Nonlinear electrostatic ion-cyclotron waves

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Nonlinear coupling of a coherent electrostatic ion-cyclotron (EIC) wave with finite amplitude compressional Alfvén modulations can give rise to localized EIC wave packets. Criteria for the wave localization are obtained analytically.

Large-amplitude coherent (over many wave periods) electrostatic ion-cyclotron (EIC) waves, propagating almost perpendicular to an external magnetic field, are found to play an important role with regard to particle acceleration and plasma heating in the auroral region of the ionosphere[1] as well as in the solar wind.[2] In particular, nonlinear EIC waves may be responsible for the large-scale potential drop in the ionosphere.[1] One of the important nonlinear effects associated with the EIC waves could be the modulational instability whose possible final state could be envelope solitons.[3] A small amplitude calculation for the latter was presented in Ref. 3 by including the linear response of the magnetohydrodynamic (MHD) perturbations to the EIC waves. However, the observed waves can by no means be considered as small amplitude.

The objective of this report is to generalize the result of Ref. 3 by incorporating fully nonlinear compressional Alfvén-wave response to the EIC waves. Criteria for the existence of finite amplitude EIC wave solitons are given analytically.

Consider the propagation of the EIC wave in the x-z plane. The external magnetic field $\vec{B}_0$ is along the z axis. Since the parallel (to $B_0\hat{z}$) phase velocity of the EIC wave is much smaller than the electron thermal velocity, the inertialess electrons maintain an equilibrium along the external magnetic field. The electron number density is thus given by

$$n_e = n_0 \exp(e\phi/T_e) \ , \tag{1}$$

where T_e is the electron temperature, n_0 is the average plasma density, and ϕ is the ambipolar potential of the EIC wave.

The ion number density in the EIC wave is determined with the help of the continuity equation, namely,

$$\partial_t \ln n_i + \vec{\nabla}_\perp \cdot \vec{v}_{i\perp} + \partial_z v_{iz} + n_0^{-1}\vec{\nabla}\cdot(n^s \vec{v}_i) = 0 \ , \tag{2}$$

where n^s is the ion number density of the compressional Alfvén wave. The perpendicular and the parallel (to $\vec{B}_0$) components of the ion fluid velocities are given by

$$(\partial_t^2+\Omega^2)\vec{v}_{i\perp} = -\frac{e}{m_i}\partial_t\vec{\nabla}_\perp\phi - \frac{v_{ti}^2}{n_0}\partial_t\vec{\nabla}_\perp n_i + \Omega\left(\frac{e}{m_i}\vec{\nabla}_\perp\phi + \frac{v_{ti}^2}{n_0}\vec{\nabla}_\perp n_i\right)\times\hat{z} \ , \tag{3}$$

$$\partial_t v_{iz} = -\frac{e}{m_i}\partial_z\phi - \frac{v_{ti}^2}{n_0}\partial_z n_i \ , \tag{4}$$

where $\Omega = eB_z/m_i c$ is the ion gyrofrequency, B_z is the total magnetic field in the z direction, $v_{ti} = (\gamma_i T_i/m_i)^{1/2}$ is the ion thermal velocity, and m_i is the ion mass. In obtaining Eqs. (3) and (4) we have noted that the compressional Alfvén waves do not have sheared magnetic fields.

Combining Eqs. (1)–(4), and making use of the quasineutrality condition, we obtain

$$\partial_t^2[\partial_t^2+\Omega_i^2-c_a^2\nabla^2-\Omega_i^2(1-B_z^2/B_0^2)]\Phi - c_a^2\Omega^2\partial_z^2\Phi - c_a^2\vec{\nabla}\cdot\left(\frac{n^s}{n_0}\partial_t^2\vec{\nabla}\Phi\right) - c_a^2\Omega^2\partial_z\left(\frac{n_s}{n_0}\partial_z\Phi\right) = 0 \ , \tag{5}$$

where

$$\Phi = e\phi/T_e, \quad \Omega_i = eB_0/m_i c, \quad c_a^2 = c_s^2 + v_{ti}^2, \quad c_s^2 = T_e/m_i \ ,$$

and $\nabla^2 = \nabla_\perp^2 + \partial_z^2$. In Eq. (5) the second-harmonic interactions are excluded.

In the absence of nonlinear interaction, linearization of Eq. (5) yields the dispersion relation describing the coupling of the EIC waves with the ion-acoustic waves in magnetized plasmas. We have

$$\omega^2(\omega^2-\omega_f^2) + k_z^2 c_a^2 \Omega_i^2 = 0 \ , \tag{6}$$

where ω is the wave frequency, $\omega_f^2 = \Omega_i^2 + k^2 c_a^2$, and $k^2 = k_x^2 + k_z^2$. Here, k_x and k_z are the wave vectors perpendicular and parallel to $B_0\hat{z}$, respectively. For $\omega \sim \Omega_i$ and $k_z^2 << k_x^2$, one encounters the EIC modes propagating almost perpendicular to the external magnetic field. In what follows, we are interested in nonlinear coupling of the EIC waves with the finite amplitude compressional Alfvén waves which satisfy $n^s = B_z$. Assuming that the amplitude of the EIC wave varies on the time and space scales of the compressional Alfvén waves, we may employ the usual WKB approximation[3–5] for the EIC wave potential

$$\Phi = \Phi(\vec{x},\tau)\exp(-i\omega_0 t + i\vec{k}_0\cdot\vec{r}) \ , \tag{7}$$

where we shall let $\omega_0^{-1}\partial_\tau << 1$ and $k_0^{-1}\partial_x << 1$.

Substitution of Eq. (7) into Eq. (5) gives an evolution equation

$$[2i\omega_0\partial_\tau + 2ik_0c_a^2\partial_x + c_a^2\partial_x^2 + \Omega_i^2(1-B^2)]\Phi = 0 \ , \tag{8}$$

where

$$B = \frac{B_z}{B_0}, \quad k_0^2\rho_s^2 < 1, \quad \rho_s^2\partial_x^2 << 1, \quad \rho_s = \frac{c_a}{\Omega_i}, \quad \partial_x >> \partial_z \ .$$

Furthermore, in deriving Eq. (8) we have also used the linear dispersion relation $[\omega_0=\omega_l(k_0)]$ of the EIC wave. The last term on the left-hand side of Eq. (8) is the leading order nonlinear term emerging from the large variation of the compressional Alfvén-wave magnetic field in the presence of the EIC waves. Note that the other nonlinear terms on the second line of Eq. (5) are smaller than the one retained in Eq. (8). The reason is that the wavelength of the compressional Alfvén waves is assumed to be longer $(\rho_s^2\partial_x^2 << 1)$ than the EIC waves $(k_0^2\rho_s^2 < 1)$.

The dynamics of fully nonlinear compressional Alfvén waves (across $\vec{B}_0$) is governed by the MHD equations[4]

$$\partial_\tau n + \partial_x(nv) = 0 \ , \tag{9}$$

$$\partial_\tau v + v\partial_x v + c_s^2\partial_x \ln n + v_A^2 n^{-1}B\partial_x B = -(c_s^2/2k_0^2\rho_s^2)\partial_x|\Phi|^2 \ , \tag{10}$$

$$\partial_\tau B + \partial_x(vb) = 0 \ , \tag{11}$$

where $\partial_\tau << \Omega_i$, $n_e^s = n^s$, $n = n^s/n_0$, and $v_A^2 = B_0^2/4\pi n_0 m_i$. The right-hand side of Eq. (10) is the ponderomotive force[3] term which originates from the averaging (over the periods of the EIC waves) of the ion convection term. We have noted that the introduction of a small but finite k_z of the compressional Alfvén waves would not alter our principal findings.

It is convenient to normalize the variables in Eqs. (8)–(11). Thus, the time and space scales are written in the units of ρ_s/v_A, and ρ_s; whereas the ion fluid velocity v and the ambipolar potential Φ are normalized by v_A and $c_s^2/2v_A^2k_0^2\rho_s^2$, respectively. We look for the stationary solutions in the form

$$\Phi = \Phi(x - M\tau)\exp[i\theta(\tau) + i\varphi(x)] \ , \tag{12}$$

$$B = B(x - M\tau), \quad n = n(x - M\tau), \quad v = v(x - M\tau) \ , \tag{13}$$

where $M = V/v_A$ is the Alfvénic Mach number.

On substituting Eq. (12) into Eq. (8), one finds

$$\frac{d^2\Phi}{dx^2} + (1 - A)\Phi - B^2\Phi = 0 \ , \tag{14}$$

where $A = 2\epsilon\partial_\tau\theta + 2k_0\rho_s\partial_x\varphi + (\partial_x\varphi)^2$ is the nonlinear frequency shift, to be determined later. Here,

$$\epsilon = v_A\omega_0/\Omega_i c_a \text{ and } \varphi(x) = 2(\epsilon M - k_0\rho_s)x \ .$$

On the other hand, inserting Eq. (13) into Eqs. (9)–(11), one gets

$$n = \frac{M}{M - v} \ , \tag{15}$$

$$\Phi^2 = 1 - n - \beta\ln n - \tfrac{1}{2}(v - M)^2 + \frac{M^2}{2} \ , \tag{16}$$

$$B = \frac{M}{M - v} \equiv n \ , \tag{17}$$

where $\beta = c_s^2/v_A^2$, and the plasma is assumed to be unperturbed ($B = 1$, $n = 1$, $\Phi = 0$, and $v = 0$) at $|x| = \infty$. For localized perturbations, we take $\Phi = \Phi_m(>0)$, $n = N$ at $x = 0$. Thus, combinations of Eqs. (15) and (16) give a nonlinear dispersion relation

$$M^2 = \frac{2N^2(\Phi_m^2 + N - 1 + \beta\ln N)}{N^2 - 1} \ . \tag{18}$$

A close inspection of Eq. (18) reveals that sub-Alfvénic solitary EIC waves consist of large amplitude density and magnetic field depressions (associated with nonlinear compressional Alfvén waves) together with localized EIC fields in a low-β plasma.

Multiplying Eq. (14) by $d\Phi/dx$, using Eqs. (15)–(17), integrating once, and using the appropriate boundary conditions ($n = 1$, $\Phi = 0$, and $dn/dx = 0$, and $d\Phi/dx = 0$) at $|x| = \infty$, we readily obtain the energy integral

$$\left(\frac{dn}{dx}\right)^2 + \Psi = 0 \ , \tag{19}$$

where the potential energy is given by

$$\Psi(n) = 4\Phi^2(n)Q^{-2}[(1 - A)\Phi^2(n) + \tfrac{1}{3}(n^3 - 1) + \frac{\beta}{2}(n^2 - 1) - M^2\ln n] \ , \tag{20}$$

with $Q = 1 + \beta n^{-1} - M^2 n^{-3}$, and

$$\Phi^2(n) = 1 - n - \beta\ln n + M^2(1 - n^{-2})/2 \ .$$

Since at the center $dn/dx = 0$, we find from $\Psi(N) = 0$ the nonlinear frequency shift

$$A = 1 - [2(N^3 - 1) + 3\beta(N^2 - 1) - 6M^2\ln N]/6\Phi^2(N) \ . \tag{21}$$

Localized solutions are ensured[5] provided that $\Psi < 0$ between the two points (say, $n = 1$ and $n = N$). Thus, near $n \approx N$, one demands $d\Psi/dn|_{n=N} > 0$, yielding:

$$A > 1 + N^2(M^2 - \beta N^2 - N^3)(N^3 + \beta N^2 - M^2)^{-1} \ . \tag{22}$$

On the other hand, a Taylor expansion of Eq. (20) near $n \approx 1$ yields $\Psi \approx -4A(n-1)^2$. It follows that $\Psi < 0$ for $A > 0$. The small amplitude result is easily obtained by retaining the terms up to order $(n-1)^3$ in the potential energy Ψ.

In summary, we have shown that finite amplitude solitary EIC waves can exist provided that the conditions (18), (21), (22), and $A > 0$ are satisfied. Since we have treated fully nonlinear compressional Alfvén-wave response to the EIC waves, our results are valid beyond the usual description of small amplitude solitary waves which are described by the cubic Schrödinger equation.[3,6] Strongly nonlinear EIC waves could have important implications in relation to auroral particle acceleration.[7] However, the particle dynamics in the solitary EIC waves have to be investigated in order to demonstrate the applicability of our work to space plasmas. This is a separate issue and is beyond the scope of this Brief Report.

In closing, we would like to point out that the present investigation can readily be generalized to include more general boundary conditions leading to periodic wave trains. However, it is expected that the basic relations between the Mach number, the EIC wave amplitude, and the corresponding density and magnetic field changes should still prevail.

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