Noise in strong laser-atom interactions: Phase telegraph noise

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We discuss strong laser-atom interactions that are subjected to jump-type (random telegraph) random-phase noise. Physically, the jumps may arise from laser fluctuations, from collisions of various kinds, or from other external forces. Our discussion is carried out in two stages. First, direct and partially heuristic calculations determine the laser spectrum and also give a third-order differential equation for the average inversion of a two-level atom on resonance. At this stage a number of general features of the interaction are able to be studied easily. The optical analog of motional narrowing, for example, is clearly predicted. Second, we show that the theory of generalized Poisson processes allows laser-atom interactions in the presence of random telegraph noise of all kinds (not only phase noise) to be treated systematically, by means of a master equation first used in the context of quantum optics by Burshtein. We use the Burshtein equation to obtain an exact expression for the two-level atom's steady-state resonance fluorescence spectrum, when the exciting laser exhibits phase telegraph noise. Some comparisons are made with results obtained from other noise models. Detailed treatments of the effects of other than phase noise are given in other papers.

I. INTRODUCTION

Observations of light absorption and scattering by atoms and molecules provide information about radiative dynamical processes as well as about atomic and molecular structure. For example, under ideal conditions intense monochromatic light tuned near a resonance line of an isolated atom can excite fluorescence radiation that reveals nonlinear properties of the light-matter interaction, most notably ac Stark splitting of the fluorescence spectral line.[1]

However, ideal atoms exist and interact with monochromatic light only in theoretical models. Real atoms experience a fluctuating environment of many perturbing interactions, and real lasers can exhibit a variety of fluctuations in phase, frequency, and amplitude.[2] Incorporation of such stochastic phenomena into the relevant atomic Liouville or Schrödinger equation, by way of empirical constants (relaxation times and bandwidths), is a common step[3] toward realistic theoretical modeling. It should, however, be possible to do better than this, even without attempting a fully microscopic treatment. One way of doing better is to incorporate stochastic processes, instead of empirical constants, into the fundamental equations to represent perturbing interactions. There is a long history of such use of stochastic processes in physics.[4]

We have developed a series of models of noisy laser-atom interactions, all based on so-called jump processes, the simplest example of which is the two-state random telegraph. These models allow convenient and very flexible manipulation of interaction parameters, while permitting nonperturbative examination of the (possibly very strongly nonlinear) noisy laser-atom interaction. In this paper we describe the characteristics of such models and use one of them to calculate features of the atomic response. For simplicity in writing, we will speak of the stochastic influences that we intend to consider by the term "laser noise," although they might equally well originate, for example, in collisions of various kinds or from other external sources (density fluctuations, etc.).

Almost all previous analyses[5] of noisy laser-atom interactions have been based on assumptions of Gaussian noise with extremely short coherence times (leading to cumulant approximations). The random telegraph models[6,7] that we will explore are subject to neither of these restrictions. We will also show that almost any kind of telegraph noise, whether associated with phase, frequency, or amplitude fluctuations, leads to equations for average values that have finite algebraic solutions. This is another significant advantage for telegraph-noise models, in the context of laser-atom interactions.

We concentrate our attention in this paper on phase telegraph noise. The technique that we use will be applied subsequently to frequency and amplitude telegraph noise.[8] We note that phase and frequency telegraphs are physically distinct, although obviously closely related. Previous treatments of so-called "phase diffusion" in laser-atom interactions are probably more accurately associated with random frequencies and so do not provide close analogs to the results obtained here. In Sec. II we derive the two-point correlation and spectrum of a laser field with random-phase telegraph noise, and we find the average equation of motion satisfied by the inversion of a two-level atom exposed to such a field of resonance in Sec. III.

Section IV is devoted to a brief review of generalized Poisson processes, of which our random telegraph is a simple example. The Chapman-Kolmogorov-Smoluchowski equation for two-time joint probabilities is used to derive our working master equation. This master equation is solved in Sec. V for the resonance spectrum of the two-level atom discussed in the previous sections. A summary of our results is given in Sec. VI. An Appendix is devoted to a detailed solution of a simple telegraph-noise problem as an example of an application of our master equation.

II. ELEMENTARY CONSIDERATIONS

Because we have in mind strong laser probes of atomic systems, that is, near-resonance excitation, only one transition in the atom will be greatly affected by the laser light, so we adopt the familiar two-level picture for the atom, and we use the rotating wave approximation (RWA).[9] The atomic equations of motion are, in the Heisenberg picture,

$$\dot{\hat{\sigma}}_{12}=-[\tfrac{1}{2}A+i\Delta]\hat{\sigma}_{12}+\tfrac{1}{2}i\Omega(t)(\hat{\sigma}_{22}-\hat{\sigma}_{11}) , \quad (2.1a)$$

$$\dot{\hat{\sigma}}_{21}=-[\tfrac{1}{2}A-i\Delta]\hat{\sigma}_{21}-\tfrac{1}{2}i\Omega^*(t)(\hat{\sigma}_{22}-\hat{\sigma}_{11}) , \quad (2.1b)$$

$$\dot{\hat{\sigma}}_{11}=A\hat{\sigma}_{22}+\tfrac{1}{2}i\Omega(t)\hat{\sigma}_{21}-\tfrac{1}{2}i\Omega^*(t)\hat{\sigma}_{12} , \quad (2.1c)$$

$$\dot{\hat{\sigma}}_{22}=-A\sigma_{22}-\tfrac{1}{2}i\Omega(t)\hat{\sigma}_{21}+\tfrac{1}{2}i\Omega^*(t)\hat{\sigma}_{12} , \quad (2.1d)$$

where $\sigma_{ij}=|i\rangle\langle j|$. We denote laser-atom detuning and the radiative decay rate by Δ and A, as usual. The field is represented classically, with a fixed amplitude and a time-dependent phase: $\vec{E}(t)=\hat{\epsilon}\mathscr{E}_0\exp[-i\omega_L t-i\phi(t)]$. The instantaneous Rabi frequency is defined to be

$$\Omega(t)=(2\vec{d}_{21}\cdot\vec{\epsilon}/h)\mathscr{E}_0\exp[-i\phi(t)] , \quad (2.2)$$

where $\vec{d}_{21}$ is the two-level transition dipole matrix element. It is the phase $\phi(t)$ that we will represent as a random telegraph in this paper.

The fundamental quantity in laser-atom interactions is the dipole autocorrelation. In dimensionless form it is given by

$$C(t,\tau)\equiv\langle\hat{\sigma}_{21}(t+\tau)\hat{\sigma}_{12}(t)\rangle . \quad (2.3)$$

Notice that the equal-time value ($\tau=0$) of this two-point function gives the upper-state population (and thus the inversion) immediately.

In Sec. IV we describe a generalized master equation appropriate to jump processes and apply it to the present problem. However, we can illustrate first the simplicity of random telegraph models by calculating directly the laser's electric field autocorrelation function and spectrum. In order to evaluate

$$\langle\vec{E}^*(t_1)\cdot\vec{E}(t_2)\rangle=|\mathscr{E}_0|^2 e^{i\omega_L(t_1-t_2)}\langle e^{i\phi(t_1)}e^{-i\phi(t_2)}\rangle \quad (2.4)$$

we simply observe that $\phi(t_2)=\phi(t_1)\pm[1-(-1)^n]a$, where a is the amount of the jump assigned to the random-phase telegraph, and where the $\pm$ indicates that the first phase value subsequent to $\phi(t_1)$ depends on whether $\phi(t_1)$ is itself associated with $+a$ or $-a$. The integer n is the number of times the telegraph changes its state between t_1 and t_2. The random telegraph is a Poisson process; the probability that its state changes n times in an interval of length Δt is given by

$$p_n=e^{-\bar{n}}(\bar{n})^n/n! , \quad (2.5)$$

where the mean number $\bar{n}$ is related to Δt through the dwell time T (i.e., the mean time between interruptions) for the telegraph:

$$\bar{n}=\Delta t/T . \quad (2.6)$$

The evaluation of the correlation function is a straightforward sum over n, the number of jumps, from 0 to ∞. We must also average over the two initial possibilities for $\phi(t_1)$. The result is

$$\langle e^{i\phi(t_1)-i\phi(t_2)}\rangle=\tfrac{1}{2}\sum_{\pm}\sum_{n}p_n(e^{\pm i[1-(-1)^n]a})=\cos^2 a+\sin^2 a\, e^{-2|t_1-t_2|/T} . \quad (2.7)$$

It is clear that the laser field is statistically stationary (i.e., it is time translation invariant, depending only on t_1-t_2) and it has a pure coherent part and a noise part, which will contribute a delta function and a broadened Lorentzian, respectively, to its Fourier spectrum [i.e., the absolute square of the Fourier transform of the correlation (2.7)]:

$$\mathscr{S}_{\text{laser}}(\omega)=2\pi|\mathscr{E}_0|^2\left\{\cos^2 a\,\delta(\omega-\omega_L)+\sin^2 a\frac{2/T}{(\omega-\omega_L)^2+(2/T)^2}\right\} . \quad (2.8)$$

Note that if $a=\pi/2$ the coherent part of (2.7) disappears. This is just the case that the phase telegraph reduces to the more familiar zero-mean amplitude telegraph, since in this case a switch of the telegraph's state is a phase change of π, equivalent to an amplitude sign change. It is well known (see S. O. Rice, Ref. 4) that the amplitude telegraph has a pure exponential correlation and Lorentzian spectrum.

III. ELEMENTARY APPROACH TO ATOMIC RESPONSE

As a second example of the application of the two-state phase random telegraph to laser-atom problems, we consider the direct calculation of the expected inversion of a two-level atom described by Eq. (2.1), in the limit of exact resonance. The dipole variables can be eliminated exactly and we find an integro-differential equation for $\hat{w}=\hat{\sigma}_{22}-\hat{\sigma}_{11}$:

$$\dot{\hat{w}}=-A(1+\hat{w})-\int_0^t dt'\exp[-(A/2)(t-t')]\times\text{Re}[\Omega^*(t)\Omega(t')]\hat{w}(t') . \quad (3.1)$$

Here the Rabi frequency carries the phase random telegraph:

$$\Omega(t)=\Omega_0 e^{-i\phi(t)} . \quad (3.2)$$

We assume that the inversion and Rabi frequency decorrelate in this case, since $t \geq t'$ and since the laser field is an external field and independent of the inversion. Under this assumption, to be justified later in the Appendix, the expected value of the inversion (w without circumflex) satisfies the equation

$$\dot{w} = -A(1+w) - \Omega_0^2 \int_0^t dt' \exp[-(A/2)(t-t')](\cos^2 a + e^{-2|t-t'|/T}\sin^2 a)w(t') \;, \tag{3.3}$$

where we have inserted the correlation (2.7) derived above. It is easy to show that (3.3) is equivalent to a third-order differential equation with constant coefficients:

$$\dddot{w} + \left[2A + \frac{2}{T}\right]\ddot{w} + \left[\Omega_0^2 + \tfrac{5}{4}A^2 + 3\frac{A}{T}\right]\dot{w} + \left[\frac{A}{2}\Omega_0^2 + \frac{2}{T}\Omega_T^2\cos^2 a + \frac{A^3}{4} + \frac{A^2}{T}\right]w + \frac{A^3}{4} + \frac{A^2}{T} = 0 \tag{3.4}$$

with initial conditions $w(0)$, $\dot{w}(0) = -A[1+w(0)]$ and $\ddot{w}(0) = A^2 + (A^2 - \Omega_0^2)w(0)$. We are mostly interested here in the effect of the phase jump on the strong-field ($\Omega_0 \gg A$) atomic response. We also take $1/T >> A$. In that case (3.4) reduces to

$$\dddot{w} + \frac{2}{T}\ddot{w} + \Omega_0^2\dot{w} + \frac{2}{T}\Omega_0^2\cos^2 a \, w = 0 \;. \tag{3.5}$$

This equation reduces in turn to a slightly simpler one if a further average over the size of the phase jump a is made. We will discuss elsewhere the relation of this equation to similar third-order equations derived by Burshtein.

Equation (3.5) can be solved exactly, of course, in terms of the roots of a cubic polynomial. It is easier to consider the two limits of very slow and very rapid switching of the phase telegraph. In the first case T is very large and the second and fourth terms in (3.5) can be ignored, leaving only the ordinary Rabi oscillation equation[9] for the inversion. That is, $w \sim \mathrm{Re}\exp(-i\Omega_0 t)\overline{w}$, where $\overline{w}$ is nearly constant. If the ignored terms are then reconsidered in order to determine the slow time dependence of $\overline{w}$, it is easily found that they contribute frictional damping of the Rabi oscillations at a rate proportional to the telegraph switching rate $1/T$. That is, in this limit the oscillation frequency and damping rate are

$$\Omega = \Omega_0 \;, \quad \gamma = \frac{1}{T}\sin^2 a \;. \tag{3.6}$$

On the other hand, if T is very short, then the first and third terms in (3.5) can be ignored. Again, the frequency and damping rate are easily found:

$$\Omega = \Omega_0 \cos a \;, \quad \gamma = \frac{\Omega_0^2 T}{4}\sin^2 a \;. \tag{3.7}$$

Note that now Ω is the average Rabi frequency, taking both telegraph states $\Omega_0 e^{ia}$ and $\Omega_0 e^{-ia}$ into account. Also, damping now occurs at a rate proportional to T, i.e., *inversely* proportional to the switching rate $1/T$.

Both of these conclusions, regarding the oscillation frequency and the damping rate, are indications that when T is quite short compared to a Rabi cycle the response of the atom enters the motional narrowing regime, well known in the theory of magnetic resonance.[10] That is, the interruptions due to the switching of the phase random telegraph become so frequent that the atom can only notice the average value of the external signal (Rabi frequency). We also note that very fast and very slow switching both lead to very weak damping. In both regimes represented by Eqs. (3.6) and (3.7), for $1/T \ll \Omega_0$ and $1/T \gg \Omega_0$, it turns out that $\gamma \ll \Omega_0$. Strong damping, which is not characterized by a single damping rate, occurs *only* for $1/T \sim \Omega_0$.

In Fig. 1 we show an example of phase random telegraph influence on atomic-level populations, taking $a = 0.4\pi$ and three different phase switching rates. The regimes of weak damping ($\Omega_0 T = 10$), strong damping ($\Omega_0 T = 1$), and motional narrowing ($\Omega_0 T = 0.1$) are clearly evident. Of course, the fact that both level populations decay to $\frac{1}{2}$ (i.e., the inversion w decays to zero, not to -1) is an indication that the phase-jump relaxation is purely "transverse," or of the T_2 type, and that the inhomogeneous terms in (3.4) have been dropped. The calcula-

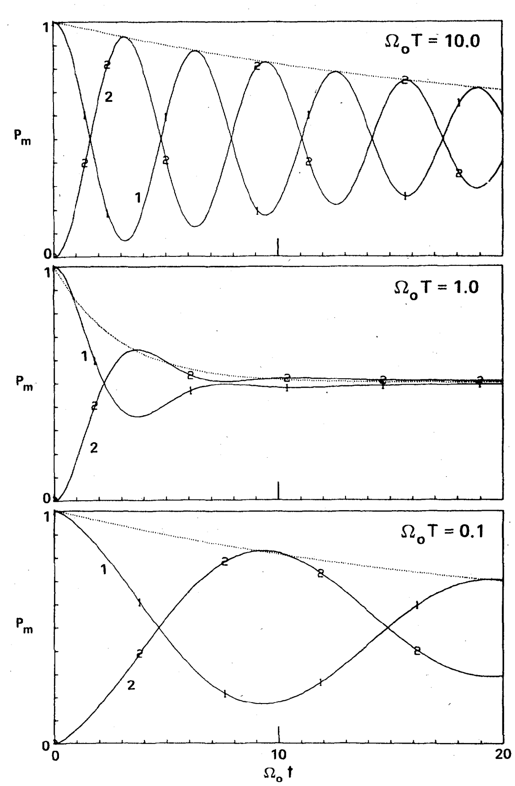

FIG. 1. Populations $P_m = \langle \sigma_{mm} \rangle$ vs time (in units of inverse Rabi frequency Ω_0) for two-state atom resonantly excited by random telegraph phase noise, with phase jumps of $a = 0.4\pi$. Successive frames show different choices for the mean interruption time T.

tions required for the figure were made according to the general methods to be described next, rather than directly from Eq. (3.5).

IV. RANDOM TELEGRAPH JUMP PROCESSES AND MASTER EQUATIONS

Our telegraph model of a random process $x(t)$ [in Secs. II and III, $x(t)=\phi(t)$, laser phase] rests on the following assumptions.

(a) The quantity $x(t)$ remains constant except during infinitesimally brief jumps, when it changes to a new constant value.

(b) The process $x(t)$ is stationary in time and the jumps occur at random.

(c) The jump process is Markovian: the value of $x(t)$ immediately after a jump depends, at most, only upon the value immediately preceding the jump, not upon any prior history.

In short, our telegraph constitutes a stationary Markov chain.

It is well known[11,12] that the dynamics of such Markov chains is completely described by two basic functions $c(\alpha;t)$ and $f(\alpha\,|\,\beta;t)$ governing all possible transitions among the different states α of the telegraph. The function $c(\alpha;t)$ is the frequency that the telegraph in the state α at time t will change its state in the interval $t,t+dt$. The function $f(\alpha\,|\,\beta;t)$ is the conditional probability that in the interval $t,t+dt$ this change takes the telegraph from state β, given that the transition *ends* in the state α. From the definition of the conditional probability $f(\alpha\,|\,\beta;t)$ it is clear that for every fixed state α and time t we have

$$\sum_{\beta} f(\alpha\,|\,\beta;t)=1\ . \tag{4.1}$$

We adopt a common simplification and specialize to the case of so-called (Ref. 12, p. 428) generalized Poisson processes. These are special Markov chains for which all $c(\alpha;t)$ are equal to the same constant value, and the functions $f(\alpha\,|\,\beta;t)$ are time independent

$$c(\alpha;t)=\frac{1}{T}\ , \tag{4.2}$$

$$f(\alpha\,|\,\beta;t)=f(\alpha\,|\,\beta)\ . \tag{4.3}$$

These definitions imply that the number of transitions within a finite interval Δt has a Poisson-like distribution [see (2.5)] with a mean value equal to $\Delta t/T$. The probability of remaining in a given telegraph state decreases with time as $\exp(-|t|/T)$.

It is well known that the joint probability distribution function $p(\alpha,t\,|\,\alpha_0 t_0)$ of a Markov chain satisfies a set of differential (backward or forward) Chapman-Kolmogorov-Smoluchowski equations which take the following form:[13]

$$\frac{\partial}{\partial t}p(\alpha,t\,|\,\alpha_0,t_0) =-\frac{1}{T}p(\alpha,t\,|\,\alpha_0,t_0)+\frac{1}{T}\sum_{\beta} f(\alpha\,|\,\beta)p(\beta,t\,|\,\alpha_0,t_0) \tag{4.4}$$

subject to the initial condition

$$p(\alpha,t_0\,|\,\alpha_0,t_0)=\delta_{\alpha,\alpha_0}\ . \tag{4.5}$$

For Markov processes the joint probability $p(\alpha,t\,|\,\alpha_0,t_0)$ and the one-fold distribution function $P_1(\alpha,t)$ describe the statistical properties completely. The time evolution of the one-fold distribution function is given by the following equation:

$$P_1(\alpha,t)=\sum_{\beta} p(\alpha,t\,|\,\beta,t_0)P_1(\beta,t_0)\ . \tag{4.6}$$

For later applications we denote the initial probability distribution of the Markov chain $x(t)$ at t_0 by

$$P_1(\alpha,t_0)=g(\alpha)\ . \tag{4.7}$$

The response of an atom to strong laser excitation in the presence of such a jump process $x(t)$ is determined by the appropriate Heisenberg or Liouville equation, of course. In the present case the relevant equations are given in (2.1). In any case, they can be written symbolically

$$\frac{\partial}{\partial t}\hat{V}(t)=-iM[\Omega_0,\Delta,\ldots;x(t)]\hat{V}(t)\ , \tag{4.8}$$

where M is effectively the Liouville operator. For example, if applied to Eqs. (2.1), $\hat{V}(t)$ is the operator vector $[\hat{\sigma}_{12},\hat{\sigma}_{21},\hat{\sigma}_{11},\hat{\sigma}_{22}]^T$ and M is the matrix of coefficients in (2.1). The random process $x(t)$ is in this case implicitly carried by $\Omega(t)$.

What is wanted is $\langle \hat{V}(t)\rangle$, that is, the solution to (4.8) averaged over the ensemble of jumps of the implicit telegraph $x(t)$. To obtain $\langle \hat{V}(t)\rangle$ one proceeds indirectly, defining a marginal average $V_\alpha(t)$ by the equation

$$V(t)=\langle \hat{V}(t)\rangle=\sum_{\alpha} g(\alpha)V_\alpha(t)\ , \tag{4.9}$$

where $V_\alpha(t)$ is the average value of $V(t)$ under the condition that x is fixed at the value α at time t. Let us assume that the matrix $M(x)$ depends only upon the current value of $x(t)$, not upon any prior values. From Eq. (4.8) we obtain the intrinsic or Hamiltonian time dependence of $V_\alpha(t)$, but the full time dependence of $\langle V(t)\rangle$ includes, in addition, the effects of the random interruptions coming from x switching from α to other values. The full equation appropriate to Markovian jump processes is the master equation which has been introduced into quantum optics first by Burshtein:[6]

$$\frac{\partial}{\partial t}V_\alpha(t)=-iM(\alpha)V_\alpha(t)-\frac{1}{T}\sum_{\beta}[\delta_{\alpha\beta}-f(\alpha\,|\,\beta)]V_\beta(t)\ . \tag{4.10}$$

The second and third terms of this equation are the Chapman-Kolmogorov-Smoluchowski (CKS) terms given in (4.4). Without M, Eq. (4.10) would be a CKS equation of traditional type, and without the CKS terms it is the nonstochastic equation obtained from (4.8) by fixing the random telegraph in the state α for all time.

Of course, the full master equation (4.10) can be written compactly as

$$\dot{V} = -iWV \,, \tag{4.11a}$$

where

$$-iW_{\alpha\beta} = -iM_{\alpha\beta} - \frac{1}{T}[\delta_{\alpha\beta} - f(\alpha \mid \beta)] \,, \tag{4.11b}$$

with appropriate matrix multiplications implied. For example, in the common case that the dynamical variable V is actually a component of the vector $(\hat{\sigma}_{12}, \hat{\sigma}_{21}, \hat{\sigma}_{11}, \hat{\sigma}_{22})^T$, then (4.8) is already a matrix equation in the atomic operator space. The existence of N possible telegraph states then means that W is a matrix in the product space of the atomic operator vector and telegraph. It is useful to see how the required matrix manipulations can be carried out explicitly in at least one case, and we include such a demonstration in the Appendix.

The important message contained in (4.11) is that no matter how many states are ascribed to the telegraph, or how many components V actually has in a given case, the full and exact solution to the stochastic dynamical equation of interest can be obtained by finite-matrix inversion and by a calculation of the roots of the proper secular equation. Of course, such matrix inversions are rather rarely possible to carry out in fully analytic form, but they are always possible to find numerically with whatever desired accuracy. What is more, in some applications, such as calculations of absorption and scattering spectra, only Laplace transforms are required. In such cases even the secular equation becomes unnecessary. From one point of view, this is the single most attractive feature following from the telegraph model of stochastic processes. In contrast to Gaussian stochastic-noise models, where infinite continued fraction[14] or other[15] methods have unsolved convergence problems, the telegraph model is extremely well defined.

We do not want to overemphasize numerical methods. Burshtein has already examined a number of important laser-atom interaction problems,[6] showing that telegraph models do not need to rely solely on numerical work. We have given another example in Sec. III, and in our paper about pre-Gaussian statistics.[16] In the following paper we will give others subsequently when we discuss frequency telegraphs and non-Lorentzian line shapes.[17] However, it remains true that the enormous flexibility of telegraph models, and their undoubted utility in theoretical modeling, is based on the ease of numerical solution of (4.11). We describe elsewhere our numerical methods[18] and devote Sec. V to an examination of some numerical results.

V. ATOMIC FLUORESCENCE SPECTRUM

The Burshtein master equation (4.10) can be applied easily to the calculation of the fluorescence spectrum of a two-level atom[1] strongly excited by a laser with telegraph noise. The stationary spectrum, conveniently normalized, is given by

$$\mathscr{S}(\omega) = 2\,\mathrm{Re} \int_0^\infty e^{-(\gamma_s + i\omega)\tau} C(\tau)\, d\tau \,, \tag{5.1}$$

where $C(\tau)$ is the dipole autocorrelation (2.3) in the stationary limit,

$$C(\tau) = \lim_{t\to\infty} \langle \hat{\sigma}_{21}(t+\tau)\hat{\sigma}_{12}(t)\rangle \,, \tag{5.2}$$

and γ is the bandwidth of the spectrometer being used to analyze the fluorescence.[19]

It is clear that $\mathscr{S}(\omega)$ is, effectively, the Laplace transform of $C(\tau)$, and that $C(\tau)$ is the first component of a vector $V(\tau)$ which is the stationary limit of the vector $V(t,\tau)$,

$$V(t,\tau) = [\langle \sigma_{12}(t+\tau)\sigma_{12}(t)\rangle, \langle \hat{\sigma}_{21}(t+\tau)\hat{\sigma}_{12}(\tau)\rangle, \langle \hat{\sigma}_{11}(t+\tau)\sigma_{12}(t)\rangle, \langle \hat{\sigma}_{22}(t+\tau)\hat{\sigma}_{12}(t)\rangle]^T \,. \tag{5.3}$$

It is known[20] that in the absence of stochastic forces $V(t,\tau)$ satisfies a first-order 4×4 matrix differential equation in τ with constant coefficients, easily obtained from Eq. (2.1). The Burshtein master equation allows stochastic telegraph-type noise to be added at the sole expense of enlarging the matrix dimensionality from 4×4 to $4N\times 4N$, where N is the number of states of the telegraph. But, because the Burshtein equation also has constant coefficients, the Laplace transform of $V(t,\tau)$ with respect to τ is trivial. Thus, in the stationary limit $(t\to\infty)$ the spectrum $\mathscr{S}(\omega)$ can be obtained exactly and explicitly in terms of finite $(4N\times 4N)$ determinants.

The Appendix is devoted to the full solution of the Burshtein equation in a simple case, including the time evolution which requires, in effect, the inverse Laplace transform and thus some $4N\times 4N$ matrix inversions. Here we present only the results for the resonance fluorescence spectrum of the atom whose inversion was calculated in Sec. III. We will not display the 8×8 matrix algebra, but only graphs of the results. The spectra can be displayed in a variety of ways. In Fig. 2 we show a set of spectra for a double range of phase values a from 0 to π. Some of the features of the spectrum can be understood on the basis of the discussion in Sec. III, but new characteristics are evident also.

Note that the spectra of Fig. 2 are taken in the motional narrowing regime of telegraph jumps: $\Omega_0 T \ll 1$. Note

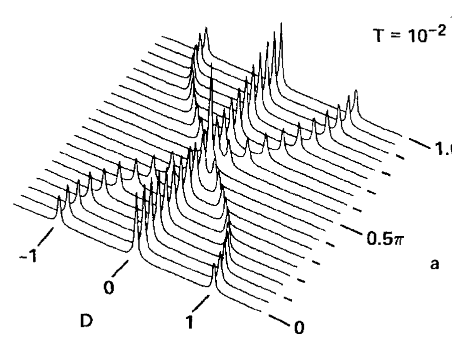

FIG. 2. Steady-state fluorescence spectrum for two-state atom resonantly excited by random telegraph phase noise vs $D = \omega_s - \omega_L$, detuning of spectrometer frequency ω_s from laser frequency ω_L (in units of Rabi frequency Ω_o), for various values of the jump parameter a and with mean interruption time $T = 10^{-2}\Omega_0^{-1}$. Spectrometer width is $\gamma_s = 0.02\Omega_0$.

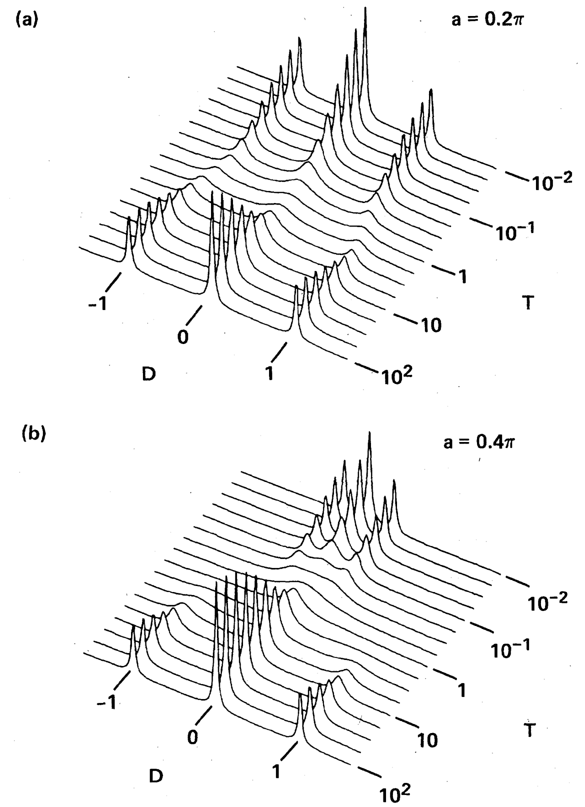

FIG. 3. Steady-state fluorescence spectrum for two-state atom resonantly excited by random telegraph phase noise vs $D=\omega_s-\omega_L$, detuning of spectrometer frequency ω_s from laser frequency ω_L (in units of Rabi frequency Ω_o), for various choices of mean interruption time T. In (a) the phase jump is 0.2π and in (b) it is 0.4π. Spectrometer width is $\gamma_s=0.02\Omega_0$.

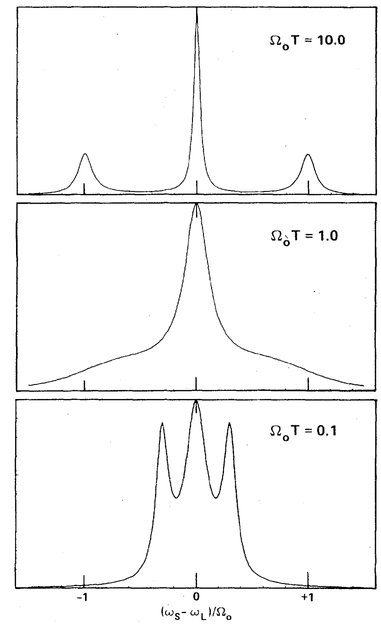

FIG. 4. Fluorescence spectrum for two-state atom resonantly excited by random telegraph phase noise with phase jumps of $a=0.4\pi$ and three choices of mean interruption time T. Spectrometer width is $\gamma_s=0.02\Omega_0$.

also that the sideband splitting, as a function of a, evidently follows the form $\Omega=\Omega_0\cos a$, as foretold in (3.6). One remarkable result is seen for $a=\pi/2$, where the phase telegraph produces a zero-mean *amplitude* fluctuation. In strong contrast to the case of Gaussian chaotic amplitude fluctuation,[21] here the spectrum shows no splitting at all.

In Fig. 3 we show spectra obtained from two of the phase telegraphs contained in Fig. 2, namely $a=0.2\pi$ and 0.4π. In this case the size of the phase jump is held fixed and the telegraph switching rate $1/T$ is varied. Note that as the relaxation rate $1/T$ gets smaller (for example, as $\Omega_0 T$ goes from 1 to 100) the spectral lines do not get correspondingly narrower. This is because the spectrometer bandwidth $\gamma=0.02\Omega_0$ is not smaller than $1/T$. In other words, we see an instrumental effect.

We showed the temporal dynamics of population inversion for $a=0.4\pi$ in Sec. III. In the corresponding spectrum [Fig. 3(b)] we see clear reflections of the population dynamics. The region around $\Omega_0 T=1$ shows no clear spectral features, for example, and in Fig. 1 it showed only overdamped decay with little oscillation. Similarly, motional narrowing is clearly evident and is consistent with the change in oscillation frequency from Ω_0 to $\Omega_0\cos a$ as T becomes sufficiently short. In Fig. 4 we show the three spectra that correspond exactly to the three population curves in Fig. 1.

VI. SUMMARY

In this paper we have discussed some of the characteristics of generalized Poisson processes, i.e., random telegraph signals (RTS), and the ways that they enter into the dynamics of strong resonant laser-atom interactions. We noted some differences with other kinds of noise models, and we stated the advantages of random telegraphs for theoretical calculations. In addition to giving the background of the formalism, we discussed in detail some new results for a two-level atom exposed to a laser modulated by a two-state phase telegraph. These included a new expression for the population equation, including spontaneous emission damping, as well as solutions for the fluorescence spectrum. Features relating to motional narrowing, spectrometer resolution, and amplitude fluctuations were pointed out. In the following paper we will draw on the formalism established here to discuss the randomly jumping frequency $\mu(t)=d\phi/dt$. We shall discuss examples which demonstrate both temporal and spectral motional narrowing, nonexponential correlations, and non-Lorentzian spectra.

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APPENDIX: THE TWO-STATE TELEGRAPH, AN EXPLICIT SIMPLE APPLICATION OF THE MASTER EQUATION

For a two-state random telegraph signal the CKS equation (4.4) takes the following form:

$$\frac{\partial}{\partial t}p(\alpha,t\mid\alpha_0,t_0) = -\frac{1}{T}p(\alpha,t\mid\alpha_0,t_0)+\frac{1}{T}p(-\alpha,t\mid\alpha_0,t_0)\ . \qquad (A1)$$

This equation can be solved exactly leading to

$$p(\alpha,t\mid\alpha_0,t_0)=\tfrac{1}{2}\delta_{\alpha,\alpha_0}(1+e^{-2|t-t_0|/T}) +\tfrac{1}{2}\delta_{\alpha,\alpha_0}(1-e^{-2|t-t_0|/T})\ . \qquad (A2)$$

The two-state ($\pm a$) jump process has the following initial probability distribution (4.7):

$$g(\alpha)=\tfrac{1}{2}\delta_{\alpha,a}+\tfrac{1}{2}\delta_{\alpha,-a}\ . \qquad (A3)$$

With the solution (A2) we can calculate the phase dependent correlation function (2.4) from the following formula:

$$\langle e^{i\phi(t_1)-i\phi(t_2)}\rangle=\sum_{\alpha_1\alpha_0} e^{i\alpha}p(\alpha,t_1\mid\alpha_0,t_2)e^{-i\alpha_0}g(\alpha_0) =\cos^2 a+\sin^2 a\, e^{-2|t_1-t_2|/T}\ . \qquad (A4)$$

This result confirms the elementary derivation performed in Sec. II.

Equation (4.11) is the symbolic master equation underlying our discussion of telegraph noise. If $\hat{V}$ is a component of the atomic operator vector, say the kth, then Eqs. (2.1) show that it is connected to the other components. That is, (4.8) could be written

$$\dot{\hat{V}}^k(t)=-iM^{kl}(x)\hat{V}^l(t)\ , \qquad (A5)$$

where repeated indices are summed, as usual. Then the master equation (4.11) takes the explicit form

$$\dot{V}^k_\alpha(t)=-iW^{kl}_{\alpha\beta}V^l_\beta(t)\ , \qquad (A6)$$

where

$$W^{kl}_{\alpha\beta}=\delta_{\alpha\beta}M^{kl}(\alpha)-\frac{i}{T}\delta_{kl}[\delta_{\alpha\beta}-f(\alpha\mid\beta)]\ . \qquad (A7)$$

It is clear that the CKS terms generally give rise to relaxation, so we can identify a relaxation matrix $\Gamma_{\alpha\beta}$ as

$$\Gamma_{\alpha\beta}=\frac{1}{T}[\delta_{\alpha\beta}-f(\alpha\mid\beta)]\ . \qquad (A8)$$

Of course, M may also include damping terms, due, for example, to radiative decay, as in (2.1).

The two-state telegraph for which $f(+,-)=f(-,+)=1$ is the simplest jump process. For this case the relaxation matrix is

$$\Gamma=\frac{1}{T}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \qquad (A9)$$

and the Burshtein equation takes the simple form

$$\frac{d}{dt}\begin{bmatrix}V_1\\ V_2\end{bmatrix}=-i\begin{bmatrix}M_1-i(1/T) & i(1/T)\\ i(1/T) & M_2-i(1/T)\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix}, \qquad (A10)$$

where $M_{1,2}=M(\pm a)$. If V_α is a vector then M is a matrix and $1/T$ is a diagonal matrix. Equation (A10) can be solved exactly in terms of Laplace transforms:

$$\begin{bmatrix}\tilde{V}_1(z)\\ \tilde{V}_2(z)\end{bmatrix}=\begin{bmatrix}\dfrac{1}{A_1A_2-1/T^2}A_2 & \dfrac{1/T}{A_1A_2-1/T^2}\\ \dfrac{1/T}{A_2A_1-1/T^2} & \dfrac{1}{A_2A_1-1/T^2}A_1\end{bmatrix} \times\begin{bmatrix}V_1(0)\\ V_2(0)\end{bmatrix}, \qquad (A11)$$

where $A_k=z+1/T+iM_k$. If M_1 and M_2 are numbers, the solutions for $V_1(t)$ and $V_2(t)$ have the form

$$\begin{bmatrix}V_1(t)\\ V_2(t)\end{bmatrix}=R\left[e^{-i\lambda_+t}\begin{bmatrix}-(1/T)^2 & -i(1/T)Q\\ -i(1/T)Q & Q^2\end{bmatrix}e^{-i\lambda_-t}\begin{bmatrix}Q^2 & -i(1/T)Q\\ -i(1/T)Q & -(1/T)^2\end{bmatrix}\right]\begin{bmatrix}V_1(0)\\ V_2(0)\end{bmatrix}, \qquad (A12)$$

where

$$Q\equiv[\tfrac{1}{4}(M_1-M_2)^2-(1/T)^2]^{1/2}-\tfrac{1}{2}(M_1-M_2)\ , \qquad (A13)$$

$$R\equiv\{2Q[Q+\tfrac{1}{2}(M_1-M_2)]\}^{-1}\ , \qquad (A14)$$

$$\lambda_\pm=\tfrac{1}{2}(M_1+M_2)-i(1/T) \pm[\tfrac{1}{4}(M_1-M_2)^2-(1/T)^2]^{1/2}\ . \qquad (A15)$$

This expression applies to any bivalued RTS, whether for phase, frequency, or amplitude fluctuations.

Two limiting regimes have particular interest. When T is sufficiently long the solutions become

$$V_1(t)\simeq e^{-i\lambda_+t}V_1(0) \text{ with } \lambda_+\simeq M_1-i(1/T) \qquad (A16a)$$

and

$$V_2(t)\simeq e^{-i\lambda_-t}V_2(0) \text{ with } \lambda_-\simeq M_2-i(1/T)\ . \qquad (A16b)$$

The stochastic average is the sum of these two independent solutions, each appropriate to a particular value of the fluctuation parameter α, but with damping rate augmented by the mean interruption rate $1/T$:

$$\langle V(t)\rangle\simeq e^{-i\lambda_+t}V_1(0)+e^{-i\lambda_-t}V_2(0)\ . \qquad (A17)$$

When T is sufficiently short the solutions approach the limit

$$\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} \cong \left[e^{-i\lambda_+ t} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + e^{-i\lambda_- t} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right] \begin{bmatrix} V_1(0) \\ V_2(0) \end{bmatrix}, \tag{A18}$$

where

$$\lambda_+ \cong \tfrac{1}{2}(M_1+M_2) - i(T/2)(M_1-M_2)^2 , \tag{A19a}$$

$$\lambda_- \cong \tfrac{1}{2}(M_1+M_2) - i(2/T) + i(T/2)(M_1-M_2)^2 . \tag{A19b}$$

We see that the interruptions occur so rapidly that the atom responds only to the average Bloch matrix (recall remarks in Sec. III). We also note that, whereas the imaginary contribution to λ_+ gets smaller linearly with decreasing T, the eigenvalue λ_- possesses a large imaginary term and hence will contribute nothing to the expression for $V_a(t)$. The stochastic average, in this limit, becomes

$$V(T) \cong e^{-i\lambda_+ t}[V_1(0)+V_2(0)] . \tag{A20}$$

We can apply the preceding analysis to obtain the third-order differential equation (3.4) in a rigorous way, i.e., without the decorrelation step performed in Eq. (3.1). By introducing the operator-valued vector $\hat{V}=(\hat{\sigma}_{12},\hat{\sigma}_{21},\hat{\sigma}_{11},\hat{\sigma}_{22})^T$ we obtain from the atomic equations (2.1) the proper M_1 and M_2 matrices required in the solution (A11). For $\sigma_{12}(0)=\sigma_{21}(0)=0$ and $\Delta=0$, from this equation we calculate the following expression for the Laplace transform of the inversion:

$$\widetilde{W}(z)=\widetilde{\sigma}_{22}(t)-\widetilde{\sigma}_{11}(z)=\left[w(0)-\frac{A}{Z}\right]\frac{\left[z+\frac{A}{2}\right]\left[z+\frac{A}{2}+\frac{2}{T}\right]}{\left[(z+A)\left[z+\frac{A}{2}\right]\left[z+\frac{A}{2}+\frac{2}{T}\right]+|\Omega|^2\left[z+\frac{A}{2}+\frac{2\cos^2 a}{T}\right]\right]} . \tag{A21}$$

Simple algebraic transformations of Eq. (A21) lead to the differential equation (3.4), thus providing a justification for the decorrelation procedure used in Sec. III.

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[1]For theory, see A. I. Burshtein, Zh. Eksp. Teor. Fiz. **49**, 1362 (1965) [Sov. Phys.—JETP **22**, 939 (1966)]; M. Newstein, Phys. Rev. **167**, 89 (1968); B. R. Mollow, *ibid.* **188**, 1969 (1969). For experiment, see F. Schuda, C. R. Stroud, and M. Hercher, J. Phys. B **7**, L198 (1974); F. Y. Wu, R. E. Groue, and S. Ezekiel, Phys. Rev. Lett. **35**, 1426 (1975); W. Hartig, W. Rasmussen, R. Schieder, and H. Walther, Z. Phys. A **278**, 205 (1976).

[2]Recent experimental work demonstrates interest in such fluctuations and possible detailed comparisons with theories which include such effects. See, for example, P. Agostini, A. T. Georges, S. E. Wheatley, P. Lambropoulos, and M. D. Levenson, J. Phys. B **11**, 1733 (1978); B. R. Marx, J. Simons, and L. Allen, *ibid.* **11**, L273 (1978); D. W. Nitz, A. V. Smith, M. D. Levenson, and S. J. Smith, Phys. Rev. A **24**, 288 (1981); D. S. Elliott, R. Roy, and S. J. Smith, *ibid.* **26**, 12 (1982); L. Allen, in *Coherence and Quantum Optics V*, proceedings of the Fifth Rochester Conference, 1983, edited by L. Mandel and E. Wolf (Plenum, New York, in press).

[3]The use of relaxation times T_1 and T_2 originated with F. Bloch in his theory of nuclear magnetic resonance. Later applications to optical resonance phenomena are summarized in M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, Mass. 1974); L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).

[4]For example, see S. O. Rice, in *Selected Topics in Noise and Stochastic Processes,* edited by N. Wax (Dover, New York, 1954); P. W. Anderson, J. Phys. Soc. Jpn. **9**, 316 (1954); and discussions and references in R. Kubo, in *Fluctuation, Relaxation and Resonance in Magnetic Systems,* edited by D. Ter Haar (Oliver and Boyd, Edinburgh, 1962); N. G. Van Kampen, Phys. Rep. **24C**, 171 (1976); R. F. Fox, *ibid.* **48C**, 179 (1978); I. Oppenheim, K. E. Shuler, and G. H. Weiss, *Stochastic Processes in Chemical Physics: The Master Equation* (MIT, Cambridge, Mass., 1977).

[5]G. S. Agarwal, Phys. Rev. Lett. **37**, 1383 (1976); J. H. Eberly, *ibid.* **37**, 1387 (1976); H. J. Kimble and L. Mandel, Phys. Rev. A **15**, 689 (1977); P. Avan and C. Cohen-Tannoudji, J. Phys. B **10**, 155 (1977); P. Zoller and P. Lambropoulos, *ibid.* **12**, L547 (1979).

[6]See the early papers of A. I. Burshtein: A. I. Burshtein, Ref. 1; A. I. Burshtein and Yu. S. Oseledchik, Zh. Eksp. Teor. Fiz. **51**, 1071 (1966) [Sov. Phys.—JETP **24**, 716 (1967)]; L. D. Zusman and A. I. Burshtein, *ibid.* **61**, 976 (1971) [*ibid.* **34**, 520 (1972)].

[7]P. Zoller and F. Ehlotzky, J. Phys. B **15**, 3023 (1977); K. Wódkiewicz, Z. Phys. B **42**, 95 (1981).

[8]An earlier description of the contents of several forthcoming papers by the present authors is given by B. W. Shore and J. H. Eberly, in *Coherence and Quantum Optics V,* proceedings of the Fifth Rochester Conference, 1983, edited by L. Mandel and E. Wolf (Plenum, New York, in press). See also Ref. 16.

[9]See, for example, L. Allen and J. H. Eberly, Ref. 3.

[10]A. Abragam, *The Principles of Nuclear Magnetism* (Oxford University, London, 1961); C. P. Slichter, *Principles of Magnetic Resonance* (Springer, Heidelberg, 1978).

[11]For a full discussion of the mathematics of Markov chains, see J. L. Doob, *Stochastic Processes* (Wiley, New York, 1967), Chap. V.

[12]A sufficient description of Markov chains and their transitions for our purposes can be found in W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York,

1958), Vol. I.
[13]See, for example, Ref. 11, p. 239; or Ref. 12, p. 523.
[14]See, for example, S. N. Dixit, P. Zoller, and P. Lambropoulos, Phys. Rev. A **21**, 1289 (1980); K. Wódkiewicz, J. Math. Phys. **23**, 2179 (1982).
[15]J. J. Yeh and J. H. Eberly, Phys. Rev. A **24**, 888 (1981).
[16]K. Wódkiewicz, B. W. Shore, and J. H. Eberly, J. Opt. Soc. Am. B **1**, 398 (1984).
[17]K. Wódkiewicz, B. W. Shore, and J. H. Eberly, following paper, Phys. Rev. A **30**, 2390 (1984).
[18]B. W. Shore, J. Opt. Soc. Am. B **1**, 176 (1984).
[19]J. H. Eberly and K. Wódkiewicz, J. Opt. Soc. Am. **67**, 1252 (1977).
[20]B. R. Mollow, Ref. 1; P. W. Milonni, Ph.D. thesis, University of Rochester, 1974; H. J. Kimble and L. Mandel, Phys. Rev. A **13**, 2123 (1976); K. Wódkiewicz and J. H. Eberly, Ann. Phys. (N.Y.) **101**, 574 (1976).
[21]A. T. Georges and P. Lambropoulos, Phys. Rev. A **20**, 991 (1979); P. Zoller, *ibid.* **20**, 1019 (1979).