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Solution of Bloch equations involving amplitude and frequency modulations

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The undamped Bloch equations are shown to have a more general analytic solution which includes the famous self-induced-transparency solution of McCall and Hahn, and the nonadiabatic excitation solution of Allen and Eberly as special cases. A generalized area characteristic for a laser pulse of modulated amplitude and frequency of the forms of hyperbolic secant and tangent is given. It resolves the difficulty of the previous area concept in the presence of frequency modulation. The solution provides an exact analytic result for the amplitude- and frequency-modulated pulse of *any* area.

The Bloch equations[1] for magnetic resonance and optical resonance problems[2] are

$$\begin{pmatrix}\dot u\\ \dot v\\ \dot w\end{pmatrix}=\begin{pmatrix}0 & \dot B & 0\\ -\dot B & 0 & \dot A\\ 0 & -\dot A & 0\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}\ , \tag{1}$$

where, in the optical resonance problems,[3] u and v are the components of the atomic dipole moment in phase and in quadrature with the incident laser field, and w is the population inversion for the atom. The matrix elements $\dot A$ and $\dot B$ have the dimension of angular frequency and are related to the generally time-dependent amplitude and detuning of the incident laser pulse by

$$\dot A=\kappa\,\mathscr{E}(z,t)\equiv\Omega(z,t)\ , \tag{2a}$$

and

$$\dot B=\omega_0-\omega(z,t)\equiv\Delta(z,t)\ , \tag{2b}$$

where $\mathscr{E}(z,t)$ and $\omega(z,t)$ are the amplitude and frequency of the incident laser pulse, ω_0 the frequency difference between the two levels of the atom, and $\kappa=2d/\hbar$, d being the atomic dipole moment.

A beautiful solution of Eq. (1) for the case of constant detuning Δ is the famous hyperbolic secant pulse of McCall and Hahn:[4]

$$\dot A=\kappa\,\mathscr{E}(z,t)=\frac{2}{\tau}\,\mathrm{sech}[(t-t_0)/\tau]\ , \tag{3a}$$

$$\dot B=\Delta(z,t)=\Delta\ , \tag{3b}$$

where τ is an arbitrary pulse length. The important concept of the envelope area of the pulse was also introduced by McCall and Hahn as

$$A(z,t)=\kappa\int_{-\infty}^{t}\mathscr{E}(z,t')\,dt'\ , \tag{4}$$

which is also identified as the dipole turning angle. The area $A(z,\infty)$ of the pulse (3a) is 2π, and it is this special 2π property which makes the pulse stable. As the pulse propagates through the atomic medium, it excites but returns the atoms to their initial state as it emerges from the medium completely unattenuated. McCall and Hahn named the phenomenon "self-induced transparency."

An interestingly related solution of Eq. (1) for the case involving both amplitude and frequency modulations is the nonadiabatic excitation solution of Allen and Eberly:[5]

$$\dot A=\kappa\,\mathscr{E}(z,t)=\frac{1}{\tau}(1+\delta^2\tau^2)^{1/2}\,\mathrm{sech}[(t-t_0)/\tau]\ , \tag{5a}$$

$$\dot B=\Delta(z,t)=-\delta\tanh[(t-t_0)/\tau]\ , \tag{5b}$$

where 2δ is the magnitude of the frequency sweep. Starting with an atom initially in the lower state (at $t=-\infty$), the pulse completely inverts the atom and leaves it in the upper state at $t=+\infty$. If the frequency modulation is zero, then $\mathscr{E}$ becomes a standard π pulse and it would naturally be expected to invert an atom. On the other hand, if the frequency modulation is substantial enough so that, for example, $\delta^2\tau^2=3$, then we have a 2π pulse which again inverts the atoms, and does not return them to their ground states. Clearly in the presence of frequency modulation, the identification of the pulse area to dipole turning angle is no longer true. The appropriate characterization of pulse area has never been made in that case.

We present, in this Brief Report, a more general analytic solution of the Bloch Eqs. (1) which includes the solution of McCall and Hahn [Eq. (3)] and the solution of Allen and Eberly [Eq. (5)] as special cases. We shall introduce a characterization of pulse area and show that the pulses given by Eqs. (3) and (5) correspond to the special cases of 2π and π pulses, respectively. Our solution provides an exact analytic result for a specific type of amplitude- and frequency-modulated pulse of *any* area.

We shall derive our solution of Eq. (1) through solving the following set of equations:

$$\begin{pmatrix}\dot C_1\\ \dot C_2\end{pmatrix}=\begin{pmatrix}0 & \frac{1}{2}i\dot A e^{-iB}\\ \frac{1}{2}i\dot A e^{iB} & 0\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix}\ . \tag{6}$$

It can be verified that given Eq. (6), Eq. (1) follows if u, v, w are related to C_1 and C_2 by

$$u=e^{-iB}C_1^*C_2+e^{iB}C_2^*C_1\ , \tag{7a}$$

$$v=-i(e^{-iB}C_1^*C_2-e^{iB}C_2^*C_1)\ , \tag{7b}$$

$$w=C_1^*C_1-C_2^*C_2=|C_1|^2-|C_2|^2\ . \tag{7c}$$

When the solutions of $|C_1|$ and $|C_2|$ are determined from Eq. (6), w can be determined from Eq. (7c), and v and u from Eq. (1) as

$$v=-\dot w/\dot A\ , \tag{8}$$

and

$$u=(-\dot{v}+\dot{A}w)/\dot{B} \quad . \tag{9}$$

The steps (7c), (8), and (9) make the determination of the phases of C_1 and C_2 unnecessary for our solution of Eq. (1).

Equations (6) are an extension of the set of equations studied by Rosen and Zener[6] in connection with the double Stern-Gerlach experiment in which they had $B=\omega t$. We shall follow the method used by them, but our extension of B to the form which we shall specify is crucial in incorporating the frequency modulation in Eq. (1).

Elimination of C_2 from Eq. (6) leads to the second-order differential equation

$$\ddot{C}_1+\left(i\dot{B}-\frac{\ddot{A}}{\dot{A}}\right)\dot{C}_1+\left(\frac{\dot{A}}{2}\right)^2 C_1=0 \quad . \tag{10}$$

We now make the choice

$$\dot{A}=\frac{\alpha}{\pi\tau}\operatorname{sech}[(t-t_0)/\tau] \quad , \tag{11}$$

$$\dot{B}=\frac{1}{\pi\tau}\{\beta_0+\beta\tanh[(t-t_0)/\tau]\} \quad , \tag{12}$$

where α, β_0, β are arbitrary constants which may be considered to have the dimension of angles in radians for our problem, and τ is an arbitrary pulse length. These functions, (11) and (12), together with the transformation

$$z=\tfrac{1}{2}\{1+\tanh[(t-t_0)/\tau]\} \tag{13}$$

reduce Eq. (10) to the hypergeometric equation

$$z(1-z)\frac{d^2C_1}{dz^2}+[c-(a+b+1)z]\frac{dC_1}{dz}-abC_1=0 \quad , \tag{14}$$

where

$$a=\frac{1}{2\pi}[(\alpha^2-\beta^2)^{1/2}+i\beta] \quad , \tag{15a}$$

$$b=\frac{1}{2\pi}[-(\alpha^2-\beta^2)^{1/2}+i\beta] \quad , \tag{15b}$$

$$c=\frac{1}{2}\left(1+i\frac{\beta_0+\beta}{\pi}\right) \quad . \tag{15c}$$

As t goes from $-\infty$ to ∞, z goes from 0 to 1. The general solution of Eq. (14), which is defined in this range, is

$$C_1=a_1F(a,b,c,z)+a_2z^{1-c}F(a+1-c,b+1-c,2-c,z) \quad . \tag{16}$$

Consider the boundary conditions

$$C_1(-\infty)=0 \quad , \tag{17a}$$

$$|C_2(-\infty)|=1 \quad , \tag{17b}$$

so that $w(-\infty)=-1$ in Eq. (1). To satisfy the boundary condition (17a), we must set $a_1=0$. Using the first of Eq. (6), we find that the boundary condition (17b) is satisfied when

$$a_2=e^{i\phi_1}\frac{\alpha}{2\pi|c|} \quad , \tag{18}$$

where ϕ_1 is an arbitrary (real) phase factor. Thus, we find

$$C_1=e^{i\phi_1}\frac{\alpha}{2\pi|c|}z^{1-c}F(a+1-c,b+1-c,2-c,z) \quad , \tag{19}$$

and similarly, we find

$$C_2=e^{i\phi_2}F(a^*,b^*,c^*,z) \quad , \tag{20}$$

where ϕ_2 is another arbitrary phase factor, and we note that

$$|C_1|^2+|C_2|^2=1 \quad . \tag{21}$$

By using the relations

$$F(a,b,c,1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad , \tag{22}$$

and

$$\Gamma(z)\Gamma(-z)=\frac{-\pi}{z\sin\pi z} \quad , \tag{23}$$

we find

$$|C_1(+\infty)|^2=\operatorname{sech}[\tfrac{1}{2}(\beta_0+\beta)]\operatorname{sech}[\tfrac{1}{2}(\beta_0-\beta)] \times[\sin^2(\tfrac{1}{2}\Phi)\cosh^2(\tfrac{1}{2}\beta)+\cos^2(\tfrac{1}{2}\Phi)\sinh^2(\tfrac{1}{2}\beta)] \tag{24}$$

and

$$|C_2(+\infty)|^2=1-|C_1(+\infty)|^2 \quad , \tag{25}$$

where

$$\Phi=(\alpha^2-\beta^2)^{1/2} \tag{26}$$

turns out to be an important parameter which we can use to characterize the "area" of the pulse given by Eqs. (11) and (12) under various conditions which we shall discuss.

Equations (24) and (25) give us the fraction of electron population inversion for the atoms at $t=+\infty$ of Eq. (1) to be

$$w(+\infty)=2\operatorname{sech}[\tfrac{1}{2}(\beta_0+\beta)]\operatorname{sech}[\tfrac{1}{2}(\beta_0-\beta)] \times[\sin^2(\tfrac{1}{2}\Phi)\cosh^2(\tfrac{1}{2}\beta)+\cos^2(\tfrac{1}{2}\Phi)\sinh^2(\tfrac{1}{2}\beta)]-1 \quad , \tag{27}$$

when the atoms, starting with $w(-\infty)=-1$, are subjected to an incident laser pulse which has a modulated amplitude and frequency given by Eqs. (11) and (12). The special cases discussed below when one or both of the parameters β_0 and β equals zero give us further insight into the nature and wide applicability of the solution which we have obtained.

Special cases

(1) The case $\beta=0$. We have from Eqs. (26) and (27),

$$\Phi=\alpha \quad , \tag{28}$$

and

$$w(+\infty)=2\sin^2(\tfrac{1}{2}\Phi)\operatorname{sech}^2(\tfrac{1}{2}\beta_0)-1 \quad . \tag{29}$$

If

$$\Phi=2n\pi, \quad n=1,2,3,\ldots \quad , \tag{30}$$

then the atoms are always returned to their initial state. The self-induced-transparency solution of McCall and Hahn corresponds to the *particular* case of Eq. (30) when $n=1$ for which a simple analytic solution of C_1 is available for all t:

$$C_1=e^{i\phi_1}\frac{1}{(1+\beta_0^2/\pi^2)^{1/2}}z^{(1/2)(1+i\beta_0/\pi)}(1-z)^{(1/2)(1-i\beta_0/\pi)} \quad (31)$$

which yields

$$u=-\frac{2\beta_0/\pi}{1+\beta_0^2/\pi^2}\operatorname{sech}\left[\frac{t-t_0}{\tau}\right], \quad (32a)$$

$$v=\frac{2}{1+\beta_0^2/\pi^2}\operatorname{sech}\left[\frac{t-t_0}{\tau}\right]\tanh\left[\frac{t-t_0}{\tau}\right], \quad (32b)$$

$$w=-1+\frac{2}{1+\beta_0^2/\pi^2}\operatorname{sech}^2\left[\frac{t-t_0}{\tau}\right]. \quad (32c)$$

Equations (32) can be written in the more familiar forms when we identify β_0/π to be $\Delta\tau$ from Eqs. (12) and (3b).

That a pulse with $\Phi=2n\pi$ always returned the atoms to their initial state was conjectured and numerically confirmed by McCall and Hahn. Equation (29), derived much earlier by Rosen and Zener, was in fact a more precise statement of that effect.

It will be noted that a pulse with $\Phi=(2n-1)\pi$, $n=1,2,\ldots$ does *not completely* invert the atoms to their upper states except when $\beta_0=0$.

(2) The case $\beta_0=0$. We have from Eqs. (26) and (27)

$$\Phi=(\alpha^2-\beta^2)^{1/2} \quad (33)$$

and

$$w(+\infty)=1-2\cos^2(\tfrac{1}{2}\Phi)\operatorname{sech}^2(\tfrac{1}{2}\beta). \quad (34)$$

If

$$\Phi=(2n-1)\pi,\quad n=1,2,3,\ldots, \quad (35)$$

then the atoms are always *fully* excited to their upper states. The nonadiabatic excitation solution of Allen and Eberly corresponds to the particular case of Eq. (35) when $n=1$ for which again simple analytic expressions of C_1 and C_2 are available for all t:

$$C_1=e^{i\phi_1}z^{(1/2)(1-i\beta/\pi)}, \quad (36a)$$

$$C_2=e^{i\phi_2}(1-z)^{(1/2)(1+i\beta/\pi)}, \quad (36b)$$

which yield

$$u=\frac{\beta}{\alpha}\operatorname{sech}\left[\frac{t-t_0}{\tau}\right], \quad (37a)$$

$$v=-\frac{\pi}{\alpha}\operatorname{sech}\left[\frac{t-t_0}{\tau}\right], \quad (37b)$$

$$w=\tanh\left[\frac{t-t_0}{\tau}\right]. \quad (37c)$$

It will be noted that in the presence of modulated detuning, the area of the pulse is no longer characterized by α but by $\Phi=(\alpha^2-\beta^2)^{1/2}$, and that for the pulse considered by Allen and Eberly [Eqs. (5)] $\alpha=\pi(1+\delta^2\tau^2)^{1/2}$, $\beta=\pi\delta\tau$ from Eqs. (11), (12), and (5), and hence

$$(\alpha^2-\beta^2)^{1/2}=[\pi^2(1+\delta^2\tau^2)-\pi^2\delta^2\tau^2]^{1/2}=\pi, \quad (38)$$

no matter what δ is. Our Eqs. (34) and (35) express a more general result not previously stated, that a pulse with $\Phi=(2n-1)\pi$ in the case $\beta_0=0$ always inverts the atoms *completely*, but that a pulse with $\Phi=2n\pi$ does not always return an atom to its initial state except when $\beta=0$.

(3) The case $\beta_0=\beta=0$. This case is of separate interest because the atomic evolutions can be expressed in terms of elementary functions for all t for an incident pulse of modulated amplitude of any area.

From Eq. (20),

$$C_2=e^{i\phi_2}F(a,-a,\tfrac{1}{2},z), \quad (39)$$

where

$$a=\frac{\alpha}{2\pi}. \quad (40)$$

Since

$$F(a,-a,\tfrac{1}{2},\sin^2x)=\cos(2ax), \quad (41)$$

we find

$$|C_1|^2=\sin^2(\tfrac{1}{2}a\theta), \quad (42a)$$

$$|C_2|^2=\cos^2(\tfrac{1}{2}a\theta), \quad (42b)$$

where

$$\sin\frac{\theta}{2}=\operatorname{sech}[(t-t_0)/\tau], \quad (43a)$$

$$\cos\frac{\theta}{2}=-\tanh[(t-t_0)/\tau]. \quad (43b)$$

Thus, we have

$$u=0, \quad (44a)$$

$$v=-\sin\left[\frac{\alpha}{2\pi}\theta\right], \quad (44b)$$

$$w=-\cos\left[\frac{\alpha}{2\pi}\theta\right], \quad (44c)$$

where $\dot\theta$ can be verified to be related to $\dot A$ of Eq. (11) by

$$\dot\theta=\frac{2\pi}{\alpha}\dot A. \quad (45)$$

Equations (44), together with Eqs. (43), express the atomic evolutions when the atoms are subjected to an incident resonant pulse given by Eq. (11) of any area α. A $(2n-1)\pi$ pulse always fully inverts the atoms and a $2n\pi$ pulse always returns the atoms to their initial state. The atomic evolutions when α is an integral multiple of π can be written down more explicitly by using the multiple angle expansions of the relevant trigonometric functions. In particular,

(a) $\alpha=\pi$,

$$v=-\operatorname{sech}[(t-t_0)/\tau], \quad (46)$$

$$w=\tanh[(t-t_0)/\tau],$$

(b) $\alpha=2\pi$,

$$v=2\operatorname{sech}[(t-t_0)/\tau]\tanh[(t-t_0)/\tau], \quad (47)$$

$$w=-1+2\operatorname{sech}^2[(t-t_0)/\tau)],$$

(c) $\alpha=3\pi$,

$$v=-3\,\mathrm{sech}[(t-t_0)/\tau]+4\,\mathrm{sech}^3[(t-t_0)/\tau]\ , \tag{48}$$

$$w=-3\tanh[(t-t_0)/\tau]+4\tanh^3[(t-t_0)/\tau]\ ,$$

(d) $\alpha=4\pi$,

$$v=8\tanh^3[(t-t_0)/\tau]\,\mathrm{sech}[(t-t_0)/\tau]-4\tanh[(t-t_0)/\tau]\,\mathrm{sech}[(t-t_0)/\tau]\ , \tag{49}$$

$$w=-1+8\tanh^2[(t-t_0)/\tau]\,\mathrm{sech}^2[(t-t_0)/\tau]\ ,$$

etc.

To summarize, we have found the population inversion of electrons in atoms [Eq. (27)] when subjected to an incident laser field involving both amplitude and frequency modulation of the forms given by Eqs. (11) and (12). The special results, Eqs. (29) and (34) for the cases when one of the parameters β and β_0 equals zero, suggest the usefulness of the parameter Φ [Eq. (26)] for characterizing the pulse area, which also unified the self-induced-transparency solution of McCall and Hahn and the nonadiabatic excitation solution of Allen and Eberly which are shown to be two special cases ($\Phi=2\pi$ and π, respectively) of our special solutions. Explicit expressions for the atomic evolutions at all times, Eqs. (44) and (43) for the case $\beta=\beta_0=0$, are also presented.

Our results are immediately applicable to the multiple soliton and multilevel excitation problems recently studied[7–9] where specific dynamic symmetries were shown to produce dynamical subspaces of the form of Eq. (1).

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[1]F. Bloch, Phys. Rev. **70**, 460 (1946).
[2]R. P. Feynman, F. L. Vernon, and R. W. Hellwarth, J. Appl. Phys. **28**, 49 (1957).
[3]L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).
[4]S. L. McCall and E. L. Hahn, Phys. Rev. **183**, 457 (1969).
[5]Reference 3, Sec. 4.6.
[6]N. Rosen and C. Zener, Phys. Rev. **40**, 502 (1932).
[7]F. T. Hioe, Phys. Rev. A **26**, 1466 (1982); **28**, 879 (1983).
[8]F. T. Hioe, in *Proceedings of the Rochester Conference in Quantum Optics and Optical Coherence,* edited by L. Mandel and E. Wolf (Plenum, New York, 1984), p. 965; F. T. Hioe and J. H. Eberly, Phys. Rev. A **29**, 1164 (1984).
[9]F. T. Hioe, Phys. Rev. A (to be published).