# Electron-pair-production cross section in the tip region of the positron spectrum

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The radial integrals for electron-pair production in a point Coulomb potential have been expressed by Sud, Sharma, and Sud in terms of the matrix generalization of the $\Gamma$ function. Two new partial differential equations in photon energy satisfied by the matrix $\underline{\Gamma}$ function are obtained. We have obtained, on integrating the partial differential equations, accurate radial integrals as a function of photon energy for the pair production by intermediate-energy photons. The cross section in the tip region of the spectrum are calculated for photons of energy 5.0 to 10.0 MeV for $^{92}$U. The new technique results in extensive saving in computer time as the basic radial integrals in terms of the hypergeometric function $F_2$ are computed at one photon energy for each pair of partial waves. The results of our calculations are compared with plane-wave Born-approximation results and with the calculations of Dugne and of Deck, Moroi, and Alling.

## I. INTRODUCTION

The distorted-wave Born approximation (DWBA) calculation of the electron-pair-production cross section in the point Coulomb field of the atomic nucleus has been performed by a number of workers.[1–6] Jaeger and Hulme[1] have expressed the DWBA radial integrals in terms of Appell's hypergeometric function $F_2$, which is a doubly infinite series, and is defined as[7]

$$F_2(a,b_1,b_2,c_1,c_2;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

where $(z)_n \equiv \Gamma(z+n)/\Gamma(z)$, with $(z)_0=1$, is the Pochhammer symbol. The condition for convergence for the $F_2$ function is $|x|+|y|<1$. The explicit DWBA expression for the pair-production cross section in terms of $F_2$ functions is given in literature.[5,6] Øverbø *et al.*[5] have calculated the pair-production cross section for a large number of atomic numbers and for photon energies up to 5.0 MeV. It has not been possible to extend the DWBA calculation for photon energy greater than 5 MeV as the number of partial waves required increases with increasing photon energy and, furthermore, the $F_2$ functions become very slowly convergent series. A new technique has been developed to handle the radial integrals involving the product of the Dirac-Coulomb functions.[8–12] In this method the radial integrals are extracted from the elements of the matrix gamma function which is a matrix generalization of the mathematical $\Gamma$ function. The recurrence relation and the partial differential equation satisfied by the matrix $\underline{\Gamma}$ function reduces the number of the radial matrix elements to be evaluated. Sud *et al.*[12] have expressed the radial integrals for the pair-production cross section in terms of the elements of the matrix $\underline{\Gamma}$ function. They have also obtained a partial differential equation in lepton energy satisfied by the matrix $\underline{\Gamma}$ function. We present in Sec. II two new partial differential equations satisfied by the matrix $\underline{\Gamma}$ function. The partial differential equation for $\underline{\Gamma}$ matrix in photon energy has been derived for a fixed electron energy, which can be integrated numerically to obtain the matrix $\underline{\Gamma}$ function for different values of the photon energies [given the matrix $\underline{\Gamma}$ function at some initial value, i.e., electron energy $E_-$ (a fixed value), positron energy $E_+$, and photon energy $k$]. The second partial differential equation for the $\underline{\Gamma}$ matrix is variable in energy, which can be integrated to obtain the accurate radial integrals for the physical value of the photon energy [given the matrix $\underline{\Gamma}$ function for fixed values of $E_-$ and $E_+$, and arbitrary energy parameter $w$]. The large value of $w$ improves the convergence of the $F_2$ functions used to form the initial $\underline{\Gamma}$ matrix. We present in Sec. III results of the calculation of the electron-pair-production cross section in the tip region ($E_- = 1.008mc^2$), of the positron spectrum for photon energies of 5.0—10.0 MeV using the differential equations (11) and (13). We have compared the results of our calculation with the PWBA calculations, the calculations of Deck *et al.*,[13] and for a few energy points with the DWBA results of Dugne.[14] In Sec. IV we discuss the utility of the theoretical formalism for the computation of the pair-production cross section for the photon in the intermediate energy range.

## II. THE PARTIAL-DIFFERENTIAL EQUATIONS OF THE MATRIX $\underline{\Gamma}$ FUNCTION IN PHOTON ENERGIES

The DWBA expression for the differential cross section for the pair-production cross section can be found in the literature and the reader is referred for details to Refs. 5 and 12. The DWBA radial integrals can be expressed in

 

terms of the elements of the matrix $\underline{\Gamma}$ function. Here we give an outline of the steps leading to the definition of the matrix $\underline{\Gamma}$ function, and the derivation of the two new partial differential equations satisfied by it.

The radial Dirac-Coulomb functions for a lepton in a point Coulomb field of the atomic nucleus satisfies a first-order matrix differential equation of the form[12]

$$\frac{d}{dx}\underline{U}_{\pm}(x)=\left[\frac{1}{x}\underline{A}_{\pm}-\underline{B}_{\pm}\right]\underline{U}_{\pm}(x)\,, \tag{1}$$

where $x=p_{\pm}r$ and the constant matrices $\underline{A}$ and $\underline{B}$ in a representation in which $\underline{B}$ is a diagonal matrix are given as

$$\underline{A}_{\pm}=\begin{bmatrix} iy_{\pm} & \gamma_{\pm}-iy_{\pm} \\ \gamma_{\pm}+iy_{\pm} & -iy_{\pm} \end{bmatrix}\,,$$
$$\underline{B}_{\pm}=p_{\pm}\underline{B}'=p_{\pm}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\,, \tag{2}$$

where $+$ $(-)$ corresponds to positron (electron). The other parameters in Eqs. (1) and (2) are $y_{\pm}=\mp\alpha ZE_{\pm}/p_{\pm}$, $\gamma_{\pm}=[\kappa_{\pm}^2-(\alpha Z)^2]^{1/2}$, where $\kappa$ is the eigenvalue of the Dirac operator $\tilde{K}=\beta(\vec{\sigma}\cdot\vec{L}+1)$, $\alpha$ is the fine-structure constant, $Z$ is the charge of the atomic nucleus, and $E$ and $p$ are the energy and momentum of the lepton.

The solution of Eq. (1) for the pair of matrices given in Eq. (2) is given as

$$\underline{U}_{\pm}=\frac{1}{(2ip_{\pm}r)^{1/2}}\begin{bmatrix} M_{-1/2-iy_{\pm},\gamma_{\pm}}(2ip_{\pm}r) & -\dfrac{(\gamma_{\pm}-iy_{\pm})}{\gamma_{\pm}+iy_{\pm}}M_{-1/2-iy_{\pm},-\gamma_{\pm}}(2ip_{\pm}r) \\ M_{1/2-iy_{\pm},\gamma_{\pm}}(2ip_{\pm}r) & M_{1/2-iy_{\pm},-\gamma_{\pm}}(2ip_{\pm}r) \end{bmatrix}\,, \tag{3}$$

where $M$ is the Whittaker function. We define a matrix function $\underline{W}(\underline{A},\underline{B};r)$ which is a matrix direct product of lepton wave functions and other $r$-dependent terms in the matrix elements

$$\underline{W}(\underline{A},\underline{B};r)=\underline{U}_{-}(p_{-}r)\otimes\underline{U}_{+}(p_{+}r)\frac{e^{-ikr}}{(kr)^{n+1}}\,, \tag{4}$$

where $k$ is the photon energy and $\otimes$ denotes the matrix direct product. The matrix function $\underline{W}(\underline{A},\underline{B};r)$ satisfies a $4\times4$ matrix differential equation (1), with the following $\underline{A}$ and $\underline{B}$ matrices:

$$\underline{A}=\underline{A}_{-}\otimes\underline{I}_2+\underline{I}_2\otimes\underline{A}_{+}-(n+1)\underline{I}_4\,,$$
$$\underline{B}=p_{-}\underline{B}'\otimes\underline{I}_2+p_{+}\underline{I}_2\otimes\underline{B}'+ik\underline{I}_4\,, \tag{5}$$

where $\underline{I}$ is a unit matrix of the dimension of the subscript level and $\underline{A}_{\pm},\underline{B}'$ are as given in Eq. (2). The integral of such an integrand [Eq. (4)] is defined as a matrix $\underline{\Gamma}$ function (for details see Sud *et al.*[9])

$$\underline{\Gamma}(\underline{A}+\underline{I},\underline{B})=\int_{(0)}^{\infty}\underline{W}(\underline{A},\underline{B};r)dr\,, \tag{6}$$

where it is assumed that the integral is convergent at the upper limit and the (0) indicates that any simple pole present at the origin has been removed. The elements of the matrix $\underline{\Gamma}$ function in Eq. (6) for the $\underline{A}$ and $\underline{B}$ matrices of Eq. (5) are Appell's hypergeometric function $F_2$.

We obtain a partial differential equation in photon energy by treating the energy of one of the leptons a fixed value and the photon energy as a variable ($k=E_{+}+E_{-}$) in Eq. (6). The partial differential equation in photon energy (electron energy constant) is given as

$$\frac{\partial}{\partial k}\underline{\Gamma}=\int_{(0)}^{\infty}\left[\left[\underline{U}_{-}(p_{-}r)\otimes\frac{\partial}{\partial p_{+}}\underline{U}_{+}\right]\left[\frac{\partial p_{+}}{\partial k}\right]\frac{e^{-ikr}}{(rk)^{n+1}}\right.$$
$$\left.+\underline{U}_{-}(p_{-}r)\otimes\underline{U}_{+}(p_{+}r)\frac{\partial}{\partial k}\times\left[\frac{e^{-ikr}}{(kr)^{n+1}}\right]\right]dr\,. \tag{7}$$

We have used the following relations to express the right-hand side of the Eq. (7) in a closed form:

$$p_{+}\frac{\partial}{\partial p_{+}}\underline{U}_{+}(p_{+}r)=r\frac{\partial}{\partial r}\underline{U}_{+}(p_{+}r) \tag{8}$$

(obtained by assuming that $\underline{A}_{+}$ and $\underline{B}'$ are constant), integrand $\underline{W}(\underline{A},\underline{B};r)$ satisfying the relation (Sud *et al.*[9])

$$r^a e^{-br}\underline{W}(\underline{A},\underline{B};r)=\underline{W}(\underline{A}+a\underline{I},\underline{B}+b\underline{I};r)\,, \tag{9}$$

and the recurrence relation satisfied by the matrix $\underline{\Gamma}$ function

$$\underline{B}\,\underline{\Gamma}(\underline{A}+\underline{I},\underline{B})=\underline{A}\,\underline{\Gamma}(\underline{A},\underline{B})\,. \tag{10}$$

We can express Eq. (7) as

$$\frac{\partial}{\partial k}\underline{\Gamma}=\left[\frac{E_{+}}{p_{+}}\underline{T}_{+}+\underline{T}_k\right]\underline{\Gamma}\,, \tag{11}$$

where

$$\underline{T}_{+}=\frac{1}{p_{+}}\underline{I}_2\otimes\underline{A}_{+}-(\underline{I}_2\otimes\underline{B}')\underline{B}^{-1}(\underline{A}+\underline{I})\,,$$
$$\underline{T}_k=-\frac{(n+1)}{k}\underline{I}_4-i\underline{I}_4\underline{B}^{-1}(\underline{A}+\underline{I})\,. \tag{12}$$

The $\underline{A}$ and $\underline{B}$ matrices are given in Eq. (5) and $\underline{A}_+$ and $\underline{B}'$ are as given in Eq. (2). In the derivation of Eq. (11) we have made an implicit assumption that the matrices $\underline{A}_+$ and $\underline{B}'$ are independent of the momentum variable $p_+$. However, the $\underline{A}_+$ matrix depends on the momentum variable through the parameter $y_+ = -\alpha Z E_+ / p_+$. In the region where $E_+ \gg m$ the parameter $y_+$ can be expressed as $y_+ \simeq -\alpha Z(1+m^2/2p_+^2)$ and may be treated as constant. In Eq. (12), aside from the explicit appearance of $k$, the only function of $k$ is the matrix $\underline{B}$ which is homogeneous and linear in $k$. So the equation is evidently homogeneous in $k$. Equation (11) can be integrated numerically [given the matrix $\underline{\Gamma}$ function at some initial value, i.e., for $E_-$ (fixed value), $E_+$, and $k$] to obtain the matrix $\underline{\Gamma}$ function for a different value of the photon energy. The radial integrals obtained by integrating Eq. (11) are very accurate for intermediate energy photons except at the lower end of the spectrum where $E_+ \simeq m$. We have used this equation to calculate the radial integrals in the tip region where the approximation is valid. It may be mentioned that an equation [similar to the Eq. (11)] obtained for $E_+ = \text{const}$ will be valid for the lower end of the spectrum.

A very useful partial differential equation in energy (a free parameter) is obtained by keeping the energies of the electron and the positron fixed. In the definition of the matrix $\underline{\Gamma}$ function [Eq. (6)] the photon energy $k$ is replaced by a parameter $w$. The partial differential equation in $w$ is obtained by following the procedure followed in deriving Eq. (11) and the required equation is given as

$$\frac{\partial}{\partial w}\underline{\Gamma} = \left[-\frac{(n+I)}{w}\underline{I}_4 - i\underline{I}_4\underline{B}^{-1}(\underline{A}+\underline{I})\right]\underline{\Gamma} . \tag{13}$$

This equation can be integrated to obtain the matrix $\underline{\Gamma}$ function of the desired value of the photon energy $k$. The parameter $w$ is taken as $w = k + k'$, where $k$ is the photon energy and $k'$ is the additonal arbitrary energy. The convergence of the $F_2$ functions required to form the $\underline{\Gamma}(w)$ is improved due to the reduction in its arguments $[x_\pm = 2p_\pm/(p_+ + p_- + w)]$.

## III. RESULTS AND DISCUSSION

We present here the results (Table I) of the calculation of the cross section in the tip region ($E_- = 1.008mc^2$) of the positron spectrum for photons of energy 5.0—10.0 MeV in steps of 0.5 MeV for $^{92}$U. We elucidate here the important features of our calculation. The matrix $\underline{\Gamma}(w)$ is formed for each partial-wave combination for $E_- = 1.008mc^2$, $E_+ = (7.5 - E_-)$ MeV, and $w = 12.5$ MeV by calculating Appell's hypergeometric functions with the improved convergence due to the parameter $w$. The differential equation [Eq. (13)] is integrated and $\underline{\Gamma}(w)$ is used as the initial value to obtain the $\underline{\Gamma}$ matrix for the physical value of the photon energy $k$ (in our case $k = 7.5$ MeV). There is an agreement up to eight significant figures between the results obtained by integrating Eq. (13) and by directly summing the $F_2$ series. Therefore the new technique [i.e., the use of Eq. (13)] gives results for the cross section as accurate as those obtained with the existing techniques. The differential equation [Eq. (11)] is integrated both upward as well as downward in photon energy by using the Runge-Kutta-Gill method, with the initial $\underline{\Gamma}$ matrix for $E_- = 1.008mc^2$, $k = 7.5$ MeV, and $E_+ = k - E_-$, to obtain the $\underline{\Gamma}$ matrix for photon energies between 5.0 and 10.0 MeV. The variation of parameter $y_+$ [treated as a constant in the derivation of Eq. (11)] is of the order of 1 part in $10^6$ over one integration step used in the calculation and is asymmetric around the initial photon energy $k = 7.5$ MeV. However, the slowly varying parameter $y_+$ changes by 0.4%, at $k = 5.0$ MeV, and by 0.12%, at $k = 10$ MeV, of its value at the initial photon energy. Therefore the approximation is better when Eq. (11) is integrated in the upward direction. The error in our calculation of the matrix $\underline{\Gamma}$ function varies with the

TABLE I. $(1/Z^2)(d\sigma/dE_+)$ differential cross section for pair production for electron energy $E_- = 1.008mc^2$ and for different photon energy $k$. $(1/Z^2)(d\sigma_{\text{DW}}/dE_+)$, $(1/Z^2)(d\sigma_{\text{PW}}/dE_+)$, and $(1/Z^2)(d\sigma_{\text{Du}}/dE_+)$ are the results of the present, PWBA, and Dugne (Ref. 14) calculations, respectively.

| $k$ (MeV) | $\frac{1}{Z^2}\frac{d\sigma_{\text{DW}}}{dE_+}$ ($10^{-4}$ b/$mc^2$) | $\frac{1}{Z^2}\frac{d\sigma_{\text{PW}}}{dE_+}$ ($10^{-5}$ b/$mc^2$) | $\frac{1}{Z^2}\frac{d\sigma_{\text{Du}}}{dE_+}$[a] ($10^{-4}$ b/$mc^2$) |
|---|---|---|---|
| 5.0 | 1.0516 | 1.1296 | 1.1264 |
| 5.5 | 0.9732 | 1.0578 | |
| 6.0 | 0.9118 | 0.9933 | |
| 6.5 | 0.8582 | 0.9353 | |
| 7.0 | 0.8135 | 0.8832 | |
| 7.5 | 0.7737 | 0.8362 | 0.7885 |
| 8.0 | 0.7310 | 0.7937 | |
| 8.5 | 0.7056 | 0.7552 | |
| 9.0 | 0.6720 | 0.7201 | |
| 9.5 | 0.6543 | 0.6879 | |
| 10.0 | 0.6363 | 0.6584 | 0.6337 |

[a]Dugne's calculations are for photon energies of $10mc^2$, $15mc^2$, and $20mc^2$. The values shown in the table are interpolated to the energies for which we have performed the calculations.

photon energy due to the variation of parameter $y_+$ and is estimated to be of the order of 1 part in $10^3$ at the photon energy $k=10$ MeV and of the order of 1 part in $10^2$ at the photon energy $k=5$ MeV. (We estimate the error in the calculation of the $\underline{\Gamma}$ function by comparing our results with the directly evaluated $\underline{\Gamma}$ matrix calculated by summing the $F_2$ series.) The error associated with the Runge-Kutta-Gill integration procedure causes no problem as the matrix $\underline{\Gamma}$ function is a smooth function of photon energy. Therefore the error estimated in the cross section due to the use of Eq. (11) is less than 10% at $k=5.0$ MeV and less than 1% at $k=10.0$ MeV. The use of the new technique results in considerable saving in computer time, as we compute the radial integrals for 11 different photon energies ($E_-$ fixed, $E_+=k-E_-$) by calculating Appell's hypergeometric series $F_2$ only once. We have written a computer code to calculate the positron spectrum [for an expression of $d\sigma/dE_+$ see Eq. (5.8) of Ref. 5] for 11 different energies simultaneously. In the computer code developed by us the radial integrals are obtained by integrating the partial differential equation and the recurrence relation satisfied by the $\underline{\Gamma}$ function is used to reduce further the computational work. In Fig. 1 we have shown $T_q$,

$$T_q=\sum_{|\kappa_-|=1}^{q}\frac{1}{Z^2}\left[\frac{d\sigma}{dE_+}\right]_{|\kappa_-|,q-|\kappa_-|+1}$$

$$(\text{with } q=|\kappa_-|+|\kappa_+|-1)$$

as a function of $q$ for three different energies. We have calculated the partial contribution $T_q$ for $q_{\max}=48$ and $\log T_q$ is quadratically extrapolated to $2q_{\max}$. In Fig. 2 the convergence of $\sum T_q$ is shown as a function $q$ for one photon energy and the limit is obtained by using the projection techniques.[15] We give in Table I the results of our DWBA calculation, a PWBA calculation[16] for photon energies of 5.0—10.0 MeV, and Dugne's calculation, available for a few energy points. It can be seen from Table I that the effect of the Coulomb distortion is quite significant in the tip region of the positron spectrum. It is also

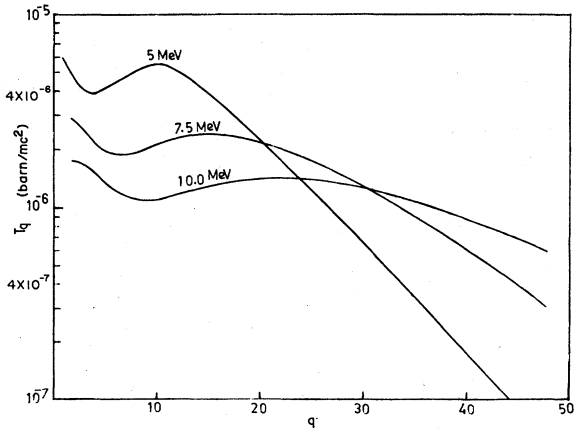


FIG. 1. The term $T_q=\sum_{|\kappa_-|=1}^{q}(1/Z^2)(d\sigma/dE_+)_{|\kappa_-|,q-|\kappa_-|+1}$ shown as a function of $q$ for $E_-=1.008mc^2$ and for three different photon energies $k$.

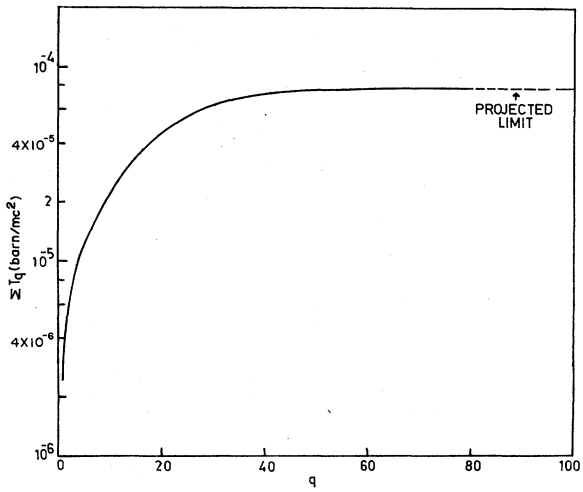


FIG. 2. The convergence of $d\sigma/dE_+=\sum_{q=1}T_q$, the differential cross section for the pair production for photon energy $k=7.5$ MeV and electron energy $E_-=1.008mc^2$, shown as a function of $q=|\kappa_-|+|\kappa_+|-1$. The projected limit using the techniques of Ref. 15 is shown.

observed that Dugne's[14] results, interpolated to our calculated energy points, are in good agreement with our results. There are also available a number of expressions in the literature for the positron spectrum using the Sommerfeld-Maue wave functions, e.g., Davies *et al.*,[17] Fink and Pratt,[18] Roche *et al.*,[19] and Deck *et al.*[13] The Davies *et al.*[17] calculation is not valid in this energy range (as it gives a negative Coulomb correction to the Bethe-Heitler results). Borie[20] has done calculations using the methods of Fink and Pratt, and Roche *et al.* which give correct behavior of the Coulomb correction. But we have not used it to compare with our results as calculations are not available for $^{92}$U for the energy range of interest. We have also compared our results with the calculation using the Deck-Moroi-Alling (referred to hereafter as DMA) expression which has been derived using the modified Sommerfeld-Maue continuum lepton wave functions. In this approximation a correction term to the Bethe-Heitler formula of the order of $\alpha Z$ is obtained for the limit $p_-=0$. We have shown in Fig. 3 the DMA calculation for $^{92}$U [performed by using Eq. (37) of Ref. 13 and by avoiding the very high energy approximation]. It may be mentioned that our calculation for $E_-=1.008mc^2$ amounts to the parameter $[Y=(E_+-1)/(k-2)]$ $Y=0.9998\simeq 1$ for $k=50$ MeV whereas the DMA result is for $Y=1$.

We have attempted to fit the calculated points by fitting the following semiempirical curves which has an energy dependence like that of the DMA (see Eq. 37, Ref. 13):

$$\sigma_D=\sigma_0\left[1+\frac{a}{k}+\frac{b}{k^2}+\frac{c}{k^3}\right], \tag{14}$$

where

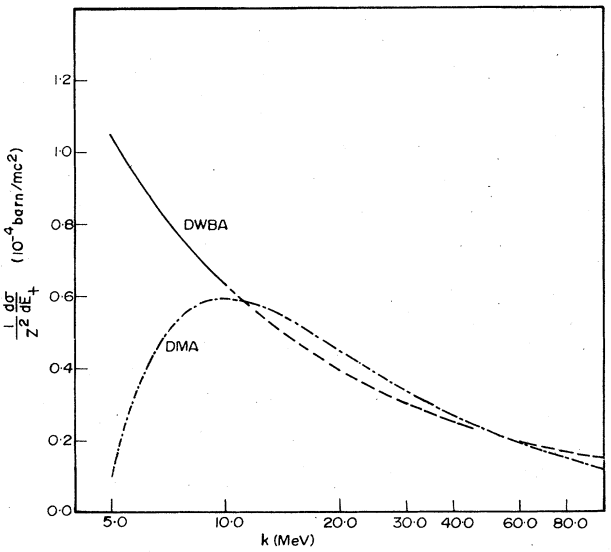


FIG. 3. $(1/Z^2)(d\sigma/dE_+)$ the differential cross section for pair production shown as a function of the photon energy $k$. DWBA results of our calculations is shown by the solid line. This calculation is in the tip region for electron energy $E_- = 1.008mc^2$. The calculation using the Deck-Moroi-Alling method (Ref. 13) is shown by the dotted dashed line, the dashed line is the extrapolated curve using the semiempirical fit of Eq. (14).

$$a = 8.8591 \times 10^1 ,$$

$$b = -2.8401 \times 10^2 ,$$

$$c = 6.6838 \times 10^2 ,$$

and

$$\sigma_0 = 4\pi Z r_0^2 \left[\frac{\alpha}{m}\right]^2 \frac{2\pi\alpha Z}{e^{2\pi\alpha Z}-1}\left(1-\tfrac{4}{15}\pi\alpha Z\right) f(Z) ,$$

where $Z$ is atomic number, $r_0$ is classical electron radius, $\alpha$ is fine-structure constant, and $m$ is electron mass. The $Z$-dependent function $f(Z)$ is given by[17]

$$f(Z) = (\alpha Z)^2\{[1+(\alpha Z)^2]^{-1} + 0.202 - 0.0369(\alpha Z)^2 + 0.0083(\alpha Z)^4 - 0.002(\alpha Z)^6\} .$$

The extrapolated curve [Eq. (14)] lies lower than the DMA results for energies between 12 and 50 MeV (see Fig. 3). The DMA results are expected to be correct for high-energy calculations, as Sommerfeld-Maue wave functions have validity in that region. We expect the exact calculation to be in between the extrapolated and the DMA curves and it should be a smooth curve joining the 10.0-MeV DWBA point with the 50.0-MeV DMA point. However, this can only be settled by extending the calculation to high-energy points.

## IV. CONCLUSIONS

We have calculated the positron spectrum in the tip region for photons of energies 5.0—10.0 MeV by using the new technique for calculating the accurate radial integral and its photon-energy dependence in approximately one-tenth of the computational time required for the existing DWBA calculations. The useful feature of Eq. (13) is that it has been derived for constant values of the parameters $y_\pm$, thus no approximations are involved in its derivation. Therefore we can use Eq. (13) to calculate accurate radial integrals even for photons of much higher energies. We would like to emphasize that the approximation involved in the differential equation (11) improves further with the increase of the photon energy. We would like to conclude, with the following remark: with our new and efficient technique one can extend the DWBA pair-production cross-section calculation into the intermediate-energy range.

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