Similarity solution of the evolution equation describing the combined effects of diffusion and recombination in plasmas

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The evolution equation accounting for effects of diffusion and recombination is transformed into a similarity equation in one, two, and three dimensions. This equation does not fulfil the Painlevé criterion and thus cannot be directly classified, or its properties analyzed, in general terms of its singularities. A particular similarity solution can, however, be constructed. It allows for a description of the evolution of plasma configurations of physical significance.

In laboratories, as well as in space, plasmas often appear in partially ionized forms. This is also true for the phase of formation of plasma in various contexts, e.g., in fusion plasma devices. In such plasmas the processes of diffusion and recombination may, simultaneously, play an important role. As a result of the nonlinear nature of the recombination process the combined effects of diffusion (which can also be nonlinear) and recombination are generally difficult to describe analytically and most efforts in this connection are numerical. Recently, some new analytical results concerning particular solutions of physical significance were obtained.[1–4] It is the purpose of this Brief Report to look for similarity solutions describing plasmas where diffusion and recombination occur simultaneously.

Consider the equation

$$\frac{\partial n}{\partial t}=D\nabla^2 n-\alpha n^2 \ , \tag{1}$$

which governs the evolution of, e.g., an electron plasma density, and where D and α denote the diffusion and recombination coefficients, respectively (D and α are here considered constants).

It is convenient to renormalize space and time; accordingly

$$(\alpha/D)^{1/2}x\rightarrow x;\ \alpha t\rightarrow t \ .$$

Then Eq. (1) becomes

$$\frac{\partial n}{\partial t}=\nabla^2 n-n^2 \ . \tag{2}$$

For symmetric situations the diffusion operator is

$$\nabla^2=\frac{1}{x^\gamma}\frac{\partial}{\partial x}\left(x^\gamma\frac{\partial}{\partial x}\right) \ , \tag{3}$$

where $\gamma=0,1,2$ corresponds to the dimension d ($d=\gamma+1$).

Looking for the simplest form of similarity solution we write

$$n(x,t)=t^\mu\phi(\xi) \ , \tag{4}$$

$$\xi=x/t^\nu \ , \tag{5}$$

where μ and ν are constants to be determined and ξ the similarity variable.

Inserting relations (4) and (5) into Eqs. (2) and (3) and matching powers of t yields

$$\mu=-1,\ \nu=\tfrac{1}{2} \tag{6}$$

and, accordingly, the similarity solution (4) of Eq. (2) is of the form

$$n_s(x,t)=t^{-1}\phi(x/t^{1/2}) \ , \tag{7}$$

where $\phi(\xi)$ satisfies the ordinary differential equation

$$\frac{d^2\phi}{d\xi^2}+\left(\frac{\gamma}{\xi}+\frac{\xi}{2}\right)\frac{d\phi}{d\xi}+\phi-\phi^2=0 \ . \tag{8}$$

The time translational invariance of Eqs. (1) and (2) implies that, given a solution of the form (7), we can generate a class of solutions of the form

$$n(x,t)=n_s(x,t+t_0)=(t+t_0)^{-1}\phi(x/(t+t_0)^{1/2}) \ , \tag{9}$$

with t_0 arbitrary.

In spite of its apparent simplicity, Eq. (8) is by no means a trivial one. Although it has, as shown below, a rather simple particular solution, the problem of finding its general solution remains very difficult because of its nonlinearity. A first step in analyzing the true complexity of this equation is to check whether or not it possesses the "Painlevé property" (absence of movable singularities other than simple poles), and whether or not it can thus be reduced to one of the fifty standard canonical forms of the theory.

Following the Ince notations [cf. Ref. 5, pp. 326–330] it is clear that Eq. (8) belongs to the so-called case (i), and has the form of the equation (G) with the special values $A=C=F=0$, $D=1$, $E=-1$, $B(\xi)=-(\gamma/\xi+\xi/2)$ for the ξ dependent coefficients. Now, by performing a standard transformation,[5] Eq. (8) can be put in the reduced form $V''=6V^2+S(Z)$, where $S(Z)$ has to be linear in Z for the Painlevé case. It is then only a matter of straightforward calculations to show that, for the present values of the coefficients, one cannot have $S(Z)$ linear in Z and, thus, that the Eq. (8) is not of the Painlevé type.

Such a result is to be considered from a general point of view of the links between the integrability of nonlinear partial differential equations and the singularities of their solutions as discussed in early works by Kowalevskaya,[6] Painlevé,[7] Gambier,[8] and more specifically, in connection with the recent conjecture of Ablowitz, Ramani, and Segur[9] for systems integrable by the inverse scattering transform (IST). If this conjecture were accepted (viz., a partial differential equation is "integrable" when all its reductions to ordinary differential equations by similarity transforms are of the Painlevé type), the result would be that the nonlinear rate equation (1) is not integrable in the sense of the IST method. Another way to deal with such a connection consists in using the "extended" definition of the Painlevé property for partial differential equations recently proposed by Weiss, Tabor, and Carnevale.[10] By applying their expansions near singularity manifolds and performing a leading order analysis it can indeed be shown that a certain compatibility condition resulting from recursion equations is not satisfied in the case of the Eq. (1). It thus follows that Eq. (1) does not possess this "extended" Painlevé property, and that it does not seem possible to define, in this way, a transform of the IST type, as it is for the standard Burgers, Korteweg–de Vries, Boussinesq, etc., equations.

In spite of the difficulties encountered when considering the general properties of Eq. (8) the particular solution of Eq. (1) obtained by Wilhelmsson,[1] which includes one, two, and three dimensions and also allows for intermediate cases of various asymmetries, may be used here to construct a similarity solution of Eq. (1). Guided by the Wilhelmsson solution we here look for particular similarity solutions of the form

$$\phi=\frac{a}{p+\xi^2}+\frac{b}{(p+\xi^2)^2}\equiv aN^{-1}+bN^{-2} \ , \tag{10}$$

where

$$N=p+\xi^2 \ . \tag{11}$$

If we insert this trial solution [Eqs. (10) and (11)] into Eq. (9) and equate successive powers of N^{-1} to zero we obtain three equations to determine the three parameters (a,b,p).

A solution where $b=0$ implies $p=0$, $a=6-2\gamma$, and corresponds to $\phi=(6-2\gamma)/\xi^2$ and a stationary solution for $n(x,t)=(6-2\gamma)/x^2$.

Consider now the more interesting case where $b\neq 0$, for which we obtain

$$a=12(4+\sqrt{6+2\gamma}) \ , \tag{12}$$

$$b=-24(30+2\gamma+10\sqrt{6+2\gamma}) \ , \tag{13}$$

$$p=30+2\gamma+10\sqrt{6+2\gamma} \tag{14}$$

[a solution with a minus sign in front of the square roots has been omitted since it does not correspond to realistic physical solutions of $n(x,t)$].

The values here determined [(12)–(14)] for the parameters (a,b,p) fully agree with the solution found in Ref. 1 and accordingly that solution corresponds to a similarity solution of Eq. (9). The solution is of a particular kind but of physical significance.

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