PHYSICAL REVIEW A VOLUME 30, NUMBER 4 OCTOBER 1984

Quantum oscillations in one-dimensional normal-metal rings

M. Büttiker
IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598

Y. Imry
IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598
and Department of Physics and Astronomy, Tel-Aviv University, Tel-Aviv 69978, Israel

M. Ya. Azbel
Department of Physics and Astronomy, Tel-Aviv University, Tel-Aviv 69978, Israel
(Received 26 April 1984)

We investigate the connection between two recent investigations on flux-periodic effects in one-dimensional normal-metal rings with inelastic diffusion length larger than the size of the ring. Büttiker, Imry, and Landauer have pointed out that closed rings, driven by an external flux, act like superconducting rings with a Josephson junction, except that $2e$ is replaced by e. Gefen, Imry, and Azbel considered such a ring connected to current leads and found a flux-periodic electric resistance. We establish a connection between these Aharonov-Bohm-like effects by demonstrating that the transmission probability of the ring, which determines the electric resistance, exhibits resonances near the energies of the electronic states of the closed ring. It is the flux dependence of the resonances which gives rise to the strong oscillatory behavior of the electric resistance.

I. INTRODUCTION

The purpose of this paper is to make a connection between two Aharonov-Bohm-type phenomena in conducting rings: The Josephson-like effects proposed in Ref. 1 for a closed ring, and the oscillations in the electric resistance of such a ring connected to current leads as suggested in Ref. 2. Reference 1 considers a small strictly one-dimensional normal-metal ring driven by a magnetic flux Φ confined to its hole. It is pointed out that the single-electron states of such a ring can be obtained by considering the band structure of a "crystal" $V(x)=V(x+L)$, where $V(x)$ is the potential around the ring and L is the circumference of the ring. The electronic states of the ring, shown in Fig. 1, are given by

$$E_n(k_0\Phi/\Phi_0)=E_n(k_0(\Phi+\Phi_0)/\Phi_0)\ , \tag{1.1}$$

where $\Phi_0=hc/e$ is the flux quantum associated with a single charge e, $k_0=2\pi/L$ is the width of the Brillouin zone of the associated crystal, and the $E_n(k)$ are the energy bands of this crystal. For weak elastic scattering, the gaps in Fig. 1 will be small. For strong elastic scattering, the gaps in Fig. 1 will be large giving rise to bands which depend only weakly on the flux Φ. The electronic states are thus periodic in the flux with period Φ_0. For a constant flux, the current at $T\rightarrow 0$, $j=(e/L)\sum_n v_n = -c\sum_n \partial E_n/\partial\Phi$ (where n numbers all occupied states up to the Fermi energy E_F), is also a periodic function of the flux with period Φ_0. For a flux which is not equal to a multiple of $\Phi_0/2$, Ref. 1 predicts a *persistent* current. If the flux increases linearly in time, the induced electromotive force $F=(1/cL)d\Phi/dt$ gives rise to an oscillating current with a *Josephson* frequency $\omega=eV/\hbar$, where $V=FL$. It is argued, in Ref. 1, that these Josephson-like effects survive even in the presence of modest inelastic scattering, i.e., when the inelastic diffusion length is much larger than the circumference of the ring. Reference 2 considers current leads connected to such a ring and employs the Landauer formula

$$R_{\rm el}(\Phi)=\frac{\hbar\pi}{e^2}\frac{1-T(E_F,\Phi)}{T(E_F,\Phi)} \tag{1.2}$$

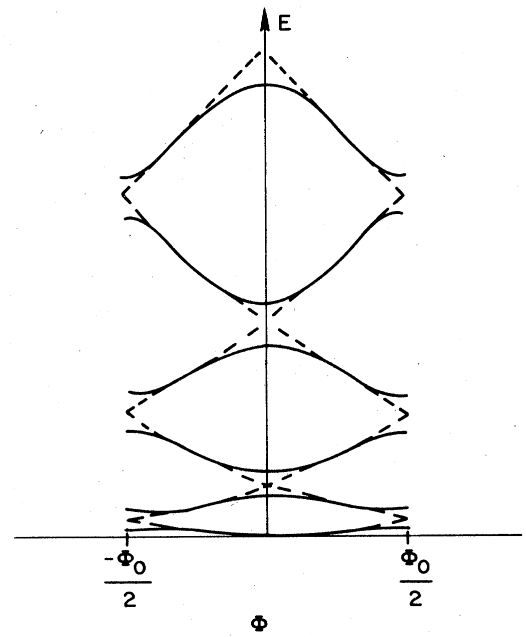

FIG. 1. One-electron energies of the closed ring as a function of flux. The case of zero scattering is represented by the dashed lines.

to calculate the electric resistance. $T(E,\Phi)$ is the quantum-mechanical transmission probability for electrons with incident energy E to traverse the ring. The main result of Ref. 2 is that this resistance is periodic in the flux, with period Φ_0,

$$R_{\rm el}(\Phi)=R_{\rm el}(\Phi+\Phi_0)\ . \qquad (1.3)$$

The flux Φ introduces phase shifts $e^{i\theta_1}$ and $e^{-i\theta_2}$ in the two branches of the wave function along the ring where $\theta_1+\theta_2=2\pi\Phi/\Phi_0$. This modifies the interference of the electron waves at the point of connection to the leads and thus leads to Aharonov-Bohm-like oscillations in the transmission probability through the ring.

In this paper we make the same restrictive assumptions as in Refs. 1 and 2, i.e., we consider strictly one-dimensional rings, and proceed to point out a connection between the two. Such a connection is established by looking at the resonances of the transmission probability $T(E,\Phi)$ of the ring. The understanding of these resonances is a necessary step in order to treat the many-channel case. The effects of resonances on electrical conduction in small systems has been extensively treated in Ref. 3. We show that sharp peaks in the transmission probability are of the Breit-Wigner form[4]

$$T(E,\Phi)=T_{\rm res}\frac{\Gamma_n^2(\Phi)}{[E-E_n(\Phi)-\Delta E_n(\Phi)]^2+\Gamma_n^2(\Phi)}\ . \qquad (1.4)$$

Here, $T_{\rm res}\leq 1$ is the value of the transmission probability at resonance. $E_n(\Phi)$ is the energy of an electronic state of the closed ring [Eq. (1.1)] and ΔE_n is a small shift away from this energy. Γ_n is the width of the resonance. In order to obtain sharp resonances, as given by Eq. (1.4), the width Γ_n has to be small compared to the gaps E_n-E_{n+1} in the electronic spectrum of the closed ring. Below we investigate these conditions in detail.

The resonant behavior of the transmission probability given by Eq. (1.4) will show up in the electric resistance[3] of the ring [Eq. (1.3)] if the Fermi energy E_F lies close to a band $E_n(\Phi)$ or more strongly if E_F falls within a band. In this case the resistance varies strongly as a function of the flux and reaches a minimum for value of flux given by $E_F=E_n(\Phi)+\Delta E_n(\Phi)$.

II. ELECTRONIC STATES OF THE CLOSED RING

In this section we derive an eigenvalue equation for the energies E_n [Eq. (1.1)] of the closed ring. Instead of a potential $V(x)$, we describe the elastic scattering in the ring with the help of a transfer or $\underline{t}$ matrix. Since we consider a one-dimensional system, the $\underline{t}$ matrix is given by

$$\underline{t}=\begin{bmatrix}1/t^* & -r^*/t^*\\ -r/t & 1/t\end{bmatrix} \qquad (2.1)$$

and relates the amplitudes β,β' of the wave function to the left of the scatterer to the amplitudes $\tilde\beta,\tilde\beta'$ to the right of the scatterer, see Fig. 2. Here

$$t=T_s^{1/2}e^{i\phi_s} \qquad (2.2)$$

is the transmission amplitude of the scatterer, T_s the transmission probability, and ϕ_s the phase change in the transmitted wave (index s for scatterer). An incoming wave, from the left of the scatterer, of amplitude 1 gives rise to a reflected wave with amplitude

$$r=e^{-i\pi/2}R_s^{1/2}e^{i\phi_s}e^{i\phi_a}\ . \qquad (2.3)$$

Here $R_s=1-T_s$ is the reflection probability. For an incoming wave to the right of the scatterer,

$$r'=e^{-i\pi/2}R_s^{1/2}e^{i\phi_s}e^{-i\phi_a} \qquad (2.4)$$

is the amplitude of the reflected wave.

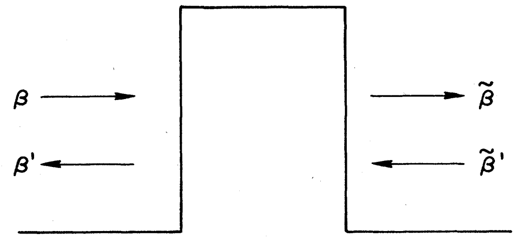

FIG. 2. Schematic representation of the potential $V(x)$ in the ring. The transfer matrix relates the amplitudes β,β' to $\tilde\beta,\tilde\beta'$.

Consider now, as in Ref. 1, a periodic arrangement of the scatterer along x with period L equal to the circumference of the ring. According to Bloch's theorem, the amplitudes of the wave functions in unit cells one lattice-period apart are given by $\tilde\beta=e^{iKL}\beta$, $\tilde\beta'=e^{iKL}\beta'$. Thus, for the periodic array of scatterers, the amplitudes of the wave function obey

$$(\underline{t}-\mathbb{1}e^{iKL})\begin{bmatrix}\beta\\ \beta'\end{bmatrix}=\underline{0}\ . \qquad (2.5)$$

Equation (2.5) has nontrivial solutions only if

$$\det(\underline{t}-\mathbb{1}e^{iKL})=0\ . \qquad (2.6)$$

To make this equation more transparent we can use Eq. (2.2) and find from Eq. (2.6)

$$\cos\phi_s=T_s^{1/2}\cos KL\ . \qquad (2.7)$$

Both ϕ_s and T_s are functions of the energy. For fixed K, $-\pi/L\leq K\leq\pi/L$, Eq. (2.7) yields a discrete sequence of eigenstates $E_n(k)$, $E_1\leq E_2\leq E_3,\ldots$ which, when considered a function of K, gives rise to bands over a Brillouin zone of width $k_0=2\pi/L$ (see Fig. 1).

For our purpose of comparing the closed ring to the connected ring, we want to consider a closed ring, as shown in Fig. 3(a), which has a scatterer in each branch, denoted by $\underline{t}_1$ and $\underline{t}_2$. $\underline{t}_1$ and $\underline{t}_2$ give the amplitudes of the wave functions to the right of the scatterers in terms of the amplitudes of the wave functions to the left of the scatterers. If we denote the transfer matrix which yields the amplitudes to the left of the scatterer in terms of the amplitudes to the right by $\underline{t}_2'$, the two transfer matrices give rise to the combined scatterer $\underline{t}=\underline{t}_2'\underline{t}_1$. The eigenvalue equation of this ring is thus given by

$$\det(\underline{t}_2'\underline{t}_1-\mathbb{1}e^{iKL})=0\ . \qquad (2.8)$$

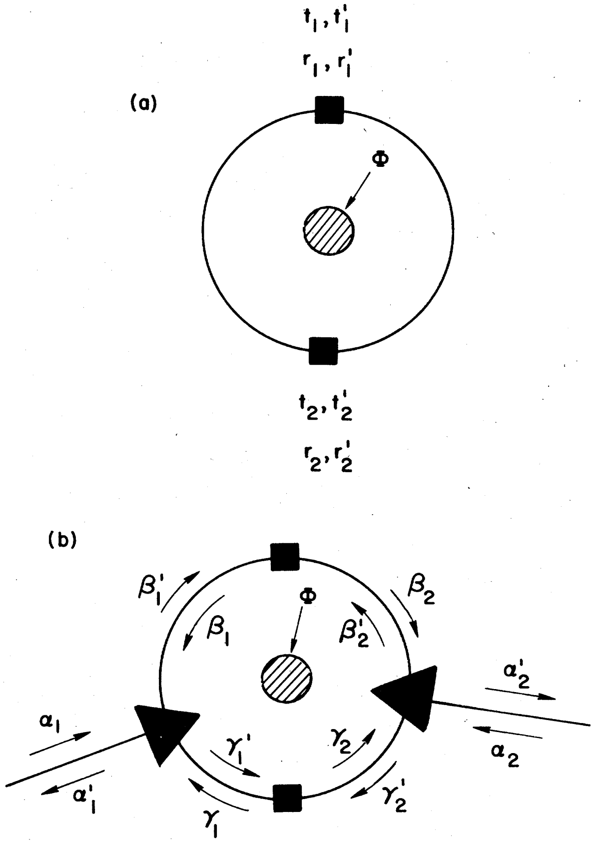

FIG. 3. (a) Closed ring with two elastic scatterers. (b) Ring connected to current leads with the same elastic scatterers as in (a).

According to Ref. 1 we obtain the energies of the electronic states of the closed ring in the presence of flux Φ by replacing K in Eqs. (2.6)–(2.8) by $k_0\Phi/\Phi_0$.

III. COUPLING LEADS TO A RING

At a junction of a lead with the ring [triangles in Fig. 3(b)], the three outgoing waves with amplitudes (α',β',γ') are related[5] by an $\underline{S}$ matrix to the three incoming waves (α,β,γ),

$$\vec{\alpha}\,'=\underline{S}\vec{\alpha}\ . \tag{3.1}$$

Current conservation implies that $\underline{S}$ is unitary. Time-reversal invariance when applicable implies, furthermore, that $\underline{S}^*=\underline{S}^{-1}$ and, therefore, that $\underline{S}$ is also symmetric. It follows that the $\underline{S}$ matrix then depends in general on five independent parameters, which is the number of symmetric generators of the Lie algebra SU(3). If we assume that $\underline{S}$ is symmetric with respect to the two branches of the circle, the number of independent parameters is three. Here we would like to go even further and assume, in addition, that $\underline{S}$ is real. These two restrictions may not be severe, since the division of the elastic scattering between the $\underline{S}$ matrix and the $\underline{t}$ matrices is arbitrary. A completely general description of the connected ring can be given by studying the combination of two general $\underline{S}$ matrices, omitting the scatterers $\underline{t}_1$ and $\underline{t}_2$ completely. Here, as in Ref. 2, we take the opposite viewpoint, keeping the $\underline{S}$ matrices as simple as possible and retaining completely general scatterers $\underline{t}_1, \underline{t}_2$.

A real $\underline{S}$ matrix symmetric with respect to the two branches of the ring is of the form

$$\underline{S}=\begin{bmatrix} -(a+b) & \epsilon^{1/2} & \epsilon^{1/2} \\ \epsilon^{1/2} & a & b \\ \epsilon^{1/2} & b & a \end{bmatrix}\ . \tag{3.2}$$

Probability (current) conservation requires

$$(a+b)^2+2\epsilon=1\ , \tag{3.3}$$

$$a^2+b^2+\epsilon=1\ . \tag{3.4}$$

Orthogonality is satisfied by the solutions of Eqs. (3.3) and (3.4). The coefficients a and b can be expressed as functions of ϵ by using Eqs. (3.3) and (3.4),

$$a_\pm=\pm\tfrac{1}{2}(\sqrt{1-2\epsilon}-1)\ , \tag{3.5}$$

$$b_\pm=\mp\tfrac{1}{2}(\sqrt{1-2\epsilon}+1)\ , \tag{3.6}$$

and

$$\tilde{a}_\pm=\pm\tfrac{1}{2}(\sqrt{1-2\epsilon}+1)\ , \tag{3.7}$$

$$\tilde{b}_\pm=\mp\tfrac{1}{2}(\sqrt{1-2\epsilon}-1)\ . \tag{3.8}$$

Equations (3.5) and (3.6) determine all real 3×3 $\underline{S}$ matrices, which are symmetric with respect to two channels, as a function of a single parameter ϵ, $0\le\epsilon\le\frac{1}{2}$.

Consider the solutions $a=a_\pm$, $b=b_\pm$. A wave of unit amplitude coming from the current lead is reflected back with probability $(a+b)^2=1-2\epsilon$ and transmitted into the two branches of the ring with equal probability ϵ. For $\epsilon=\frac{1}{2}$, the junction is completely transparent for incoming electrons and the current lead is strongly coupled to the ring. It is this strong coupling limit with $a=a_-$, $b=b_-$ which has been considered in Ref. 2. On the other hand, for $\epsilon=0$, electrons from the current lead are totally reflected and thus there is no coupling between the current leads and the ring. In this case the transmission probability from one branch of the ring into the other is $b^2=1$ and electrons in the ring do not see the junction. Thus ϵ is a coupling parameter and the branches $a=a_\pm$, $b=b_\pm$ describe the transition from the strong coupling limit $\epsilon=\frac{1}{2}$ to the weak[6] or zero coupling limit $\epsilon=0$.

The branches $a=\tilde{a}_\pm$, $b=b_\pm$ describe a transition from strong coupling $\epsilon=\frac{1}{2}$ to a situation where the current leads and the two branches are completely decoupled. For $\epsilon=0$ we find $a=\tilde{a}_+=-\tilde{a}_-=1$, $b=\tilde{b}_+=-\tilde{b}_-=0$. These latter two branches are of no further interest here.

IV. RESONANCES IN THE TRANSMISSION PROBABILITY OF THE RING

A. Transmission probability

In this section we derive an expression for the transmission probability for the ring connected to current leads, shown in Fig. 3(b), which will illuminate the connection with the resonances. Consider a wave of unit amplitude $\alpha_1=1$, incident from the left. To find the transmission probability of the ring $T=|\alpha_2'|^2$ we have to determine the β's and γ's under the condition $\alpha_2=0$. The reflection probability is then given by $R=|\alpha_1'|^2=1-T$. From the link to the right we find with the help of Eqs. (3.1) and (3.2)

$$\alpha_2'=\sqrt{\epsilon}(\beta_2+\gamma_2)\ , \tag{4.1}$$

$$\beta_2'=a\beta_2+b\gamma_2\ , \tag{4.2}$$

$$\gamma_2'=b\beta_2+a\gamma_2\ . \tag{4.3}$$

Using Eqs. (4.2) and (4.3) to express the γ's in terms of the β's yields

$$\begin{bmatrix}\gamma_2'\\ \gamma_2\end{bmatrix}=\underline{t}_l\begin{bmatrix}\beta_2\\ \beta_2'\end{bmatrix}\ , \tag{4.4}$$

with a matrix $\underline{t}_l$ (index l for link) given by

$$\underline{t}_l=\frac{1}{b}\begin{bmatrix}(b^2-a^2) & a\\ -a & 1\end{bmatrix}\ . \tag{4.5}$$

Note that $\det(\underline{t}_l)=1$, but $\underline{t}_l$ is not unitary for $\epsilon\neq0$. For the junction to the left we find from Eqs. (3.1) and (3.2), using $\alpha_1=1$,

$$\alpha_1'=-(a+b)+\sqrt{\epsilon}(\beta_1+\gamma_1)\ , \tag{4.6}$$

$$\beta_1'=\sqrt{\epsilon}+a\beta_1+b\gamma_1\ , \tag{4.7}$$

$$\gamma_1'=\sqrt{\epsilon}+b\beta_1+a\gamma_1\ . \tag{4.8}$$

Using Eqs. (4.7) and (4.8) to express the β's as function of the γ's yields

$$\begin{bmatrix}\beta_1'\\ \beta_1\end{bmatrix}=\frac{\sqrt{\epsilon}}{b}\begin{bmatrix}b-a\\ -1\end{bmatrix}+\underline{t}_l\begin{bmatrix}\gamma_1\\ \gamma_1'\end{bmatrix}\ . \tag{4.9}$$

As we follow the wave function around the ring, the phase of the wave function changes by $2\pi\Phi/\Phi_0$, where Φ is the applied flux. The phase changes θ_1,θ_2 both taken in a counterclockwise sense, along the two branches of the ring, depend on the length of these branches or, if we assume the ring to be circular, on the position of the links. But $\theta_1+\theta_2=2\pi\Phi/\Phi_0$, in any case. Thus the amplitudes in the upper branch are transferred according to

$$\begin{bmatrix}\beta_2\\ \beta_2'\end{bmatrix}=e^{-i\theta_1}\underline{t}_1\begin{bmatrix}\beta_1'\\ \beta_1\end{bmatrix}\ , \tag{4.10}$$

and the amplitudes of the lower branch are related by

$$\begin{bmatrix}\gamma_1\\ \gamma_1'\end{bmatrix}=e^{-i\theta_2}\underline{t}_2'\begin{bmatrix}\gamma_2'\\ \gamma_2\end{bmatrix}\ . \tag{4.11}$$

Note that here we have used the $\underline{t}$ matrix which transfers the amplitudes from right to left. Using Eqs. (4.4) and (4.9)—(4.11) yields an equation for β_1',β_1 alone,

$$\underline{\Pi}\begin{bmatrix}\beta_1'\\ \beta_1\end{bmatrix}=-\frac{\sqrt{\epsilon}}{b}\begin{bmatrix}b-a\\ -1\end{bmatrix}\ , \tag{4.12}$$

with

$$\underline{\Pi}=(\underline{t}_l e^{-i\theta_2}\underline{t}_2'\underline{t}_l e^{-i\theta_1}\underline{t}_1-\underline{\mathbb{1}})\ . \tag{4.13}$$

The transmitted amplitude is found by eliminating γ_2 in Eq. (4.1) with the help of Eq. (4.2). This yields

$$\alpha_2'=\frac{\sqrt{\epsilon}}{b}[(b-a)\beta_2+\beta_2']\ . \tag{4.14}$$

With Eqs. (4.10) and (4.12) we obtain

$$\alpha_2'=-\frac{\epsilon}{b^2}e^{-i\theta_1}\frac{h}{\det(\underline{\Pi})}\ , \tag{4.15}$$

where

$$h=\det(\underline{\Pi})(\pm1,1)\underline{t}_1\underline{\Pi}^{-1}\begin{bmatrix}\pm1\\ -1\end{bmatrix}\ . \tag{4.16}$$

Here we have used the relationship $b-a=b_\pm-a_\pm=\pm1$ [see Eqs. (3.4) and (3.5)]. The transmission probability of the ring is thus given by

$$T(E,\Phi,\epsilon)=|\alpha_2'|^2=\frac{\epsilon^2}{b^4}\frac{|h|^2}{|\det(\underline{\Pi})|^2}\ . \tag{4.17}$$

The resonant behavior of the transmission probability is determined by the poles of the transmission amplitude, Eq. (4.15). The poles of Eq. (4.15) are the solutions of

$$\det[\underline{\Pi}(E,\Phi,\epsilon)]=0\ . \tag{4.18}$$

In Secs. IV B—D we will discuss various limits of Eq. (4.18).

B. Weak coupling limit

For $\epsilon=0$ the current leads and the ring are decoupled. We have $a=a_\pm=0$, $b=b_+=-b_-=1$, and thus

$$\underline{t}_l=\pm\underline{\mathbb{1}}\ . \tag{4.19}$$

In this case

$$\det[\underline{\Pi}(\epsilon=0)]=\det(e^{-i2\pi\Phi/\Phi_0}\underline{t}_2'\underline{t}_1-\underline{\mathbb{1}})\ , \tag{4.20}$$

and Eq. (4.18) is just the eigenvalue equation of the closed ring [Eq. (2.8)]. Thus the poles of the transmission amplitude are the eigenvalues of the closed ring. Therefore, for $\epsilon=0$, the denominator of Eq. (4.17) is proportional to $[E-E_n(\Phi)]^2$ for $E\approx E_n$. For small ϵ we can expand Eq. (4.12) [Eq. (4.17)] in powers of ϵ ($\sqrt{\epsilon}$). To lowest order we obtain

$$T(E,\Phi,\epsilon)=\frac{\epsilon^2}{b^4}\frac{|h(\epsilon=0)|^2}{|\det[\underline{\Pi}(\epsilon=0)]|^2}\ . \tag{4.21}$$

Since $|\det[\underline{\Pi}(\epsilon=0)]|^2\propto(E-E_n)^2$ for $E\approx E_n$, Eq. (4.21) is singular at the eigenvalues of the closed ring. As we increase the coupling ϵ, the solutions of Eq. (4.18) move away from the real axis into the complex energy plane and are of the form

$$E_n(\Phi)+\Delta E_n(\Phi,\epsilon)+i\Gamma_n(\Phi,\epsilon) \tag{4.22}$$

giving rise to a Breit-Wigner expression for the transmission probability. Since the numerator in Eq. (4.17) is to lowest order in ϵ proportional to ϵ^2, comparison with the Breit-Wigner formula, Eq. (1.4), shows that $\Gamma_n \propto \epsilon$. $\Gamma_n/\hbar$ is the probability per unit time of an electron in an eigenstate of the closed ring to leave this state (i.e., the ring). In the weak coupling limit the lifetime of an electron in an eigenstate of the ring is thus proportional to $\hbar/\Gamma_n \sim 1/\epsilon$. Below we find that the shift ΔE_n away from E_n is an order of magnitude smaller than Γ_n and given by $\Delta E_n \propto \epsilon^2$ to lowest order in ϵ. Expansion of Eq. (4.17) to higher powers in ϵ removes these singularities of Eq. (4.21) and yields instead a transmission probability T_{res} of order 1 for $E \cong E_n$.

C. Transition from weak to strong coupling

If the coupling of the link to the ring is increased, the width Γ_n and the shift ΔE_n will generally increase and the intensity of the resonances will decrease. We illustrate this behavior by studying the simple example of the perfect symmetric ring, without scattering on the branches, $t_1=t_2=e^{i\phi_s}$, $r_1=r_2=r_1'=r_2'=0$, $\theta_1=\theta_2=\theta=\pi\Phi/\Phi_0$. The spectrum of the closed perfect ring, using Eq. (2.8), is

$$\cos^2\phi_s=\cos^2\theta \ . \tag{4.23}$$

This is the free-electron spectrum indicated in Fig. 1 by dashed lines. The transmission amplitude found from Eq. (4.15) is given by

$$\alpha_2'(\phi_s,\Phi,\epsilon) = \frac{i\epsilon\sin\phi_s(1+e^{2i\theta})}{a^2+b^2\cos2\theta-(1-\epsilon)\cos2\phi_s+i\epsilon\sin2\phi_s} \ , \tag{4.24}$$

and yields a transmission probability

$$T(\phi_s,\Phi,\epsilon)=\frac{4\epsilon^2\sin^2\phi_s\cos^2\theta}{[a^2+b^2\cos2\theta-(1-\epsilon)\cos2\phi_s]^2+\epsilon^2\sin^2 2\phi_s} \ . \tag{4.25}$$

Equation (4.25) is illustrated in Fig. 4. Note that Eq. (4.22) is valid for both branches of $\underline{S}$ matrices given by Eqs. (3.4) and (3.5). For this case Eq. (4.18), i.e., the denominator of Eq. (4.24), is zero for $\phi_s=\phi_r+i\phi_i$. The real and imaginary part of the phase are determined by

$$\cos^2\phi_r=\frac{1-\epsilon+\sqrt{1-2\epsilon}}{2\sqrt{1-2\epsilon}}\cos^2\theta \ , \tag{4.26}$$

$$\phi_i=\tfrac{1}{4}\ln(1-2\epsilon) \ , \tag{4.27}$$

for $\epsilon<\epsilon^*(\theta)$. Here ϵ^* is the maximum coupling strength such that the right-hand side of Eq. (4.26) is smaller than 1 and this yields

$$\epsilon^*(\theta)=\frac{2\,|\sin\theta\,|}{\cos^4\theta}(1-|\sin\theta\,|)^2 \ . \tag{4.28}$$

Note that the argument of the ln in Eq. (4.27) is given by the probability $(a+b)^2=1-2\epsilon$ that an electron in the current lead is reflected at the junction [see Eq. (3.2)]. The larger the transmissivity of the junction, i.e., the coupling between the wire and the ring, the larger is the imaginary part of the pole. For a coupling strength ϵ exceeding ϵ^*, the denominator of Eq. (4.24) is zero for

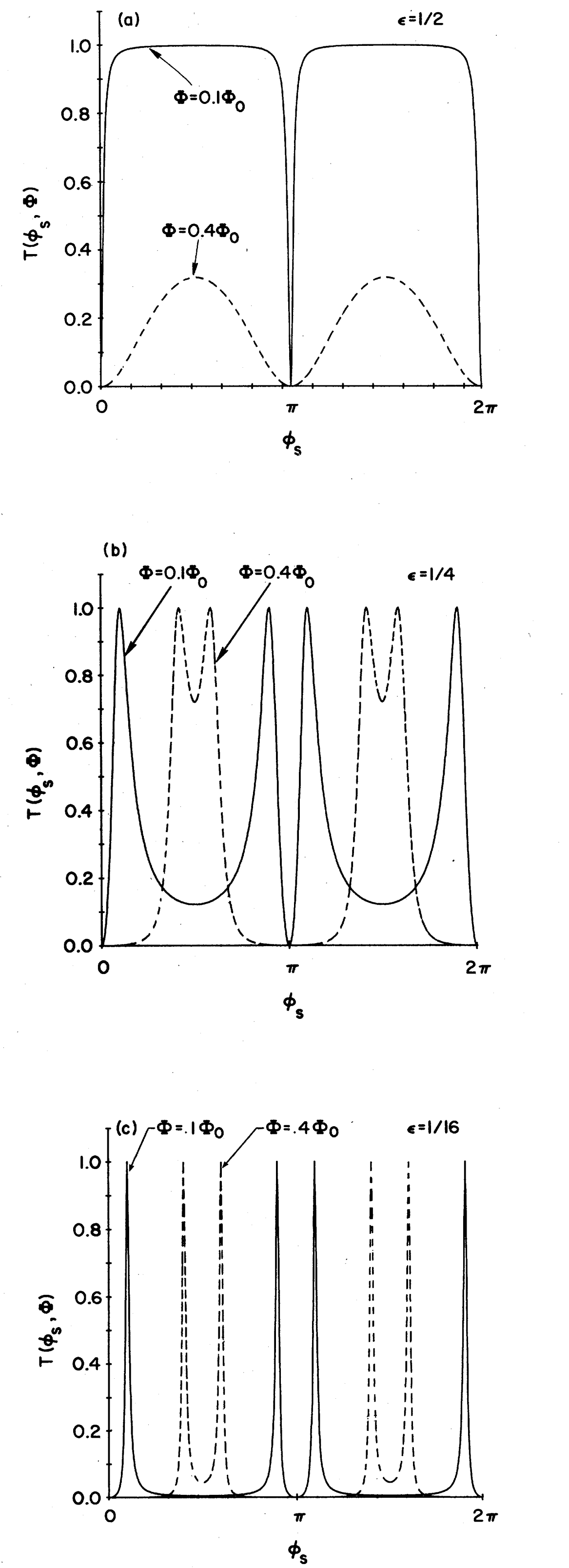

FIG. 4. Transmission probability of a symmetric ring with no elastic scattering on its branches in the presence of a flux $\Phi=0.1\Phi_0$ and $\Phi=0.4\Phi_0$ for different degrees of coupling between the current leads and the ring. (a) The strong coupling limit $\epsilon=\frac{1}{2}$, (b) $\epsilon=\frac{1}{4}$, (c) $\epsilon=\frac{1}{16}$.

$$\phi_r = n\pi \,, \tag{4.29}$$

$$\phi_{i,\pm} = \tfrac{1}{2}\ln[\,(a^2+b^2\cos 2\theta) \pm\sqrt{(a^2+b^2\cos2\theta)^2-(1-2\epsilon)}\,] \,. \tag{4.30}$$

The behavior of these solutions as a function of ϵ is illustrated in Fig. 5. For small ϵ, Eqs. (4.26) and (4.27) can be expanded in ϵ,

$$\phi_r = \theta - \frac{1}{8}\frac{\cos\theta}{\sin\theta}\epsilon^2 + O(\epsilon^3) \,, \tag{4.31}$$

$$\phi_i = -\tfrac{1}{2}[\epsilon+\epsilon^2+O(\epsilon^3)] \,. \tag{4.32}$$

The poles move away from the real axis proportional to ϵ. Poles associated with eigenvalues θ in the interval $0,\pi/2$ move toward $\phi_r=0$ proportional to ϵ^2. Poles associated with eigenvalues θ in the interval $\pi/2,\pi$ move toward $\phi_r=\pi$. The real part of the pole of the eigenvalue $\theta=\pi/2$ is independent of ϵ. To order ϵ^2 in the numerator and the denominator, Eq. (4.25) is given by

$$T(\phi_s,\theta,\epsilon) = \frac{\epsilon^2\sin^2 2\theta}{(2\sin 2\theta\,\Delta\phi)^2+\epsilon^2\sin^2 2\theta} \,, \tag{4.33}$$

where $\Delta\phi=\phi_s-\theta$. At resonance $\Delta\phi=0$, the transmission probability is $T_{\text{res}}=1-O(\epsilon^2)$. As ϵ increases further, the poles move further away from the real axis. At $\epsilon=\epsilon^*$ two poles coalesce and their real part is located at $\phi_r=n\pi$. For $\epsilon>\epsilon^*$, the poles are far away from the eigenvalues of the closed ring (see Fig. 5) and the transmission probability can no longer be related in a simple way to the energy spectrum of the closed ring. For later reference we note here only that in the strong coupling limit $\epsilon=\frac{1}{2}$, Eqs. (4.29) and (4.30) yield one pole at $\phi_r=n\pi$, with a finite imaginary part

$$\phi_{i,+} = \tfrac{1}{2}\ln\cos^2\theta \,. \tag{4.34}$$

and a pole whose imaginary part,

$$\phi_{i,-} = \tfrac{1}{2}\ln\left[\frac{1-2\epsilon}{2\cos^2\theta}\right] , \tag{4.35}$$

tends to infinity as ϵ approaches $\frac{1}{2}$. In Sec. III D we investigate how elastic scattering in the two branches of the ring modifies this picture.

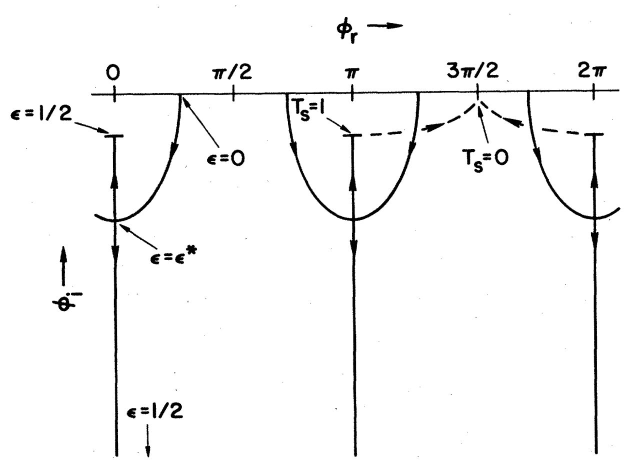

FIG. 5. Poles of the transmission amplitude in the complex ϕ_s plane. The full lines give the poles of the perfect symmetric ring (Sec. IV C) for $\theta=\pi/4,(\Phi=\Phi_0/4)$, as a function of the coupling parameter ϵ. In the zero coupling limit $\epsilon=0$, the poles are on the real axis corresponding to eigenstates of the closed ring. With increasing ϵ, the poles move away from the real axis. For ϵ^*, two poles coalesce. For $\epsilon>\epsilon^*$, the real part of the poles is $\phi_r=n\pi$ and one pole moves deeper into the complex plane, whereas the other one moves toward the real axis as ϵ increases. $\epsilon=\frac{1}{2}$ indicates the strong coupling limit. The dashed lines give the poles of the symmetric ring with elastic scattering in the two branches (Sec. IV D) as a function of the transmission probability T_s of the scatterers. The arrow points in the direction of increasing elastic scattering (decreasing transmission probability T_s).

D. Strong coupling limit with scattering

Sharp resonances in the strong coupling limit do occur if the states of the closed ring are sufficiently localized. Such states are rather insensitive to the perturbation caused by coupling the current leads to the ring. We demonstrate this by studying again a simple example and consider a symmetric ring with two equal scatterers: $t_1=t_2=T_s^{1/2}e^{i\phi_s}$, $r_1=r_2=r_1'=r_2'=e^{-i\pi/2}R_s^{1/2}e^{i\phi_s}$, $R_s=1-T_s$, and $\theta_1=\theta_2=\theta$. For this case Eq. (2.8) becomes

$$\cos^2\phi_s = T_s\cos^2\theta \,. \tag{4.36}$$

For $T_s=1$, Eq. (4.36) is identical with the free-electron spectrum given by Eq. (4.23) and shown in Fig. 1. For $T_s<1$, the spectrum obtained from Eq. (4.36) exhibits gaps at $\theta=n\pi(k=0)$. However, since the ring with two equal scatterers corresponds to a "crystal" with two identical "atoms" per unit cell, no gaps appear at the Brillouin zone boundary, $\pm k_0/2$, or $\theta=\pm\pi/2(2n+1)$. Here, we are interested in the limit of strong scatterers, $T_s\ll1$. In this case the gaps at $k=0$ are large and of the order of the level spacing. For $T_s\ll1$, the solutions of Eq. (4.36) are

$$\phi_E = (2n+1)\pi/2 \pm \Delta\phi_E \,, \tag{4.37}$$

where

$$\Delta\phi_E = T_s^{1/2}\cos\theta + \tfrac{1}{6}T_s^{3/2}\cos^3\theta + O(T_s^{3/2}) \,. \tag{4.38}$$

Equation (4.15) yields an amplitude for the transmitted wave

$$\alpha_2' = \frac{T_s^{1/2}(\sin\phi_s \pm R_s^{1/2})\cos\theta}{(\sin\phi_s \pm R_s^{1/2}) + \frac{1}{2}iT_s e^{i\phi_s}\sin^2\theta} \tag{4.39}$$

giving rise to a transmission probability

$$T(\phi_s,\theta)=\frac{T_s(\sin\phi_s \pm R_s^{1/2})^2\cos^2\theta}{(\sin\phi_s \pm R_s^{1/2})^2 - T_s\sin\phi_s\sin^2\theta(\sin\phi_s \pm R_s^{1/2}) + \frac{1}{4}T_s^2\sin^4\theta} . \tag{4.40}$$

In Eqs. (4.39) and (4.40) the upper signs are obtained by using the $\underline{S}$ matrix specified by $a=a_+=-\frac{1}{2}$, $b=b_+=\frac{1}{2}$, and the lower signs from the $\underline{S}$ matrix given by $a=a_-=\frac{1}{2}$, $b=b_-=-\frac{1}{2}$. For simplicity we will concentrate on the first case (upper signs). Note that in the limit $T_s=1$, Eqs. (4.39) and (4.40) are identical with Eqs. (4.24) and (4.25) for $\epsilon=\frac{1}{2}$.

To find the resonances of Eq. (4.40), we have to search for the zeros of Eq. (4.18), i.e., of the denominator of Eq. (4.39). We find for the real part,

$$\sin\phi_r = -\frac{R_s^{1/2}}{(1-T_s\sin^2\theta)^{1/2}} , \tag{4.41}$$

and for the imaginary part

$$\phi_i = \tfrac{1}{2}\ln(1-T_s\sin^2\theta) . \tag{4.42}$$

For $T_s=1$, the poles determined by Eqs. (4.41) and (4.42) are the same as the poles given by Eq. (4.34). For $T_s=1$, the real part of these poles is $\phi_r=n\pi$. As the elastic scattering is increased, T_s decreased, the real part of the pole with $\phi_r=\pi$ at $T_s=1$ increases to $\phi_r=3\pi/2$ at $T_s=0$. The pole with $\phi_r=2\pi$ at $T_s=1$ decreases to $\phi_r=3\pi/2$ at $T_s=0$ (see Fig. 5). The imaginary part of these poles decreases with increasing scattering and becomes zero in the limit $T_s=0$. For $T_s \ll 1$, the solutions of Eqs. (4.41) and (4.42) are given by

$$\phi_{r,\pm} = \tfrac{3}{2}\pi + 2\pi n \pm [\, T_s^{1/2}\cos\theta + \tfrac{1}{6}T_s^{3/2}(1+2\sin^2\theta) + O(T_s^{5/2})] , \tag{4.43}$$

$$\phi_i = -\tfrac{1}{2}[T_s\sin^2\theta + \tfrac{1}{2}T_s^2\sin^4\theta + O(T_s^3)] . \tag{4.44}$$

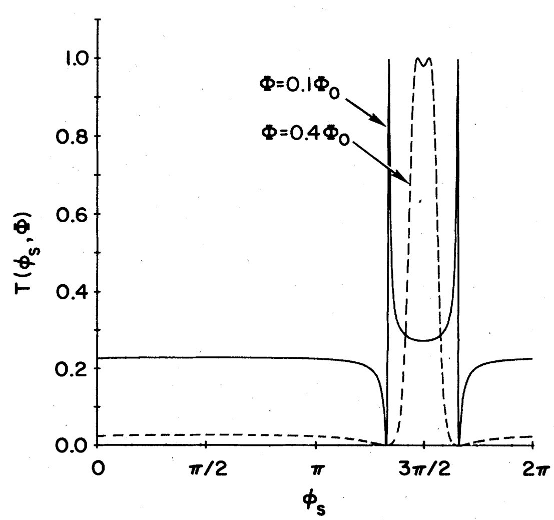

FIG. 6. Transmission probability of a symmetric ring with equal elastic scattering in both branches in the presence of a flux $\Phi=0.1\Phi_0$ and $\Phi=0.4\Phi_0$. Here the scatterers have a transmission probability $T_s=0.25$.

To order $T_s^{1/2}$, Eq. (4.43) is identical with the solution of the eigenvalue equation, Eq. (4.37). The difference, $\phi_r=\phi_{E_n}$, with n odd, i.e., the energy shift ΔE, is proportional to $T_s^{3/2}$. For $\phi_s \approx 3\pi/2+2\pi n$, the denominator of Eq. (4.39) is thus of the form $(\phi_s-\phi_{r,+}-i\phi_i)(\phi_s-\phi_{r,-}-i\phi_i)$. To order T_s^2, Eq. (4.40) exhibits the resonant structure

$$T=\frac{T_s\cos^2\theta(\Delta\phi - T_s^{1/2})^2(\Delta\phi+T_s^{1/2})^2}{[(\Delta\phi - T_s^{1/2}\cos\theta)^2+\Gamma^2][(\Delta\phi+T_s^{1/2}\cos\theta)^2+\Gamma^2]} , \tag{4.45}$$

where $\Delta\phi=\phi_s-3\pi/2-2\pi n$ and $\Gamma=\frac{1}{2}T_s\sin^2\theta$ is the width of the resonances. The resonances are very narrow for $\theta=\pi n$, i.e., in the center of the Brillouin zone, and broadest for fluxes $(n+\frac{1}{2})\phi_0$, i.e., $\theta=(n+1/2)\pi$, at the boundary of the Brillouin zone. The resonant structure of Eq. (4.45) is shown in Fig. 6. At resonance, the transmission probability is given by

$$T_{\rm res}=\frac{16\cos^2\theta}{16\cos^2\theta+T_s\sin^4\theta} . \tag{4.46}$$

Since $T_s \ll 1$, $T_{\rm res}$ is close to 1 except near the boundary of the Brillouin zone. Thus for the $\underline{S}$ matrix given by $a=a_+=-\frac{1}{2}$, $b=b_+=\frac{1}{2}$, the eigenstates near $\phi_s=3\pi/2+2\pi n$ exhibiting nodes at the links give rise to sharp resonances, whereas the eigenstates near $\phi_s=\pi/2+2\pi n$ do not give rise to a recognizable feature in the transmission probability. For the $\underline{S}$ matrix given by $a=a_-=\frac{1}{2}$, $b=b_-=-\frac{1}{2}$, the eigenstates near $\phi_s=\pi/2+2\pi n$ exhibiting maxima at the links give rise to sharp resonances, but the eigenstates near $\phi_s \cong 3\pi/2+2\pi n$ do not give rise to resonances. This example demonstrates that strong elastic scattering can keep certain poles near the real axis giving rise to sharp resonances. The complete discrimination between the two classes of eigenstates appears to be an artifact of the limiting case $\epsilon \to \frac{1}{2}$ and the symmetry of this example.

V. CONCLUSIONS

We have shown that when sharp resonances exist in the transmission probability they are associated with the electronic eigenstates E_n of the closed ring. The periodic dependence[1] of E_n on Φ in the closed ring is then related, via the resonances, to the oscillations in the transmission coefficient (and hence to the oscillations in the conductance suggested in Ref. 2). We have identified two mechanisms which give rise to sharp resonances. If the ring is poorly coupled to the leads, an electron entering the ring will spend a long time in the ring before being reflected or transmitted. Electrons which spend a long time[7,8] in the ring must necessarily be in an eigenstate of the closed ring. Only in this case is interference constructive and the probability to find the electron in the ring large. The second mechanism which we have identified is strong elastic scattering. The small transmission probability of these scatterers leads to multiple reflection of the electron in the ring and constructive interference is again only possible in the case that the electron has an energy E_n. Both in the case of poor coupling and large elastic scattering the nonresonant transmission will also show oscillations as a function of the flux Φ. Since, however, the nonresonant transmission probability is very small compared to 1, these oscillations will also be small (see Figs. 4 and 6). In the case of strong coupling *and* weak elastic scattering, we could not relate the oscillations in the transmission probability of the ring to the eigenstates of the closed ring. In this case electrons traverse the ring without much scattering. The wires and the ring act in this case almost like "wave guides" and the oscillations in the transmission probability are closely related to the Aharonov-Bohm effect in a vacuum experiment.

The total resistance R_{el} [Eq. (1.2)] of the parallel quantum resistors $R_{1,\text{el}}=(\hbar\pi/e^2)(1-T_1)/T_1$ with $T_1=|t_1|^2$, $R_{2,\text{el}}=(\hbar\pi/e^2)(1-T_2)/T_2$ with $T_2=|t_2|^2$ [see Fig. 3(b)] is in general *not* given by the classical composition law for parallel resistors.[2] For the case of poor coupling, or the case of strong elastic scattering, this deviation is extremely marked, when the conditions for being on (or near) a resonance are satisfied, as is also the case for series addition of resistors.[8,9]

ACKNOWLEDGMENTS

The research at Tel-Aviv University was partially supported by the U.S.-Israel Binational Science Foundation (BSF), Jerusalem, Israel.

[1] M. Büttiker, Y. Imry, and R. Landauer, Phys. Lett. **96A**, 365 (1983).

[2] Y. Gefen, Y. Imry, and M. Ya. Azbel, Phys. Rev. Lett. **52**, 139 (1984).

[3] M. Ya. Azbel and P. Soven, Phys. Rev. B **27**, 831 (1983); M. Ya. Azbel, Solid State Commun. **45**, 527 (1983); Phys. Rev. B **27**, 3852 (1983).

[4] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1977).

[5] B. Shapiro, Phys. Rev. Lett. **50**, 747 (1983).

[6] The case of weak coupling of a current lead to a reservoir has been considered by H. L. Engquist and P. W. Anderson, Phys. Rev. B **24**, 1151 (1981).

[7] M. Büttiker and R. Landauer, Phys. Rev. Lett. **49**, 1739 (1983); M. Büttiker, Phys. Rev. B **27**, 6178 (1983).

[8] B. Ricco and M. Ya. Azbel, Phys. Rev. B **29**, 1970 (1984).

[9] R. Landauer, Philos. Mag. **21**, 863 (1970).