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Nonlinear propagation of electromagnetic waves in magnetized plasmas

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The nonlinear propagation of an electromagnetic wave along an external magnetic field in a uniform plasma is reconsidered in order to include the combined effects of the relativistic-mass and ponderomotive-force nonlinearities. The results of previous works on this subject are improved significantly.

In the past, there have been several attempts[1–3] to describe the nonlinear propagation of electromagnetic waves in magnetized plasmas. Hasegawa[2] introduced the reductive perturbation method in order to derive a pair of nonlinear equations governing the coupling of an electron cyclotron wave with magnetohydrodynamic oscillations. Further work pointing out the existence of a new time-dependent ponderomotive force,[4] which is of significant importance for the dynamical study of self-modulation of electromagnetic waves in magnetoplasmas, have later emerged.[5]

Recently, powerful gyrotrons and laser beams have been employed for heating of fusion plasmas as well as for accelerating electrons to very high energies.[6] However, when the intensity of the electromagnetic wave is very large, nonlinear effects cannot be ignored. Berezhiani and Tskhakaya[7] thus investigated the combined effects of the relativistic-electron-mass and ponderomotive-force nonlinearities on the propagation of finite amplitude electron cyclotron waves. For wave frequencies close to the electron gyrofrequency, nonlinearities associated with the relativistic mass variation[8–11] were then found to be of fundamental importance. Unfortunately, however, Ref. 7 has omitted the contribution of the parallel electron flow[5] in the derivation of the nonlinear frequency shift which is caused by the nonlinear interaction of the cyclotron waves with the background plasma. In the present Brief Report, we will include this free streaming term and in addition generalize the results of Ref. 7 to arbitrary electromagnetic wave frequencies. The role of the nonlinear ion motion as well as the effects of charge separation are also taken into account.

Let us consider a right-hand circularly polarized electromagnetic wave

$$\vec{E} = E(\hat{x} + i\hat{y})\exp(ikz - i\omega t) + \text{c.c.} ,$$

which propagates along a constant magnetic field $B_0\hat{z}$ in a plasma where any equilibrium drift velocity is zero. The frequency ω is supposed to be much higher than the lower-hybrid frequency and it is also assumed that $|\omega-\omega_{ce}|/k$ exceeds the thermal velocity of the electrons. We have here denoted the electron gyrofrequency $|q_e|B_0/m_e$ by ω_{ce}, where q_e is the electron charge, $m_e = m_{e0}(1-v^2/c^2)^{-1/2}$ is the relativistic electron mass, and $v^2 = q_e^2|E|^2/m_e^2(\omega-\omega_{ce})^2$.

It is then well known from linear theory that the frequency ω and wave number k are related by

$$\omega^2 = k^2c^2 + \omega\omega_{pe}^2/(\omega-\omega_{ce}) , \quad (1)$$

where $\omega_{pe} = (n_0q_e^2/\epsilon_0 m_e)^{1/2}$ is the electron plasma frequency.

The wave amplitude E will, however, vary slowly in space and time due to the nonlinear interaction with low-frequency electrostatic field-aligned perturbations. This process can be described by the equation[5]

$$i(\partial_t + v_g\partial_z)E + \tfrac{1}{2}v_g'\partial_z^2E - \Delta E = 0 , \quad (2a)$$

where $v_g = \partial\omega/\partial k$ stands for the group velocity and $v_g' = \partial v_g/\partial k$ represents the group dispersion of the wave. The frequency shift Δ discussed for the nonrelativistic case by Karpman and Washimi[5] can easily be generalized by including weak relativistic effects. One then finds

$$\Delta = \frac{\omega\omega_p^2}{2(\omega-\omega_c)}\frac{v_g}{kc^2}\left[N + \frac{kv_{ez}\omega_c}{\omega(\omega-\omega_c)} - \frac{\omega}{c^2}\frac{q_e^2}{m_{e0}^2}\frac{|E|^2}{(\omega-\omega_c)^3}\right] , \quad (2b)$$

where $\omega_p^2 = n_0q_e^2/\epsilon_0 m_{e0}$, $\omega_c = |q_e|B_0/m_{e0}$; $N \equiv \delta n_e/n_0$ and v_{ez} are the relative electron density and field aligned velocity perturbations associated with the low-frequency motion. Those quantities are related to each other by the electron continuity equation:

$$\partial_t N + \partial_z v_{ez} = 0 . \quad (3)$$

The low-frequency (phase velocity much smaller than the electron thermal velocity) electrostatic modulations must, of course, also satisfy the electron momentum equation:

$$(1+N)^{-1}\partial_z N = -\frac{q_e}{T_e}\partial_z\phi - \frac{\omega_p^2}{\omega(\omega-\omega_c)}\left[\partial_z - \frac{k\omega_c}{\omega(\omega-\omega_c)}\partial_t\right]W , \quad (4)$$

where ϕ is the ambipolar potential, T_e the electron temperature, and $W = \epsilon_0|E|^2/n_0T_e$. The last term in Eq. (4) is the ponderomotive-force contribution.[4,5] The ion motion is described by the ion continuity and momentum equations

$$\partial_t\delta n_i + \partial_z[(n_0+\delta n_i)v_{iz}] = 0 , \quad (5a)$$

and

$$\partial_t v_{iz} + v_{iz}\partial_z v_{iz} = -\frac{q_i}{m_i}\partial_z\phi . \quad (5b)$$

On using the Poisson equation

$$\partial_z^2\phi = -(q_e\delta n_e + q_i\delta n_i)/\epsilon_0 \quad (6)$$

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and eliminating δn_i, v_{iz}, and ϕ from the system of Eqs. (4)–(6), we then obtain our low-frequency equation:

$$\partial_t^2 N - c_s^2 \partial_z^2 (N + N^2 + \lambda_D^2 \partial_z^2 N) = \frac{c_s^2 \omega_p^2}{\omega(\omega-\omega_c)} \left[\partial_z^2 - \frac{k\omega_c}{\omega(\omega-\omega_c)} \partial_{tz} \right] W \ , \quad (7)$$

where $c_s = (T_e/m_i)^{1/2}$ is the ion sound speed and λ_D the electron Debye length.

We thus have derived a system of three coupled equations, namely, (2), (3), and (7), which describe the nonlinear coupling of electron cyclotron waves with field aligned electrostatic density perturbations. In what follows, we shall present some conditons under which wave localization is possible.

Let us now look for solutions in a coordinate system moving with a velocity[12] V, where V is sufficiently far away from c_s, and introduce the coordinate $\xi \equiv z - Vt$. In Eq. (7) we then replace ∂_t by $-V\partial_\xi$ and neglect the N^2 and $\lambda_D^2 \partial_\xi^2 N$ terms. By means of Eqs. (3) and (7), we can then write Eq. (2b) as $\Delta = -Q|E|^2$, where Q is defined as

$$Q \equiv \frac{1}{2} \frac{q_e^2}{m_{e0}^2} \frac{v_g}{kc^4} \frac{\omega_p^2}{(\omega-\omega_c)^2} \times \left[\frac{\omega^2}{(\omega-\omega_c)^2} - \frac{m_{e0}}{m_i} \frac{c^2}{(V^2 - c_s^2)} \left(1 + \frac{kV\omega_c}{\omega(\omega-\omega_c)} \right)^2 \right] , \quad (8)$$

and then write Eq. (2a) in the form

$$i[\partial_t + (v_g - V)\partial_\xi]E + \tfrac{1}{2} v_g' \partial_\xi^2 E + Q|E|^2 E = 0 \ . \quad (9)$$

Equation (9), together with (8), stands for the main result of the present paper.

Berezhiani and Tskhakaya[7] have previously derived an equation for the amplitude of a transverse electron cyclotron wave propagating along an external magnetic field. In order to compare our Eq. (9) with their Eq. (8), we put $V = v_g >> c_s$ and $\omega - \omega_c \approx -\omega_p^2 \omega_c / k^2 c^2$. Our expression for Q then reduces to

$$Q_c = \frac{1}{2} \frac{q_e^2}{m_{e0}^2} \frac{v_g k^7 c^4}{\omega_c^2 \omega_p^6} \left(1 - \frac{m_{e0}}{4m_i} \frac{k^2 c^2}{\omega_c^2} \right) \ . \quad (10)$$

Inspecting the factor $1 - (m_{e0}/4m_i) k^2 c^2 / \omega_c^2$ in (10), we note the interesting fact that the nonlinearity due to the relativistic-electron-mass variation is more important than the nonlinearity from the radiation pressure if $m_{e0}/4m_i < \omega_c^2 / k^2 c^2$. Furthermore, Q_c becomes negative if the inequality is reversed. When comparing with Ref. 7, we note that these authors have instead studied the factor $1 + (m_{e0}/8m_i) k^2 c^2 / \omega_c^2$ which is always positive. The reason[13] for this discrepancy is that in Ref. 7 the $kv_{ez}\omega_c / \omega(\omega-\omega_c)$ term in our Eq. (2b) is missing in their paper.

Finally, we present the stationary soliton solution of Eq. (9). It is given by

$$|E| = E_m \operatorname{sech}[E_m (Q/v_g')^{1/2} \xi] \ , \quad (11)$$

which is valid when Q and

$$v_g' = [1 - (v_g^2/c^2)(1 - \omega_p^2 \omega_c / (\omega-\omega_c)^3] v_g / k$$

have the same sign. Here, E_m is the soliton amplitude.

When V is close to c_s, all the nonlinear and dispersive terms in Eq. (7) must be kept. In this case, Eq. (7) reduces to

$$\partial_t N + c_s N \partial_\xi N + \frac{c_s}{2} \lambda_D^2 \partial_\xi^3 N = -\frac{c_s}{2\omega} \frac{\omega_p^2}{(\omega-\omega_c)} \left[1 + \frac{kc_s\omega_c}{\omega(\omega-\omega_c)} \right] \partial W / \partial \xi \ . \quad (12)$$

Equations (2a) and (12) are for this case the coupled set whose solution can be found by standard techniques.[3]

In addition, we ought to mention that in strongly magnetized tenuous cold plasmas, nonlinear processes may occur on the time scale of the electron plasma period. Neglecting ions motion as well as the electron thermal pressure term, one thus finds from (4) and (6) that the ambipolar potential is balanced by the radiation pressure, which means that

$$\Delta = \frac{q_e^2}{2m_{e0}^2 (\omega-\omega_c)^2} \frac{v_g}{kc^2} \times \left[\left(1 + \frac{kV\omega_c}{\omega(\omega-\omega_c)} \right)^2 \frac{\partial^2 |E|^2}{\partial \xi^2} - \frac{\omega^2 \omega_p^2 |E|^2}{c^2 (\omega-\omega_c)^2} \right] \ . \quad (13)$$

The solution of Eq. (2a) with (13) is well known.[14]

Summarizing, we have incorporated the contribution of parallel electron flow to the nonlinear frequency shift caused by the coupling of the electron cyclotron wave with the field-aligned electrostatic density modulations. The inclusion of this extra term has some important consequences. We have thus found that the coupling coefficient Q_c can change sign at k values equal to $(m_{e0}/4m_i)^{1/2} \omega_c / c$, which means that the cyclotron waves can be modulationally unstable. A possible final state of that instability could be a supersonic bright solitonlike structure. In the subsonic regime, the waves are modulationally stable and one may then encounter dark solitons.[15] Also considered is the case of transonic ($V \approx c_s$) propagation by including the ion nonlinearities and charge separation effects. We then found a coupled set of Schrödinger–Korteweg de Vries equations whose solution has been given elsewhere.[13]

It is, of course, straightforward to generalize the results above to a plasma with equilibrium drift motion and to include three-dimensional perturbations, but the algebra will then turn out to be considerably more complex.

The results presented in this Brief Report can have applications in many branches of physics. First, in space and astrophysical plasmas, the whistler modes are often of considerable interest. They are described by our formalism in the limit $\omega_c >> \omega$. Second, our nonlinear effects should also play an important role in studies of beat-wave particle accelerators[6] as well as for electron-cyclotron-resonance heating[7] of fusion plasmas.

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