Photon statistics and squeezing properties of a free-electron laser

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It is shown that the field emitted by a free-electron laser (FEL) in the low-gain regime exhibits antibunching over a wide range, for an initial Poisson distribution of probability amplitudes. We also investigate the variance of the momentum coordinate, and find the FEL state to be squeezed.

I. INTRODUCTION

Recently some interesting characteristics of free-electron lasers (FEL) such as photon antibunching[1,2] and squeezing[3] have been reported. These features can only be explained quantum mechanically. This has revived interest in quantum-mechanical treatments of FEL as the earlier explanations of FEL fail to describe them.

In this Brief Report, we investigate the photon statistics and squeezing properties of FEL, retaining the quantum nature of both the laser and wiggler fields, whereas in some earlier works,[1] the wiggler has been approximated to a classical field. We work in the low-gain, weak beam, single-particle approach, in the Bambini-Renieri frame.[4] We assume an initial Poisson distribution of probability amplitudes for the FEL state and study the time evolution of this state using perturbation theory. The FEL state evolves from the vacuum state by spontaneous emission buildup into the laser mode. During this buildup we investigate antibunching and squeezing under conditions discussed in Secs. III and IV.

II. FEL STATE AND ITS PERTURBATION

We start from a state with electron energy $p_0^2/(2m)$ and with laser and wiggler photon numbers n_L and n_W, respectively. The most general state at any time t (Ref. 5) can be given by the superposition

$$|\psi\rangle = \exp\left[i\left(\frac{p_0^2}{2m\hbar}+\omega n_L+\omega n_W\right)t\right] \times \sum_{l=n_L}^{-n_W} c_l|p_0+2l\hbar k, n_L-l, n_W+l\rangle \ . \tag{2.1}$$

Momentum and photon number conservation enable the state to be labeled by index l only. They are

$$p+\hbar k(n_L-n_W)=\text{const} \ , \tag{2.2a}$$

$$\Delta n_L = -\Delta n_W = -\frac{\Delta p}{2\hbar k} \ . \tag{2.2b}$$

In Eq. (2.1), c_l's are unknown coefficients. The equation of motion for the c_l's may be obtained from Schrödinger's equation,

$$H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle \ . \tag{2.3}$$

The FEL Hamiltonian in the Bambini-Renieri frame is

$$H=H_F+H_A+H_{\text{int}} \ , \tag{2.4}$$

where

$$H_F=\hbar\omega(a_L^\dagger a_L+a_W^\dagger a_W), \quad H_A=\frac{p^2}{2m} \ ,$$

and

$$H_{\text{int}}=\hbar\Lambda(a_L^\dagger a_W e^{-2ikz}+a_L a_W^\dagger e^{2ikz}) \ .$$

Here,

$$\Lambda = e^2/(2m\omega\epsilon_0 V_W)$$

is the coupling constant. Now

$$H_F|\psi\rangle = \hbar\omega(n_L+n_W)\exp\left[i\left(\frac{p_0^2}{2mk}+\omega n_L+\omega n_W\right)t\right]\sum_{L=n_L}^{-n_W} c_l|p_0+2l\hbar k, n_L-l, n_W+l\rangle \ , \tag{2.5a}$$

and

$$H_A|\psi\rangle = \sum_{l=n_L}^{-n_W}\frac{(p_0+2l\hbar k)^2}{2m}c_l|p_0+2l\hbar k, n_L-l, n_W+l\rangle \exp\left[i\left(\frac{p_0^2}{2m\hbar}+\omega n_L+\omega n_W\right)t\right] \ . \tag{2.5b}$$

$$H_{\text{int}}|\psi\rangle = \hbar\Lambda\exp\left[i\left(\frac{p_0^2}{2m\hbar}+\omega n_L+\omega n_W\right)t\right]\left[\exp(-2ikz)\sum_{l=n_L}^{-n_W} c_l|p_0+2l\hbar k, n_L-l+1, \ n_W+l-1\rangle[(n_L-l+1)(n_W+l)]^{1/2}\right.$$
$$\left. +\exp(2ikz)\sum_{l=n_L}^{-n_W} c_l|p_0+2l\hbar k, \ n_L-l-1, n_W+l+1\rangle[(n_L-l)(n_W+l+1)]^{1/2}\right] \ , \tag{2.5c}$$

and

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = i\hbar\left[i\left(\frac{p_0^2}{2m\hbar}+\omega n_L+\omega n_W\right)\sum_{l=n_L}^{-n_W} c_l|p_0+2l\hbar k, n_L-l, n_W+l\rangle\right.$$

$$\left.+\sum_{l=n_L}^{-n_W}\dot{c}_l|p_0+2l\hbar k, n_L-l, n_W+l\rangle\right]\exp\left[i\left(\frac{p_0^2}{2m\hbar}+\omega n_L+\omega n_W\right)t\right] . \quad (2.5d)$$

Comparing coefficients on both sides of Eq. (2.3), we get the equation of motion as

$$i\dot{c}_l = (W_0 l+\epsilon l^2)c_l + \Lambda\{c_{l+1}[(n_L-l)(n_W+l+1)]^{1/2} + c_{l-1}[(n_L-l+1)(n_W+l)]^{1/2}\} , \quad (2.6)$$

where

$$W_0 = \frac{2kp_0}{m} , \quad (2.7a)$$

and

$$\hbar\epsilon = \frac{(2\hbar k)^2}{2m} . \quad (2.7b)$$

It is not possible to solve this equation exactly. We use perturbation theory to solve for the c_l's. We assume that the zeroth-order values of the probability amplitudes c_l have a Poissonian distribution. This in effect is starting from a coherent state[6] of the FEL. We take

$$c_l^{(0)} = \left(\frac{e^{-\lambda}\lambda^l}{l!}\right)^{1/2} . \quad (2.8)$$

The equation of motion (2.6) now becomes

$$i\dot{c}_l^{(1)} = Ac_l^{(1)} + B , \quad (2.9)$$

where

$$A = W_0 l + \epsilon l^2 , \quad (2.10a)$$

and

$$B = \Lambda\left[\left(\frac{e^{-\lambda}\lambda^{l+1}}{(l+1)!}\right)^{1/2}[(n_L-l)(n_W+l+1)]^{1/2} + \left(\frac{e^{-\lambda}\lambda^{l-1}}{(l-1)!}\right)^{1/2}[(n_L-l+1)(n_W+l)]^{1/2}\right] . \quad (2.10b)$$

Integrating (2.9) we get

$$c_l^{(1)} = \frac{B}{A}(e^{-iAt}-1) . \quad (2.11)$$

This gives us an expression for the first-order probability amplitudes $c_l^{(1)}$ as a function of time.

III. PHOTON STATISTICS

The probability $p(n)$ of having n photons in the FEL state $|\psi\rangle$ [Eq. (2.1)], gives the photon statistics of the FEL. It is

$$p(n) = |\langle n|\psi\rangle|^2 , \quad (3.1)$$

where $|n\rangle$ is the number state of the laser field. Knowing the c_l's from Eq. (2.11) we can obtain an expression for the photon statistics.

$$p(n) = |c_l|^2 = |c_{n_L-n}|^2 \text{ since } n_L - l = n . \quad (3.2)$$

The first-order probabilities are

$$|c_l^{(1)}|^2 = \frac{4|B|^2\sin^2[(W_0 l+\epsilon l^2)t/2]}{(W_0 l+\epsilon l^2)^2} . \quad (3.3)$$

To investigate bunching properties, we calculate the first and second-order moments of the photon number:

$$\langle n\rangle = \sum_0^\infty np(n) = n_L - \sum_{l=n_L}^{-n_W} l|c_l^{(1)}|^2 , \quad (3.4)$$

and

$$\langle n^2\rangle = \sum_0^\infty n^2 p(n) = \sum_{l=n_L}^{-n_W} l^2|c_l^{(1)}|^2 + n_L^2 - 2n_L\sum_{n_L}^{-n_W} l|c_l^{(1)}|^2 . \quad (3.5)$$

Bunching is determined by the parameter[7]

$$\Delta = \langle n^2\rangle - \langle n\rangle^2 - \langle n\rangle . \quad (3.6)$$

Photons are bunched when Δ is positive and antibunched when Δ is negative. For the FEL, in our case,

$$\Delta = \sum_l l^2|c_l^{(1)}|^2 + \sum_l l|c_l^{(1)}|^2 - \left(\sum_l l|c_l^{(1)}|^2\right)^2 - n_L . \quad (3.7)$$

Analytic determination of the moments does not appear possible. We have evaluated the moments numerically on a computer and obtained Δ for a range of values of the wiggler and laser photon numbers, the parameter λ of the Poisson statistics of the coefficients $c_l^{(0)}$, and time (see Table I). Some of the experimental parameters for our numerical evaluation were taken from data reported in Ref. 8 (see appendix).

Under these conditions, we find that when we start from the vacuum state ($n_L=0$), the photon statistics depend strongly on the parameter of the Poisson statistics λ and on the wiggler field strengths (n_W) as shown in the table. When λ is either very small ($\lambda\sim 1$) or quite large ($\lambda\sim 100$) antibunching occurs irrespective of wiggler field strengths. In the region around ($\lambda\sim 10$), the photons are antibunched when the wiggler field is weak (n_W small) and

TABLE I. The table depicts the values for Δ—the bunching parameter and $(\Delta p)^2$—the values for the squeezing variable. $\Delta -ve$ indicates antibunching while $\Delta +ve$ indicates bunching. $(\Delta p)^2 < \frac{1}{2}\times 1.6\times 10^{-34}$ indicates squeezing.

		$n_L=0$		
λ	Time	n_W	Δ	$(\Delta p)^2$ (10^{-53})
1	1	10	−0.9822	0.1407
		100	−0.9781	0.1410
		10 000	−0.9777	0.1411
1	10	10	−0.9811	0.1414
		100	−0.9767	0.1418
		10 000	−0.9763	0.1418
1	100	10	−0.9664	0.1543
		100	−0.9609	0.1556
		10 000	−0.9604	0.1557
10	1	10	−0.3966	0.1690
		100	+0.9875	0.1815
		10 000	+0.1348	0.1825
10	10	10	−0.2014	0.1767
		100	+0.6988	0.1970
		10 000	+0.7599	0.1984
10	100	10	−0.6262	0.2169
		100	+0.8892	0.2423
		10 000	+0.9499	0.2442

bunched when the wiggler field strength increases (n_W large). When we start from a state where $n_L > 0$, we predict antibunching will always occur. These results are valid for both short times and long times.

Becker and Zubairy[1] calculate photon statistics using an electron field number state assuming the wiggler to be classical. Starting from the field vacuum, they predict bunching or antibunching to occur depending upon whether the electron momentum is $p>0$ or $p<0$ in the moving frame. Our results do not depend on whether the electron velocity is greater or less than the velocity of the Bambini-Renieri frame since in our expressions this parameter appears much less significant compared to the other summed variables. Sibilia *et al.*,[2] predict that antibunching will occur at short times and it will depend on the initial phase between the laser and wiggler fields. There is no dependence on electron momentum since they eliminate the electron variable and consider the radiation field to be in a combined laser and wiggler coherent state. Our results differ from those of the Refs. 1 and 2 due to differences in our assumptions and initial conditions. We take a generalized, fully quantum electron, laser, wiggler state but consider a model where the zeroth-order probability amplitudes for the state are Poissonian, i.e., initially we start from a coherent state. Using conservation laws, the state is labeled by one index l only, which gives the change in photon number. Thus in calculating the variance of the photon number, the initial phase difference between the laser and wiggler fields does not arise as a parameter.

IV. SQUEEZING

Squeezing is purely a quantum-mechanical phenomenon in which the fluctuations of one variable is reduced below its symmetrical quantum limit at the expense of the other conjugate one so that the uncertainty relation is not violated.

The uncertainty relation for two conjugate variables A and B states

$$\Delta A\, \Delta B \geqslant \hbar|2 \ , \tag{4.1}$$

where

$$\Delta A = [\langle A^2\rangle - \langle A\rangle^2]^{1/2} \ , \text{ etc.}$$

A state is squeezed if

$$(\Delta A)^2 \ \text{ or } (\Delta B)^2 < \frac{\hbar}{2} \ . \tag{4.2}$$

We have studied the squeezing properties of the states defined in Eq. (2.1) with the coefficients defined in Eq. (2.11) starting from the coherent state defined in Eq. (2.8). The expectation values of the electron coordinate x and momentum p are determined as follows:

$$\langle x\rangle = \langle\psi|x|\psi\rangle = \sum_{l=n_L}^{-n_W} |c_l|^2\langle l|x|l\rangle \ , \tag{4.3}$$

where

$$|l\rangle = |p_0+2l\hbar k, \ \ n_L-l, \ \ n_W+l\rangle \tag{4.4}$$

and

$$\langle x^2\rangle = \sum_{l=n_L}^{-n_W} |c_l|^2\langle l|x^2|l\rangle \ . \tag{4.5}$$

Similarly, for p,

$$\langle p\rangle = \sum_{l=n_L}^{-n_W} |c_l|^2\langle l|p|l\rangle \ , \tag{4.6}$$

where

$$p|l\rangle=(p_0+2l\hbar k)|l\rangle \ . \tag{4.7}$$

Therefore

$$\begin{aligned}\langle p\rangle &= \sum_{l=n_L}^{-n_W}|c_l|^2(p_0+2l\hbar k)\ ,\\ &= p_0+2\hbar k\sum_{l=n_L}^{-n_W} l|c_l|^2\ ,\end{aligned} \tag{4.8}$$

and

$$\langle p^2\rangle = p_0^2+2\hbar kp_0\sum_l l|c_l|^2+4\hbar^2k^2\sum_l l^2|c_l|^2\ . \tag{4.9}$$

So the variance is

$$\begin{aligned}(\Delta p)^2 &= \langle p^2\rangle-\langle p\rangle^2\\ &= 4\hbar k^2\left[\sum_l l^2|c_l|^2-\left(\sum_l l|c_l|^2\right)^2\right]-2\hbar kp_0\sum_l l|c_l|^2\ .\end{aligned} \tag{4.10}$$

We have calculated the variance $(\Delta p)^2$ numerically for the range of values of parameters given in the appendix. Our results indicate that the FEL state is squeezed in the momentum coordinate p, for the values considered.

APPENDIX

These values have been calculated using experimental data of Ref. 8. The following constants are in SI units.

$$\Lambda=\frac{e^2}{2mW\epsilon_0V_W}=1.108\times10^{-7}\ .$$

λ is the parameter of Poisson statistics; range 1, 10, 100. n_L is the number of initial laser photons; range 0, 10, 100. n_W is the number of wiggler photons; range 1, 10, 100, 1000, 10 000. t is time; range 1, 10, 100.

$$W_0=\frac{2kp_0}{m}=0.1305\times10^{-6},\ -0.1305\times10^{-6}\ .$$

$$\epsilon=\frac{2\hbar k^2}{m}=0.1347\times10^5\ .$$

$$p_0=8.5584\times10^{-25}\ .$$

Computer program available on request.

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