Relativistic charged-particle interactions in a chaotic laser field

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Interactions of charged particles in the presence of a chaotic laser field are considered for the case where the quadratic A^2 term in the Volkov solution must be retained. This is necessary whenever the kinematics are relativistic or the masses of the particles involved change during the interaction, such as in a decay, even though the kinematics might be nonrelativistic. Spin terms of the Volkov solution are also considered. The problem reduces to the solution of a stationary Gaussian stochastic process. This allows for an explicit evaluation if the two-time correlation function of the laser field is exponential with an arbitrary correlation time, i.e., for an Ornstein-Uhlenbeck process. The evaluation is performed by two alternative methods, one relying on path integrals and the other one on functional methods. Various limiting cases are discussed, notably that of an infinite correlation time. In the latter case, a cross section in a chaotic field is obtained from that in a coherent field by integrating the latter over the intensity with an exponential weight function. This prescription was already known to hold in the nonrelativistic case. As an application, we discuss high-intensity Compton scattering. A specialization of the present results yields an ensemble average, which is needed in the evaluation of a recent $g-2$ experiment for the anomalous magnetic moment of the electron.

I. INTRODUCTION

There has been continuous interest over the last two decades in the influence of a very intense laser field on decays or scattering processes of elementary particles.[1] In all of these investigations the laser field has been represented by a classical coherent field with definite amplitude and phase. The wave functions of the charged particles were then given by Volkov solutions corresponding to this specific field. The classical description of the laser field is very well justified for the extremely high intensities which are necessary in order to yield sensible effects. However, high-power multimode lasers tend to produce chaotic rather than coherent fields. Laser photon statistics is known to play a vital role in laser-induced multiphoton processes in atomic and molecular physics,[2] where the rates for a nonresonant n-photon transition can differ by as much as a factor of $n!$ for a chaotic versus a coherent laser field. Formally, in the case of a chaotic field, its rapid amplitude and phase fluctuations must be accounted for by an appropriate ensemble average.[3] When this is carried out, the A^2 term in the particle-field interaction, which gives rise to an effective mass, is usually neglected. This is only justified if two assumptions are met. (1) The kinematics of all particles must be nonrelativistic. If the field is strong enough, the situation will always be relativistic, no matter how nonrelativistic it was in the absence of the field. (2) The process considered must be a pure scattering process, i.e., the masses of the particles in the initial and final state must be identical. Hence, for example, for a decay process the A^2 term is of vital importance even for a field which is sufficiently weak in order to allow for nonrelativistic kinematics. On the other hand, for electron scattering in the presence of a not too strong laser field[4] the A^2 term can be neglected; in fact, it cancels in the nonrelativistic limit.

In this paper we will carry out the above-mentioned ensemble average for the case of the fully relativistic Volkov solution, keeping track of the relativistic A^2 term. The laser field will be described by a circularly polarized plane-wave field that satisfies a stationary Gaussian stochastic process. The outline of this paper is as follows: In Sec. II we formulate the problem and set up the basic notation, in Sec. III we calculate the ensemble average of the decay rate or the cross section by means of functional methods.[5,6] This procedure yields immediately the structure of the result for an arbitrary field correlation function. In order to obtain an explicit answer the latter must

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be specified. We assume an Ornstein-Uhlenbeck process, i.e., the field correlation function decreases exponentially with the absolute time difference between the two fields on the scale of an arbitrary correlation time. In Sec. IV we rederive the same results by using a different method which has been applied previously[7] to the solution of a nonlinear Langevin equation with driving fields of the same type considered here. The method relies on a path-integral representation[8] of the coherent state propagator. Sec. V deals with various limits of the general result, which are of particular interest. (1) The limit of an infinite correlation time. In this case, a cross section can be represented as an integral with respect to intensity with a certain weight function over the cross section for a coherent field. This result has already been known to hold in the nonrelativistic case.[4] We have now proved that it applies as well to the fully relativistic situation, where the quadratic effective mass term and spin terms are essential. (2) The nonrelativistic limit when the quadratic A^2 term is neglected, both for a finite and an infinite correlation time. In this case we can make contact with previously published results.[4,9] In conclusion, we point out qualitatively the physical consequences of applying a chaotic instead of a coherent field to the processes of interest.

This paper is formulated within the physical context of charged particles interacting with each other in the presence of an intense chaotic laser field. However, the formal task to be solved, namely, to carry out the ensemble average of the exponential of a stochastic quadratic form, might show up in entirely different physical problems. An example is provided by the analysis of the $g-2$ experiment for the determination of the anomalous magnetic moment of the electron.[10] In this experiment the electron is subject to a large constant magnetic field plus a superimposed small one, which varies quadratically with the axial position. Hence, the cyclotron frequency includes a small term which depends quadratically on the axial position of the electron. The detailed analysis of the experiment requires that an ensemble average be carried out over the axial position of the electron, which fluctuates due to Brownian motion. The solution of this problem is a special case of the results presented in this paper; see also Refs. 7 and 11.

II. BASIC NOTATION

We assume that the laser field propagates in the positive z direction with circular polarization, so that it is represented by the vector potential

$$\vec{A}=(A_x(u),A_y(u),0)$$
$$=a(u)(\cos[\omega u+\phi(u)],-\sin[\omega u+\phi(u)],0) \quad (2.1)$$

with

$$u=t-z/c\ . \quad (2.2)$$

The amplitude $a(u)$ and phase $\phi(u)$ are stochastic quantities such that for

$$A_x(u)-iA_y(u)=A(u)e^{i\omega u}\ , \quad (2.3)$$

$$\langle A(u_1)\cdots A(u_n)A^*(u_1')\cdots A^*(u_m')\rangle$$
$$=\delta_{nm}\sum_P\prod_{i=1}^n\Delta(u_i,u'_{p_i})\ , \quad (2.4)$$

where we introduced the abbreviation

$$\Delta(u,u')=\langle A(u)A^*(u')\rangle\ . \quad (2.5)$$

For the time being we will not specify $\Delta(u,u')$ but only note that it satisfies

$$\Delta^*(u,u')=\Delta(u',u)\ . \quad (2.6)$$

The sum in Eq. (2.4) extends over all permutations P of the variables u_i'. Averages with an unequal number of A's and A^*'s vanish due to phase fluctuations. Equations (2.3)–(2.5) describe a complex stationary Gaussian stochastic process.

The Volkov solution for a Dirac particle with asymptotic four-momentum $p=(E/c,\vec{p})$ in the presence of the plane-wave field (2.1) is[12]

$$\psi_p(x)=e^{-ip\cdot x/\hbar}V_p(u)D_p(u)u_p\ , \quad (2.7)$$

where

$$V_p(u)=\exp\left[\frac{ie}{2p\cdot n\hbar c}\int^u d\tau[2c\vec{p}\cdot\vec{A}(\tau)-e\vec{A}^2(\tau)]\right]\ , \quad (2.8)$$

$$D_p(u)=1-\frac{eA\!\!\!/ n\!\!\!/}{2cp\cdot n}\ , \quad (2.9)$$

and u_p is a free Dirac spinor so that $(p\!\!\!/-mc)u_p=0$. The four-vector $n=(1,0,0,1)$ is the propagation vector of the field (2.1). For the Klein-Gordon wave function, the spinor part D_pu_p in Eq. (2.6) is missing.

We shall consider a situation where n particles with charges e_i and initial asymptotic momenta p_i $(i=1,\ldots,n)$ are due to a pointlike interaction, possibly mediated by a potential $V(\vec{x})$, scattered into $\bar{n}$ particles with charges $\bar{e}_j$ and final asymptotic momenta $\bar{p}_j$ $(j=1,\ldots,\bar{n})$ in the presence of a laser field as specified by Eq. (2.1). For $n\neq\bar{n}$ this includes particle creation and decay processes. The corresponding matrix element is

$$M\sim\int d^4x\exp\left[-i\left[\sum_i p_i-\sum_j\bar{p}_j\right]\cdot x/\hbar\right]$$
$$\times\prod_i V_{p_i}(u)\prod_j V^*_{\bar{p}_j}(u)\prod_j\bar{u}_{\bar{p}_j}\bar{D}_{\bar{p}_j}G\prod_i D_{p_i}u_{p_i}\ . \quad (2.10)$$

The matrix G couples the various Dirac spinors and includes the potential $V(x)$; it also depends on the potential $A(u)$, if Klein-Gordon particles take part in the process. The cross section or decay rate is proportional to $|M|^2$. Hence, in order to account for the field fluctuations, field averages like

$$\Big\langle[A(u)]^k[A^*(u)]^l[A(u')]^{k'}[A^*(u')]^{l'}$$
$$\times\prod_i V_{p_i}(u)V^*_{p_i}(u')\prod_j V^*_{\bar{p}_j}(u)V_{\bar{p}_j}(u')\Big\rangle \quad (2.11)$$

will have to be evaluated. The explicit powers of $A(u)$ and $A(u')$ in the quantity (2.11) originate from the Dirac spinors (2.9) and possibly from the coupling G. The process $\langle \ldots \rangle$ is defined by Eqs. (2.4) and (2.5).

Henceforth, we will be concerned with the ensemble average of the quantity

$$I(u,u')=\exp\left[i\int_{u'}^{u}d\tau[f(\tau)A(\tau)+f^*(\tau)A^*(\tau)+QA(\tau)A^*(\tau)]\right], \qquad (2.12)$$

i.e., $\langle I(u,u')\rangle$. We will leave the function $f(\tau)$ arbitrary, so that the powers of $A(u)$, $A^*(u)$, $A(u')$, and $A^*(u')$ in the expression (2.11) can be generated from Eq. (2.11) by functional differentiation with respect to $f(u)$, $f^*(u)$, $f(u')$, and $f^*(u')$, respectively. After all functional derivatives have been carried out we will have to make the substitution

$$f(\tau)\to P_+e^{i\omega\tau},\quad f^*(\tau)\to P_-e^{-i\omega\tau}, \qquad (2.13)$$

where

$$P_\pm=\frac{1}{2\hbar}\left[\sum_i\frac{(p_{i,1}\pm ip_{i,2})\cdot e_i}{p_i\cdot n}-\sum_j\frac{(\bar p_{j,1}\pm i\bar p_{j,2})\cdot\bar e_j}{\bar p_j\cdot n}\right], \qquad (2.14)$$

and

$$Q=-\frac{1}{2\hbar c}\left[\sum_i\frac{e_i^2}{p_i\cdot n}-\sum_j\frac{e_j^2}{\bar p_j\cdot n}\right]. \qquad (2.15)$$

We notice that for a scattering process ($n=\bar n$, $e_i=\bar e_i$, $m_i=\bar m_i$, where m_i and $\bar m_i$ denote the mass of the ith incident and scattered particle, respectively) we have in the limit of nonrelativistic kinematics $p_i\cdot n=\bar p_i\cdot n=m_ic$, so that $Q=0$. This holds no longer true in the relativistic case. Even for nonrelativistic kinematics the quantity Q is nonzero if we consider a decay process, e.g., neutron beta decay in the presence of a laser field. In this case, it is of vital importance,[13] and dropping it would lead to entirely misleading results such as the prediction of field-induced enhancements of the decay which are nonexistent.

The next two sections will deal with the evaluation of $\langle I(u,u')\rangle$ via functional or, alternatively, path-integral methods.

III. ENSEMBLE AVERAGE: FUNCTIONAL APPROACH

We will in what follows make use of a condensed operator notation, viz.,

$$fA=\int_{u'}^{u}d\tau f(\tau)A(\tau),$$
$$f^*\Delta f=\int_{u'}^{u}d\tau\, d\tau' f^*(\tau)\Delta(\tau,\tau')f(\tau'), \qquad (3.1)$$
$$f^*\Delta^2 f=\int_{u'}^{u}d\tau\, d\tau'\, d\tau'' f^*(\tau)\Delta(\tau,\tau')\Delta(\tau',\tau'')f(\tau''),$$

etc., so that the quantity to be evaluated reads

$$\langle I(u,u')\rangle=\langle\exp[i(fA+f^*A^*+QA^*A)]\rangle. \qquad (3.2)$$

If we rewrite this expression as

$$\langle I(u,u')\rangle=\exp(-iQ^{-1}f^*f)\times\langle\exp[iQ(A+f^*Q^{-1})(A^*+fQ^{-1})]\rangle, \qquad (3.3)$$

we can apply the functional shift operator,

$$\exp\left[h\frac{\delta}{\delta g}F[g]\right]=F[g+h], \qquad (3.4)$$

or explicitly

$$\exp\left[\int dx'h(x')\frac{\delta}{\delta g(x')}\right]F[g(x)]=F[g(x)+h(x)],$$

so as to obtain

$$\langle I(u,u')\rangle=\exp(-iQ^{-1}f^*f)\times\left\langle\exp\left[QA\frac{\delta}{\delta f^*}\right]\exp\left[QA^*\frac{\delta}{\delta f}\right]\right\rangle\times\exp(iQ^{-1}f^*f). \qquad (3.5)$$

The ensemble average can now be carried out easily: Because of the Gaussian character of the stochastic process considered here, as it is expressed in Eq. (2.4), we have

$$\langle e^{gA}e^{g^*A^*}\rangle=e^{g\Delta g^*}, \qquad (3.6)$$

where Δ is the correlation function of the process as defined in Eq. (2.5). Hence,

$$\langle I(u,u')\rangle=\exp(-iQ^{-1}f^*f)\exp\left[Q^2\frac{\delta}{\delta f^*}\Delta\frac{\delta}{\delta f}\right]\times\exp(iQ^{-1}f^*f). \qquad (3.7)$$

This can be further evaluated with the help of the functional formula

$$\exp\left[\frac{\delta}{\delta g}B\frac{\delta}{\delta g^*}\right]\exp(g^*Cg)=\exp[g^*C(1-BC)^{-1}g]\times\exp[-\mathrm{Tr}\ln(1-BC)]. \qquad (3.8)$$

Here B and C are arbitrary Hermitian operators [for the application to Eq. (3.7) C will be proportional to the unit operator], and the trace of an operator is defined by

$$\mathrm{Tr}\,B=\int_{u'}^{u}d\tau B(\tau,\tau).$$

For a proof of Eq. (3.8) see Appendix A. We then obtain

$$\langle I(u,u')\rangle=e^{-iQ^{-1}f^*f}e^{iQ^{-1}f^*(1-iQ\Delta)^{-1}f}e^{-\mathrm{Tr}\ln(1-iQ\Delta)}$$
$$=e^{-\mathrm{Tr}\ln(1-iQ\Delta)}e^{-f^*\Delta(1-iQ\Delta)^{-1}f}. \qquad (3.9)$$

Equation (3.9) is exact and applies for any Gaussian process with an arbitrary correlation function Δ. It must be emphasized, however, that the representation (3.9) is highly implicit since it contains functions of operators and inverses. In some special cases, which will be discussed in

Sec. V, e.g., for an infinite correlation time or for $Q=0$, it can be trivially evaluated. In general, without specification further evaluation of Eq. (3.9) is impossible.

Henceforth, we shall concentrate on the particular field correlation function

$$\Delta(\tau,\tau')=\frac{2\Gamma}{\tau_c}e^{-|\tau-\tau'|/\tau_c}=\Delta(\tau',\tau)\ , \tag{3.10}$$

which is real and symmetric. The quantity 2Γ measures the variance of the field fluctuations and τ_c denotes the (retarded) coherence time of the field. In this case, which should suffice for all applications, Eq. (3.9) can be explicitly evaluated. We shall make use of the notation $B(\tau,\tau')=\langle\tau|B|\tau'\rangle$ to denote the matrix elements of an operator B (this should not be confused with the ensemble average $\langle\ldots\rangle$). The quantity of interest is then

$$L(\tau,\tau')=\int_{u'}^{u}d\tau''\Delta(\tau,\tau'')\langle\tau''|(1-iQ\Delta)^{-1}|\tau'\rangle$$
$$=L(\tau',\tau)\ , \tag{3.11}$$

where the symmetry of L is a consequence of the symmetry of Δ. In Appendix B we show that

$$L(\tau,\tau')=\frac{2\Gamma\tilde{\tau}}{\tau_c^2}e^{-|\tau-\tau'|/\tilde{\tau}}+\frac{2\Gamma\tilde{\tau}}{\varphi\tau_c^2}\left[(C-1)e^{-(u-u')/\tilde{\tau}}\cosh[(\tau-\tau')/\tilde{\tau}]-\frac{2iQ\Gamma\tilde{\tau}}{\tau_c}\cosh[(u+u'-\tau-\tau')/\tilde{\tau}]\right]\ , \tag{3.12}$$

where

$$\varphi=\varphi(u,u')=\cosh[(u-u')/\tilde{\tau}]+C\sinh[(u-u')/\tilde{\tau}]\ , \tag{3.13}$$

$$\tilde{\tau}=(1-4iQ\Gamma)^{-1/2}\tau_c\ , \tag{3.14}$$

and

$$C=\tilde{\tau}(1-2iQ\Gamma)/\tau_c\ . \tag{3.15}$$

Furthermore,

$$\mathrm{Tr}\ln(1-iQ\Delta)=\ln\varphi-(u-u')/\tau_c\ . \tag{3.16}$$

As mentioned at the end of Sec. II, obtaining the averages of the type (2.11) still requires performing various functional differentiations on $\langle I(u,u')\rangle$. In order to given an example,

$$\left\langle A(u)A^*(u')\left[\prod_i V_{p_i}(u)V_{p_i}^*(u')\right]\left[\prod_j V_{\bar{p}_j}^*(u)V_{\bar{p}_j}(u')\right]\right\rangle$$
$$=\left[-i\frac{\delta}{\delta f(u)}\right]\left[-i\frac{\delta}{\delta f^*(u')}\right]\left\langle\prod_i\cdots\prod_j\cdots\right\rangle$$
$$=\left[L(u,u')-\int_{u'}^{u}d\tau'L(u',\tau')f(\tau')\int_{u'}^{u}d\tau f^*(\tau)L(\tau,u)\right]\langle I(u,u')\rangle\Bigg|_{f(\tau)=P_+e^{i\omega\tau},\ f^*(\tau)=P_-e^{-i\omega\tau}} \tag{3.17}$$

and in the end the functions f and f^* have to be replaced according to Eq. (2.13). In this process various integrals will be encountered, all of which can be carried out owing to the simple exponential form of $L(\tau,\tau')$ as given in Eq. (3.12) and of $f(\tau)$ as given in Eq. (2.13). We shall here be content with writing down the explicit form of $L(\tau,\tau')$ after the substitution (2.13) has been made:

$$\int_{u'}^{u}d\tau d\tau' f^*(\tau)L(\tau,\tau')f(\tau')\Bigg|_{f(\tau)=P_+e^{i\omega\tau},f^*(\tau)=P_-e^{i\omega\tau}}\equiv\bar{\Delta}(u,u')$$
$$=\frac{4\Gamma\tilde{\tau}^2}{\tau_c^2}\frac{u-u'}{1+\omega^2\tilde{\tau}^2}P_+P_-+\frac{4\Gamma\tilde{\tau}^3P_+P_-}{(1+\omega^2\tilde{\tau}^2)^2\tau_c^2\varphi}$$
$$\times\left[-\frac{\tilde{\tau}}{\tau_c}(1-\omega^2\tau_c^2)(\cosh[(u-u')/\tilde{\tau}]-\cos[\omega(u-u')])\right.$$
$$\left.-(1-\omega^2\tilde{\tau}^2)\sinh[(u-u')/\tilde{\tau}]-2\omega\tilde{\tau}\sin[\omega(u-u')]\right]\ . \tag{3.18}$$

Hence, after the substitution (2.13) has been made, the final answer is

$$\langle I(u,u')\rangle=\varphi^{-1}e^{(u-u')/\tau_c}e^{-\bar{\Delta}(u,u')}\ , \tag{3.19}$$

where we have compiled Eqs. (3.9), (3.11)–(3.13), and (3.16).

An even simpler, though less explicit derivation of the result (3.9) starts from the observation that the ensemble average is generated by

$$\langle I(u,u')\rangle = \exp\left[\frac{\delta}{\delta A}\Delta\frac{\delta}{\delta A^*}\right] I(u,u')\Big|_{A=A^*=0} , \tag{3.20}$$

where A and A^* are set equal to zero after all functional derivatives have been carried out. The functional operator

$$\exp[(\delta/\delta A^*)\Delta(\delta/\delta A)]$$

when applied to a function of A and A^* performs all possible two-time "contractions" as prescribed by Eq. (2.4). This is done by picking in all possible ways one $A(\tau)$ and one $A^*(\tau')$ and replacing them by $\Delta(\tau,\tau')$. The exponential prevents multiple counting of identical contributions. This is completely analogous to the functional form of Wick's theorem, see, e.g., Ref. 5. Since $I(u,u')$ is quadratic in A and A^*, Eq. (3.20) can be immediately evaluated with the help of a trivial generalization of Eq. (3.8), viz., Eq. (A8) of Appendix A. This gives again Eq. (3.9). The form (3.20) will turn out to be useful later.

IV. ENSEMBLE AVERAGE: PATH-INTEGRAL APPROACH

In this section we will give an alternative derivation of Eq. (3.18) using path integrals[8] instead of functional methods. The derivation will rely heavily on previously published material.[7] We will restrict ourselves to the case of the field correlation function (3.10). Also, for simplicity, we will let $u'=0$ and assume $u>0$. The results of the preceding section will then be recovered by replacing u by $u-u'$ in the end. We will also ignore spin terms in this section.

It follows from Eqs. (2.3) and (3.10) that the real and imaginary parts A_1 and A_2 of $A(u)$ are statistically independent and their correlation functions are given by $(i=1,2)$

$$\langle A_i(\tau)A_i(\tau')\rangle = \frac{\Gamma}{\tau_c}e^{-|\tau-\tau'|/\tau_c} = \tfrac{1}{2}\Delta(\tau,\tau') . \tag{4.1}$$

It can then be shown from Eq. (3.2) that

$$\langle I(u,0)\rangle = \langle I_1(u,0)\rangle\langle I_2(u,0)\rangle , \tag{4.2}$$

and $I_1(u,0)$ and $I_2(u,0)$ satisfy the following nonlinear Langevin equations ($i=1,2$; no summation over repeated indices):

$$\frac{d}{du}\langle I_i(u,0)\rangle = i[A_i(u)G_i(u)+A_i^2(u)H_i(u)] \times\langle I_i(u,0)\rangle , \tag{4.3}$$

where

$$G_1(u)=P_+e^{i\omega u}+P_-e^{-i\omega u} , \tag{4.4a}$$

$$G_2(u)=i(P_+e^{i\omega u}-P_-e^{-i\omega u}) , \tag{4.4b}$$

$$H_1(u)=H_2(u)=Q . \tag{4.4c}$$

Here we have substituted for $f(u)$ and $f^*(u)$ from Eq. (2.13).

An exact solution of the Langevin equations of the form (4.3) with the correlation functions of the Gaussian driving fields A_1 and A_2 given by Eq. (4.1) was given in Ref. 7 using a path-integral representation of the coherent state propagator:[8]

$$\langle I_i(u,0)\rangle = \exp\left[\int_0^u d\tau R_i(\tau)\right] , \tag{4.5}$$

where

$$R_i(u) = -2iH_i(u)X_i(u) - iH_i(u)Z_i^2(u) + G_i(u)Z_i(u) + i\frac{\Gamma}{\tau_c}H_i(u) \tag{4.6}$$

and the functions $X_i(u)$ and $Z_i(u)$ $(i=1,2)$ satisfy the following differential equations with the initial values $X_i(0)=Z_i(0)=0$:

$$\frac{dX_i}{du} = \frac{2}{\tau_c}[-1+2i\Gamma H_i(u)]X_i - 4iH_i(u)X_i^2 - \frac{i\Gamma^2}{\tau_c^2}H_i(u) , \tag{4.7a}$$

$$\frac{dZ_i}{du} = \left[-\frac{1}{\tau_c}+\frac{2i\Gamma}{\tau_c}H_i(u)-4iH_i(u)X_i\right]Z_i - \frac{\Gamma}{\tau_c}G_i(u) + 2G_i(u)X_i . \tag{4.7b}$$

The set of nonlinear differential equations (4.7a) and (4.7b) can be solved for the given values of $G_i(u)$ and $H_i(u)$ from Eqs. (4.4a)–(4.4c). On substituting these solutions in Eqs. (4.5) and (4.6) we obtain an exact solution for $\langle I_i(u,0)\rangle$ $(i=1,2)$. The solution for $\langle I(u,0)\rangle$ is then given by Eq. (4.2). The calculations are rather lengthy but straightforward. The details are given in Appendix C. The expression for $\langle I(u,0)\rangle$ is identical to that given by Eq. (3.19).

V. SPECIAL CASES

We will now consider various special cases of the general result of the preceding sections. If the A^2 term in the Volkov solution is neglected, i.e., for $Q=0$, we can make contact with previously published results.[4,9] In this case we have $\tilde{\tau}=\tau_c$, $C=1$, and the Tr ln of Eq. (3.9) vanishes. Hence, the general result reduces to

$$L(\tau,\tau') = \frac{2\Gamma}{\tau_c}e^{-|\tau-\tau'|/\tau_c} \tag{5.1}$$

and [cf. Eq. (3.18)]

$$\bar{\Delta}(u,u') = \frac{4\Gamma P_+P_-\tau_c}{(1+\omega^2\tau_c^2)^2}\left[\frac{1}{\tau_c}(1+\omega^2\tau_c^2)(u-u')+\omega^2\tau_c^2-1+e^{-(u-u')/\tau_c}\{(1-\omega^2\tau_c^2)\cos[\omega(u-u')]-2\omega\tau_c\sin[\omega(u-u')]\}\right] . \tag{5.2}$$

The additional limit of an infinite correlation time when $\tau_c \to \infty$, $\Gamma \to \infty$ such that

$$2\Gamma/\tau_c \to a^2 \tag{5.3}$$

is finite, leaves the simple result

$$\bar{\Delta}(u,u')=\frac{4a^2P_+P_-}{\omega^2}\sin^2[\omega(u-u')/2] \;. \tag{5.4}$$

In the presence of the quadratic term the limit (5.3) of an infinite correlation time is still fairly simple. It follows from the definitions (3.14) and (3.15) that in this limit $\bar{\tau}\to\infty$, $C\to\infty$, $\bar{\tau}/\tau_c\to 0$, whereas $C/\bar{\tau}\to -iQa$ and $c\bar{\tau}/\tau_c \to \frac{1}{2}$ are finite. As a consequence we have

$$L(\tau,\tau')=\frac{a^2}{1-iQa^2(u-u')} \;, \tag{5.5}$$

$$\bar{\Delta}(u,u')=\frac{4P_+P_-a^2}{\omega^2[1-iQa^2(u-u')]}\sin^2[\omega(u-u')/2] \;, \tag{5.6}$$

and

$$\mathrm{Tr}\ln(1-iQ\Delta)=\ln[1-iQa^2(u-u')] \;. \tag{5.7}$$

We notice in passing that Eqs. (5.5)–(5.7) can be obtained immediately from Eq. (3.9) since for an infinite correlation time the correlation function is just a constant, viz., $\Delta(\tau,\tau')=a^2$, so that all the integrations implicit in Eq. (3.9) are trivial.

Another case, where we can relate our present results to independently obtained ones, is when the ensemble average does not contain the linear terms, i.e., $P_+=P_-=0$. The general result (3.19) then reduces to

$$\langle I(u,u')\rangle=\exp[(u-u')/\tau_c]\varphi^{-1} \;, \tag{5.8}$$

where φ is defined in Eq. (3.13). This expression is needed for the improved analysis of the $g-2$ experiment for the anomalous magnetic moment of the electron,[10] i.e., in the context of a physical problem completely different from that one considered in this paper. Equation (5.8) leads to the expression that is given in Ref. 10, if our constants τ_c and $\bar{\tau}$ are identified with γ^{-1} and γ'^{-1}, respectively, of Ref. 10.

In what follows we shall be concerned with the limit of the infinite correlation time. In this case there is a simple recipe for how to obtain cross sections and decay rates if they are known for a coherent field, viz.,

$$d\sigma_{\mathrm{inc}}(a)=\int_0^\infty d\left[\left[\frac{\bar{a}}{a}\right]^2\right]e^{-(\bar{a}/a)^2}d\sigma_{\mathrm{coh}}(\bar{a}) \;. \tag{5.9}$$

Here $d\sigma_{\mathrm{coh}}(\bar{a})$ denotes a differential transition rate in the presence of a coherent field (2.1) with $a(u)=\bar{a}$ and a constant phase $\phi(u)=\bar{\phi}$ [$d\sigma_{\mathrm{coh}}(\bar{a})$ is independent of $\bar{\phi}$], and $d\sigma_{\mathrm{inc}}(a)$ is the corresponding transition rate in the presence of a fluctuating field with infinite correlation time, where $\Delta(\tau,\tau')=2\Gamma/\tau_c=a^2$. The simple relation (5.9) between the two quantities is known to hold[4] for a nonrelativistic situation where the quadratic term QA^*A in the Volkov solution is absent. We shall now prove that it still holds in the presence of the quadratic term.

The differential transition rate for the chaotic field is

$$d\sigma_{\mathrm{inc}}(a)=\int d^4x\,d^4x' \times\exp\left[-i\left[\sum_i p_i-\sum_j \bar{p}_j\right](x-x')/\hbar\right] \times\langle I(u,u')\rangle\prod_i dp_i\prod_j d\bar{p}_j \;. \tag{5.10}$$

It is convenient to transform

$$d^4x\,d^4x'=du\,du'\,dv\,dv'\,d^2x_i\,d^2x_i' =d(u-u')d[\tfrac{1}{2}(u+u')]dv\,dv'\,d^2x_i\,d^2x_i' \;, \tag{5.11}$$

where $u=t-z/c$, $v=ct+z$, $x_i=(x,y)$, and analogously for the primed variables. The expression for $d\sigma_{\mathrm{coh}}(\bar{a})$ differs from Eq. (5.10) in that $\langle I(u,u')\rangle$ is replaced by $\bar{I}(u,u')$ as given in Eq. (2.12) with fixed values $a(u)=\bar{a}$, $\phi(u)=\bar{\phi}$ for amplitude and phase of the field. Carrying out the integral in Eq. (2.12) with (2.13) and expanding in terms of Bessel functions yields

$$\bar{I}(u,u')=\sum_{k,l}i^kJ_k\left[\frac{2\bar{a}}{\omega}(P_++P_-)\sin\left[\frac{\omega}{2}(u-u')\right]\right] \times J_l\left[\frac{2i\bar{a}}{\omega}(P_+-P_-)\sin\left[\frac{\omega}{2}(u-u')\right]\right] \times e^{i(k-l)[\omega(u+u'/2+\bar{\phi}]}e^{iQa^2(u-u')} \;. \tag{5.12}$$

If this is integrated over $(u+u')/2$, the terms with $k\neq l$ vanish and the remaining sum can be evaluated so that

$$\int d[\tfrac{1}{2}(u+u')]\bar{I}(u,u') =T_u e^{iQ\bar{a}^2(u-u')} \times J_0\left[\frac{4\bar{a}}{\omega}\sin\left[\frac{\omega}{2}(u-u')\right]\sqrt{P_+P_-}\right] \;, \tag{5.13}$$

where

$$T_u=\int d[\tfrac{1}{2}(u+u')]$$

is an (infinite) constant, which cancels, when a transition rate per unit time is calculated. With the help of the integral[14]

$$\int_0^\infty dx\,x^{\nu+1}e^{-\alpha x^2}J_\nu(\beta x)=\beta^\nu(2\alpha)^{-\nu-1}e^{-\beta^2/4\alpha} \tag{5.14}$$

we obtain

$$\int_0^\infty d\left[\frac{\bar{a}^2}{a^2}\right]e^{-(\bar{a}/a)^2}\int d\left[\frac{u+u'}{2}\right]\bar{I}(u,u')=T_u[1-iQa^2(u-u')]^{-1}\exp\left[-\frac{4a^2P_+P_-\sin^2[\frac{1}{2}\omega(u-u')]}{\omega^2[1-iQa^2(u-u')]}\right] =\int d\left[\frac{u+u'}{2}\right]\langle I(u,u')\rangle \;. \tag{5.15}$$

In the last line of Eq. (5.15) we have written down the relation to the case of the chaotic field, using Eqs. (3.19), (5.6), and (5.7). This establishes the validity of Eq. (5.9).

This derivation ignored possible explicit powers of A and A^* as exhibited in Eq. (2.11) which will invariably be present whenever particles with spin take part in the process, due to the spin terms (2.9) in the Volkov solution. The preceding proof can be generalized so as to include this case. In Appendix D, we shall give a slightly different argument based on Eq. (3.20). The conclusion will be that Eq. (5.9) still applies in the presence of spin terms. Hence, whenever the field coherence time is long compared with all other characteristic times in a particular problem, Eq. (5.9) can be used to infer the transition rate in the presence of a chaotic field from the corresponding one in the presence of a coherent field.

VI. CONCLUSIONS

The main results of this paper are the following. We have obtained all ingredients which are necessary in order to recalculate cross sections and decay rates of charged particles in the presence of an intense circularly polarized plane wave which is chaotic rather than coherent. Presently available high-energy lasers produce chaotic fields to a good approximation. Our calculation is fully relativistic, i.e., it incorporates the quadratic effective mass term in the Volkov solution as well as spin terms. Both effects are vital in a relativistic situation. We assumed a field correlation function which decreases exponentially in time on the scale of an arbitrary coherence time. The results are summarized in Eqs. (3.9), (3.12)–(3.16), and (3.18).

For most practical applications to cross sections and decays of elementary particles and nuclei, the coherence time of the field will be long in comparison with all other characteristic times. In this case the limit of an infinite correlation time can be invoked which simplifies things considerably: the transition rate in the presence of the chaotic field with average intensity I can then be obtained from the rate in a coherent field with intensity $\bar{I}$ by integrating the latter over the intensity with a weight function $\exp(\bar{I}/I)$, viz., Eq. (5.9). This result was known for the nonrelativistic case:[4] here we have extended it to the relativistic case where both the fluctuations of the effective mass and spin terms become important.

For the case where the limit of the infinite correlation time applies we can draw the following conclusions regarding the difference in the effects caused by a chaotic versus a coherent field. (1) It is known that total cross sections and decay rates of charged particles in the presence of a coherent field are unaffected by the field as long as its field strength is, loosely speaking, small compared with the (quantum-mechanical) critical field strength $E_c=m^2c^3/(e\hbar)$.[15] Consequently, the same holds true in the presence of a chaotic field, as long as its *average* intensity is small compared with the critical intensity. (2) Differential transition rates are significantly distorted even by fields which are well below the critical field. This effect will be noticeably enhanced by a chaotic field, since its actual intensity can be much larger than its average intensity. (3) When, for a coherent field, its field strength E approaches the critical field, total rates are modified (most often enhanced) by intensity-dependent contributions which normally are proportional to $(E/E_c)^2$.[1] Again, in view of Eq. (D8), there is no difference between the chaotic and the coherent field. (4) If, however, the intensity-dependent contribution starts with $(E/E_c)^{2n}$ and $n>1$ or when, for $E\sim E_c$, higher-order contributions modify the leading $(E/E_c)^2$ behavior, a glance at Eq. (D8) makes clear that the notorious factors of $n!$, well known from multiphoton ionization, show up. (5) Spectacular enhancements can only be expected when the leading term is proportional to $(E/E_c)^{2n}$ with $n\gg 1$. This would be the case, e.g., in nuclear beta decay for a nucleus where the decay is energetically forbidden in the absence of the field but becomes energetically possible after absorption of n photons from the laser field. The decay rate will then be enhanced by the factor of $n!$. Because of the factor of $(E/E_c)^{2n}$, however, fields very close to the critical one are required. Also, if just the minimum number of photons which is necessary in order to render the decay energetically possible, is absorbed from the field, the available phase space is extremely small.

Finally, we will briefly discuss high-intensity Compton scattering[16] for a chaotic incident field. For a coherent incident field, the cross section is a sum over partial cross sections for emission into (approximately) the stimulating frequency and harmonics thereof. If $\bar{a}$ denotes the amplitude of the vector potential of the stimulating field and ω its frequency, the emitted frequencies are ($n\geq 1$)

$$\omega_n=n\omega\left[1+\left[\bar{v}^2+2n\frac{\hbar\omega}{mc^2}\right]\sin^2(\vartheta/2)\right]^{-1}, \qquad (6.1)$$

where $\bar{v}=e\bar{a}/mc^2$ and ϑ is the angle between the incident and the emitted photons. The partial cross section for emission of the nth harmonic is for $\bar{v}^2\ll 1$ proportional to

$$J_n^2(2n\bar{v}\sin(\vartheta/2)\cos(\vartheta/2)[1+\bar{v}^2\sin^2(\vartheta/2)]^{-1}) . \qquad (6.2)$$

The so-called intensity-dependent frequency shift, i.e., the term proportional to $\bar{v}^2$ in Eq. (6.1), can be traced back to the quadratic term Q in the Volkov solution.

In order to obtain the corresponding results for the chaotic field with infinite coherence time the integration indicated in Eq. (5.9) need to be carried out. Because of the ubiquitous dependence of $d\sigma_{\text{coh}}$ on $\bar{v}^2$ this cannot be done in closed form. Two features, however, will certainly evolve: (1) since $J_n^2(z)$ is proportional, z^{2n} its leading term will pick up the factor of $n!$, so that higher harmonic emission will be enhanced; and (2) in the coherent case, the frequencies ω_n emitted at a fixed angle ϑ are sharp. Because of their intensity dependence, this will no longer be true in the chaotic case. Indeed, for ever more increasing intensity of the stimulating field, the line shapes of the individual harmonics will for $\vartheta\neq 0$ become broader and broader until they will finally overlap so that a separation becomes impossible.

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APPENDIX A: PROOF OF EQ. (3.8)

For the reader's convenience we shall here reproduce the standard proof[5,6] of Eq. (3.8). Consider the quantity

$$F(\lambda)=\exp\left[\lambda\frac{\delta}{\delta g}B\frac{\delta}{\delta g^*}\right]\exp(g^*Cg)\ , \tag{A1}$$

so that

$$F'(\lambda)=\frac{\delta}{\delta g}B\frac{\delta}{\delta g^*}F(\lambda)\ . \tag{A2}$$

We make the ansatz

$$F(\lambda)=e^{g^*\chi(\lambda)g}e^{\psi(\lambda)}\ , \tag{A3}$$

where $\psi(\lambda)$ and $\chi(\lambda)$ are independent of g. Taking the derivative with respect to λ directly of the ansatz (A3) and comparing it with the same quantity calculated from Eq. (A2) we obtain the two differential equations

$$\chi'(\lambda)=\chi(\lambda)B\chi(\lambda) \tag{A4}$$

and

$$\psi'(\lambda)=\mathrm{Tr}\,[B\chi(\lambda)]\ . \tag{A5}$$

A comparison of Eq. (A3) with (A1) yields the initial conditions $\chi(0)=C$ and $\psi(0)=0$. Hence, Eq. (A4) is solved by

$$\chi(\lambda)=C(1-\lambda BC)^{-1}=(1-\lambda CB)^{-1}C\ , \tag{A6}$$

and consequently

$$\psi(\lambda)=-\mathrm{Tr}\ln(1-\lambda BC)\ . \tag{A7}$$

If we insert this into Eq. (A3) and let λ be equal to one, we recover Eq. (3.8). A trivial generalization of Eq. (3.8) which is needed for the evaluation of Eq. (3.20) is obtained by letting $g\to g+C^{-1}h$, where we now assume that C is Hermitian, viz., $C^*(\tau,\tau')=C(\tau',\tau)$. It reads

$$\exp\left[\frac{\delta}{\delta g}B\frac{\delta}{\delta g^*}\right]\exp(g^*Cg+h^*g+g^*h)=\exp[g^*C(1-BC)^{-1}g]\exp[h^*(1-BC)^{-1}Bh]$$
$$\times\exp[h^*(1-BC)^{-1}g+g^*(1-CB)^{-1}h]$$
$$\times\exp[-\mathrm{Tr}\ln(1-BC)]\ . \tag{A8}$$

APPENDIX B: EXPLICIT EVALUATION OF $\langle I(u,u')\rangle$ FOR THE FIELD CORRELATION FUNCTION (3.10)

We start by noticing that the derivatives of

$$\Delta(\tau,\tau')=\frac{2\Gamma}{\tau_c}e^{-|\tau-\tau'|/\tau_c} \tag{B1}$$

are

$$\frac{\partial\Delta(\tau,\tau')}{\partial\tau}=-\frac{1}{\tau_c}\epsilon(\tau-\tau')\Delta(\tau,\tau') \tag{B2}$$

and

$$\frac{\partial^2\Delta(\tau,\tau')}{\partial\tau^2}=-\frac{4\Gamma}{\tau_c^2}\delta(\tau-\tau')+\frac{1}{\tau_c^2}\Delta(\tau,\tau')\ , \tag{B3}$$

where the distribution $\epsilon(\tau)$ is defined by $\epsilon(\tau)=\pm1$ for $\tau\gtrless0$ so that $d\epsilon/d\tau=2\delta(\tau)$. Consequently, we have

$$\frac{\partial^2L(\tau,\tau')}{\partial\tau^2}=\frac{1}{\tau_c^2}L(\tau,\tau')-\frac{4\Gamma}{\tau_c^2}\langle\tau|(1-iQ\Delta)^{-1}|\tau'\rangle\ .$$

With the help of the relation

$$\frac{1}{1-iQ\Delta}=1+iQ\frac{\Delta}{1-iQ\Delta} \tag{B4}$$

we obtain the differential equation

$$\frac{\partial^2L}{\partial\tau^2}-\frac{1}{\tilde{\tau}^2}L=-\frac{4\Gamma}{\tau_c^2}\delta(\tau-\tau')\ , \tag{B5}$$

where

$$\tilde{\tau}=(1-4iQ\Gamma)^{-1/2}\tau_c\ . \tag{B6}$$

The general solution of Eq. (B5) is

$$L(\tau,\tau')=\frac{2\Gamma\tilde{\tau}}{\tau_c^2}e^{-|\tau-\tau'|/\tilde{\tau}}$$
$$+\gamma_1\sinh(\tau/\tilde{\tau})+\gamma_2\cosh(\tau/\tilde{\tau})\ , \tag{B7}$$

where the constants γ_i may depend on τ', u, and u'. Incorporating the symmetry of L we may write

$$L(\tau,\tau')=\frac{2\Gamma\tilde{\tau}}{\tau_c^2}e^{-|\tau-\tau'|/\tilde{\tau}}+\alpha\sinh(\tau/\tilde{\tau})\sinh(\tau'/\tilde{\tau})$$
$$+\beta\cosh(\tau/\tilde{\tau})\cosh(\tau/\tilde{\tau})$$
$$+\gamma\sinh[(\tau+\tau')/\tilde{\tau}]\ , \tag{B8}$$

where the constants α, β, and γ still depend on u and u'. In order to determine them we need initial conditions.

For this purpose we will consider the functions

$$M(u)=L(u,u),\quad \bar{M}(u')=L(u',u'),\quad N(u)=L(u,u')\ . \tag{B9}$$

Note that $L(\tau,\tau')$ depends implicitly on u and u' via the limits of the integrations. Hence, contrary to appearance, $M(u)$ also depends on u', and $\bar{M}(u')$ also depends on u. In order to derive a differential equation for M it is convenient to rewrite it with the help of Eq. (B4) as

$$\begin{aligned}M(u)&=\Delta(u,u)+iQ\langle u\,|\,\Delta(1-iQ\Delta)^{-1}\Delta\,|\,u\rangle\\&=\Delta(u,u)+iQ\int d\tau d\tau'\Delta(u,\tau)\langle\tau\,|\,(1-iQ\Delta)^{-1}\,|\,\tau'\rangle\\&\qquad\times\Delta(\tau',u)\ .\end{aligned} \tag{B10}$$

Because of Eq. (B1),

$$\Delta(u,u)=2\Gamma/\tau_c \tag{B11}$$

is independent of u. In differentiating Eq. (B10) with respect to u we encounter two types of contributions. Differentiating $\Delta(u,\tau)$ just provides a factor of $-1/\tau_c$ [see Eq. (B2) and notice that $u>\tau$ for all τ in the integration interval]. Apart from the explicit dependence on u, $M(u)$ depends on u via the limits of the integrations in the definition of the operator products. By means of a power series expansion it can easily be established that

$$\frac{\partial}{\partial u}\Delta(1-iQ\Delta)^{-1}\Delta=\frac{\Delta}{1-iQ\Delta}\,|\,u\rangle\langle u\,|\,\frac{\Delta}{1-iQ\Delta}\ . \tag{B12}$$

Hence, we obtain the differential equation

$$\frac{dM}{du}=-\frac{2}{\tau_c}\left[M-\frac{2\Gamma}{\tau_c}\right]+iQM^2 \tag{B13}$$

with the initial condition

$$M(u)\Big|_{u=u'}=2\Gamma/\tau_c\ . \tag{B14}$$

Similarly we find

$$\frac{d\bar{M}}{du'}=\frac{2}{\tau_c}\left[\bar{M}-\frac{2\Gamma}{\tau_c}\right]-iQ\bar{M}^2 \tag{B15}$$

and

$$\frac{dN}{du}=-\frac{1}{\tau_c}N+iQMN \tag{B16}$$

with the initial conditions

$$\bar{M}(u')\Big|_{u'=u}=N(u)\Big|_{u=u'}=2\Gamma/\tau_c\ . \tag{B17}$$

The solution of Eq. (B13) with (B14) is

$$M(u)=\frac{2\Gamma\tilde{\tau}}{\tau_c^2}\,\frac{\sinh[(u-u')/\tilde{\tau}]+(\tau_c/\tilde{\tau})\cosh[(u-u')/\tilde{\tau}]}{\cosh[(u-u')/\tilde{\tau}]+C\sinh[(u-u')/\tilde{\tau}]}\ , \tag{B18}$$

where $\tilde{\tau}$ is defined in Eq. (B6) and

$$C=\tilde{\tau}(1-2iQ\Gamma)/\tau_c\ . \tag{B19}$$

It is immediately obvious that the solution of Eq. (B15) with (B17) is

$$\bar{M}(u')=M(u)\ . \tag{B20}$$

Finally, if we note that the function $M(u)$ can be written as a total derivative, viz.,

$$M(u)=\frac{i}{q}\left[\frac{d}{du}\ln\{\cosh[(u-u')/\tilde{\tau}]+C\sinh[(u-u')/\tilde{\tau}]\}-\frac{1}{\tau_c}\right]\ , \tag{B21}$$

we can integrate Eq. (B16) with (B17). The result is

$$N=\frac{2\Gamma}{\tau_c}\,\frac{1}{\cosh[(u-u')/\tilde{\tau}]+C\sinh[(u-u')/\tilde{\tau}]}\ . \tag{B22}$$

With explicit expressions for the three functions (B9) now at hand we can determine the three constants α, β, and γ in Eq. (B8). A rather tedious but straightforward calculation then yields Eq. (3.12).

In order to complete evaluation of $I(u,u')$ [Eq. (3.9)] the function $\mathrm{Tr}\ln(1-iQ\Delta)$ still has to be computed. We expand the logarithm into a power series and take advantage of the cyclic property of the trace, writing

$$\begin{aligned}\frac{\partial}{\partial u}\mathrm{Tr}\ln(1-iQ\Delta)&=\frac{\partial}{\partial u}\mathrm{Tr}[-iQ\Delta-\tfrac{1}{2}(iQ)^2\Delta^2-\tfrac{1}{3}(iQ)^3\Delta^3-\cdots]\\&=\mathrm{Tr}[-iQ\Delta\,|\,u\rangle\langle u\,|\,-(iQ)^2\Delta^2\,|\,u\rangle\langle u\,|\,-(iQ)^3\Delta^3\,|\,u\rangle\langle u\,|\,-\cdots]\\&=-iQ\langle u\,|\,\Delta(1-iQ\Delta)^{-1}\,|\,u\rangle=-iQM(u)\ .\end{aligned} \tag{B23}$$

Hence, in view of Eq. (B21), we obtain the result (3.16).

APPENDIX C: DETAILS OF THE PATH-INTEGRAL APPROACH

In the present problem [cf. Eqs. (4.4a)—(4.4c)], Eqs. (4.7a) and (4.7b) reduce to ($i=1,2$)

$$\frac{dX_i}{du}=\frac{2}{\tau_c}(-1+2iQ\Gamma)X_i-4iQX_i^2-i\frac{\Gamma^2}{\tau_c^2}Q \ , \tag{C1}$$

$$\frac{dZ_i}{du}=\left[-\frac{1}{\tau_c}+\frac{2iQ\Gamma}{\tau_c}-4iQX_i\right]Z_i-G_i(u)\left[\frac{\Gamma}{\tau_c}-2X_i\right] . \tag{C2}$$

The solutions of Eqs. (C1) and (C2), subject to the initial conditions $X_i(0)=Z_i(0)=0$, are

$$X_1(u)=X_2(u)=\frac{1}{4iQ}\frac{\dot{\varphi}}{\varphi}-\frac{C}{4iQ\tilde{\tau}} \ , \tag{C3}$$

$$Z_i(u)=-\frac{\tilde{\tau}^2}{2iQ\varphi(1+\omega^2\tilde{\tau}^2)}\left[\dot{\varphi}\frac{G_i}{\tau_c}-\frac{\dot{G}_i\varphi}{\tau_c}-\frac{G_i\varphi}{\tilde{\tau}^2}+\ddot{G}_i\varphi+\frac{2iQ\Gamma}{\tau_c}\left[\dot{G}_i(0)-\frac{G_i(0)}{\tau_c}\right]\right] , \tag{C4}$$

where $\varphi(u,0)$ and $\tilde{\tau}$ are given by Eqs. (3.13), (3.14), and (3.15), respectively. Substituting Eqs. (C3) and (C4) into Eq. (4.6) we obtain

$$\begin{aligned}R(u)&=R_1+R_2\\&=-2iQ(X_1+X_2)-iQ(Z_1^2+Z_2^2)+G_1Z_1+G_2Z_2+\frac{2iQ\Gamma}{\tau_c}\\&=-\frac{d}{du}\ln\varphi(u,0)+\frac{4P_+P_-Q\Gamma\tilde{\tau}^2}{\tau_c^2(1+\omega^2\tilde{\tau}^2)}\left[\frac{\tilde{\tau}^2(1-\omega^2\tau_c^2)}{\tau_c(1+\omega^2\tilde{\tau}^2)}\frac{d}{du}\left[\frac{\cos(\omega u)}{\varphi(u,0)}\right]+2\omega\frac{d}{du}\left[\frac{\sin(\omega u)}{\varphi(u,0)}\right]+1\right.\\&\qquad\left.+\frac{2iQ\Gamma\tilde{\tau}^2(1+\omega^2\tau_c^2)}{\varphi^2(u,0)\tau_c^2(1+\omega^2\tilde{\tau}^2)}\right]+\frac{1}{\tau_c} .\end{aligned} \tag{C5}$$

Finally, in order to evaluate

$$\langle I(u,0)\rangle=\exp\int_0^u d\tau[R_1(\tau)+R_2(\tau)] \tag{C6}$$

we need to integrate $R(t)$. This can be done in a straightforward way. The resulting solution agrees with Eq. (3.19).

APPENDIX D: PROOF OF EQ. (5.12) FOR THE CASE WHERE SPIN TERMS ARE PRESENT

In the most general case we will have to evaluate the average of quantities like (2.11), i.e.,

$$G_{kl,k'l'}=[A(u)]^k[A^*(u)]^l[A(u')]^{k'}[A^*(u')]^{l'}I(u,u')\exp\{i\omega[(k-l)u+(k'-l')u']\} \ , \tag{D1}$$

where we have included the appropriate phases. If we notice that Eq. (3.20) does not just apply to $I(u,u')$ but to any function of A and A^* we may write

$$\langle G_{kl,k'l'}\rangle=\exp\frac{\delta}{\delta A}\Delta\frac{\delta}{\delta A^*}G_{kl,k'l'}\bigg|_{A=A^*=0} . \tag{D2}$$

The functional $I(u,u')$ can be expanded into a power series so that

$$\begin{aligned}G_{kl,k'l'}=&[A(u)]^k[A^*(u)]^l[A(u')]^{k'}[A^*(u')]^{l'}\exp[i\omega(k-l)u+(k'-l')u')]\\&\times\sum_{n,n'}\int d\tau_1\cdots d\tau_n\,d\tau_1'\cdots d\tau_{n'}'g(\tau_1,\ldots,\tau_n;\tau_1',\ldots,\tau_{n'}')\\&\times A(\tau_1)\cdots A(\tau_n)A(\tau_1')\cdots A(\tau_{n'}') .\end{aligned} \tag{D3}$$

Since spin terms are explicitly taken care of we are free to replace the functions f and f^* which are inherent in $I(u,u')$ according to (2.13). As a consequence the frequency dependence of the function g in Eq. (D3) is given by

$$g(\tau_1,\ldots,\tau_{n'}')=\exp[i\omega(\tau_1+\cdots+\tau_n-\tau_1'-\cdots-\tau_{n'}')]\tilde{g}(\tau_1,\ldots,\tau_{n'}') . \tag{D4}$$

Since owing to Eq. (D2) we shall finally let $A=A^*=0$, only those terms in Eq. (D3) with

$$k+k'+n=l+l'+n' \tag{D5}$$

will contribute. Because of the infinite correlation time we have

$$\frac{\delta}{\delta A}\Delta\frac{\delta}{\delta A^*}=a^2\int d\tau\frac{\delta}{\delta A(\tau)}\int d\tau'\frac{\delta}{\delta A^*(\tau')} . \tag{D6}$$

If we expand the exponential in Eq. (D2),

$$\exp\left[\frac{\delta}{\delta A}\Delta\frac{\delta}{\delta A^*}\right]=\sum_\nu\frac{1}{\nu!}\left[\frac{\delta}{\delta A}\Delta\frac{\delta}{\delta A^*}\right]^\nu ,$$

and apply it to Eq. (D3) letting finally $A=A^*=0$, only the term with $\nu=k+k'+n$ will contribute. Without loss of generality the function $g(\tau_1,\ldots,\tau'_{n'})$ can be symmetrized with respect to permutations of the variables τ_i and $\tau'_{i'}$ among each other. Then, the net effect of carrying out the functional derivatives is the following: in view of Eq. (D6) the derivatives $(\delta/\delta A)^\nu$ and $(\delta/\delta A^*)^\nu$ introduce a factor of $\nu!$ each, and all A's and A^*'s are replaced by a. Hence,

$$\langle G_{kl,k'l'}\rangle=\sum_\nu \nu! a^{2\nu}\int_{u'}^{u} d\tau_1\cdots d\tau_{\nu-k-k'}d\tau'_1\cdots d\tau'_{\nu-l-l'}\exp[i\omega(\tau_1+\cdots+\tau_{\nu-k-k'}-\tau'_1-\cdots-\tau'_{\nu-l-l'})]$$
$$\times\exp\{i\omega[(k-l)u+(k'-l')u']\}\tilde{g}(\tau_1\ldots,\tau'_{\nu-l-l'}) . \tag{D7}$$

The function $\tilde{g}$ depends on τ_i and $\tau'_{i'}$ only in as much as it may contain products of δ functions $\delta(\tau_i-\tau'_{i'})$, each of which leads to the cancellation of the corresponding exponentials in Eq. (D7) and in turn to a factor of $u-u'$ after integration over τ_i and $\tau'_{i'}$. It can then be readily shown that $\langle G_{kl,k'l'}\rangle$ depends only on the difference $u-u'$.

Now, if we repeat the same procedure for a coherent field, i.e., choose a particular field (2.1) with a definite amplitude $a(u)=\bar{a}$ and phase $\Phi(u)=\bar{\Phi}$, and calculate $\int d(u+u')/2G_{kl,k',l'}$ this will reproduce Eq. (D7) with the only difference that a is replaced by $\bar{a}$ and the factor of $\nu!$ is missing. Since

$$\int_0^\infty d\left[\frac{\bar{a}}{a}\right]^2 e^{-(\bar{a}/a)^2}\bar{a}^{2\nu}=\nu! a^{2\nu} , \tag{D8}$$

we have again the relation

$$\int d\left[\frac{\bar{a}}{a}\right]^2 e^{-(\bar{a}/a)^2}\int d\left[\frac{u+u'}{2}\right]G_{kl,k'l'}(\bar{a})=\int d\left[\frac{u+u'}{2}\right]\langle G_{kl,k'l'}\rangle \tag{D9}$$

analogous to Eq. (5.15) but now shown to hold for arbitrary spin terms. This finishes the proof of the general validity of Eq. (5.9). The preceding argument also should have made clear that a relation like Eq. (5.9) only holds in the case of an infinite correlation time when $\Delta(\tau,\tau')$ is independent of τ and τ'.

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