip_1q_1/\hbar}dp_1\ , \tag{25}$$

with the minus sign arising from the fact that we are dealing with fermions. This equation will be simplified by expressing the product of the two σ matrices on the left side,

$$[\sigma(\kappa_1,\kappa_2)]_{m_1'm_2';m_1m_2}=\sigma^{\kappa_1}_{m_1'm_1}\sigma^{\kappa_2}_{m_2'm_2}\ , \tag{26a}$$

by the product of the σ matrices on the right side,

$$[\tau(\kappa_1,\kappa_2)]_{m_1'm_2';m_1m_2}=\sigma^{\kappa_1}_{m_1'm_2}\sigma^{\kappa_2}_{m_2'm_1}\ . \tag{26b}$$

We can then write

$$\tau(\kappa_1,\kappa_2)=\sum_{\lambda_1,\lambda_2}B(\kappa_1\kappa_2;\lambda_1\lambda_2)\sigma(\lambda_1,\lambda_2)\ , \tag{27a}$$

and since the $\sigma(\kappa_1,\kappa_2)$ as functions of the m are linearly independent of each other (in fact orthogonal), we can write, instead of Eq. (25),

$$\int P(q+q_1,p+p_1,\kappa_1;q-q_1,p-p_1,\kappa_2)e^{4ip_1q_2/\hbar}dp_1 =-\sum_{\lambda_1,\lambda_2}B(\lambda_1,\lambda_2;\kappa_1,\kappa_2)\int P(q+q_2,p+p_1,\lambda_1;q-q_2,p-p_1,\lambda_2)e^{4ip_1q_1/\hbar}dp_1\ . \tag{27b}$$

Equation (27b) is the basic result sought, being a consequence of the antisymmetry of the wave function for spin-$\frac{1}{2}$ particles. The matrix B is much simplified by the fact that the rotational transformation properties of the σ and τ matrices are the same—they represent either scalars or vectors or tensors. Hence, for example, the matrix element of B which connects the scalar component of $\sigma(\lambda_1,\lambda_2)$ with the vector components of $\tau(\kappa_1,\kappa_2)$ vanishes. It may be stated additionally, that B is unitary and its square is the unit matrix so that its characteristic values are all 1 or -1. It is not difficult to calculate B—its elements are, of course, the same for the different component parts of the vector (tensor) components of $\sigma(\lambda_1,\lambda_2)$ which are connected to the vector (tensor) components of $\tau(\kappa_1,\kappa_2)$.

As can be easily verified, the explicit form of the relation (27a) is

$$\tau(0,0)=\tfrac{1}{2}\sigma(0,0)+\tfrac{1}{2}[\sigma(x,x)+\sigma(y,y)+\sigma(z,z)] \equiv \tfrac{1}{2}[\sigma(0,0)+\sigma(\vec{r},\vec{r})] ; \quad (28a)$$

$$\tau(\vec{r},\vec{r})\equiv\tau(x,x)+\tau(y,y)+\tau(z,z) =\tfrac{3}{2}\sigma(0,0)-\tfrac{1}{2}[\sigma(x,x)+\sigma(y,y)+\sigma(z,z)] ; \quad (28b)$$

$$\tau(x,y)+\tau(y,x)=\sigma(x,y)+\sigma(y,x);\tau(y,z)+\tau(z,y) =\sigma(y,z)+\sigma(z,y), \ldots ; \quad (28c)$$

$$\tau(0,x)+\tau(x,0)=\sigma(0,x)+\sigma(x,0), \ldots , \tau(0,z)+\tau(z,0) =\sigma(0,z)+\sigma(z,0) ; \quad (28d)$$

$$\tau(0,x)-\tau(x,0)=-i[\sigma(y,z)-\sigma(z,y)], \ldots ; \quad (28e)$$

$$\tau(y,z)-\tau(z,y)=i[\sigma(0,x)-\sigma(x,0)], \ldots ; \quad (28f)$$

$$2\tau(x,x)-\tau(y,y)-\tau(z,z) =2\sigma(x,x)-\sigma(y,y)-\sigma(z,z), \ldots ; \quad (28g)$$

$$\tau(x,x)=\tfrac{1}{2}[\sigma(x,x)-\sigma(y,y)-\sigma(z,z)]+\tfrac{1}{2}\sigma(0,0) \equiv\sigma(x,x)-\tfrac{1}{2}\sigma(\vec{r},\vec{r})+\tfrac{1}{2}\sigma(0,0) . \quad (28h)$$

Clearly, these equations are not all independent of each other—Eqs. (28b) and (28c) follow from Eq. (28h) and show only the effects of the invariances more explicitly.

Evidently, the consequences of the wave functions antisymmetry for spin-$\frac{1}{2}$ particles are much more complicated than for the density matrix even though the preceding equations could be given a more simple form. Nevertheless, it should not be truly difficult to give Eq. (27b)—the basic equation—a more explicit form and also a more simple one by introducing, instead of the indices κ and λ the indices referring to the left sides of the equations (28). Another possibility is to accept the matrix B as a basic quantity. One way this could be done would be to decompose B into three parts: those referring to scalar, vector, and irreducible-tensor expressions in terms of the τ appearing in Eq. (28). The first part of B would be two dimensional, the second part would contain three separate three-dimensional matrices [referring to $\tau(0,x)$, $\tau(x,0)$, $\tau(y,z)-\tau(z,y)$, and the other two similar triplets], and the last one would be a five-dimensional unit matrix referring to the five components of the five-dimensional representation of the rotation group [cf. Eqs. (28c) and (28g)]. But even if this is done, it must be admitted that the equations will be more complicated than the equations postulating Fermi statistics for the density matrix.

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