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Stationary, harmonic, and pulsed operations of an optically bistable laser with saturable absorber. II

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In the preceding paper we have analyzed the bifurcation diagram of the steady and time-periodic solutions of the lasers with saturable absorbers (LSA) equations. However, a study of the experimental results presented in the literature indicates that, in general, the control parameter is a slowly varying function of time. In this second paper we analyze the influence of this time dependence on the bifurcation diagram of the LSA. We show that the stability changes of the slowly varying steady-state solutions do not correspond to their bifurcation or limit points in the case where all parameters are constant. In particular, we show that the zero-intensity state can be stablized during a certain interval of time and that this stabilization can be controlled by the initial value of the time-dependent bifurcation parameter.

I. INTRODUCTION

This paper is the second of a pair of papers dealing with lasers with saturable absorbers (LSA). In the first paper[1] we have analyzed the LSA equations in a standard way. Assuming time-independent control parameters, we have studied numerically and analytically the bifurcation diagram of the LSA equations for a rather simple model (two-level atoms, homogeneous broadening, single running mode, semiclassical theory in the mean-field limit). We have shown the occurrence of solutions, periodic in time, which can be either pulsed solutions (describing passive Q switching) or harmonically modulated solutions of small amplitude.

However, a study of the experimental results presented in the literature[2–5] indicates that in general the control parameter is swept across the domain to be studied. The purpose of this second paper is to analyze the influence of this time dependence on the stability properties of the LSA. As we shall demonstrate, the effect of slowly varying parameters may have important consequences.

For systems governed by nonlinear ordinary differential equations—such as the LSA equations—the response due to slow variation in the bifurcation parameter has recently been examined.[6,7] Suppose, for example, that the system is initially in a slightly perturbed steady state, such as that corresponding to point A_0 in Fig. 1. If the bifurcation parameter A is slowly changing, it is expected that the system will, more or less, follow the branch of steady-state solutions $z=z_s(A)$ until a critical point (bifurcation or limit point) is reached. Then a transition to a new branch of solutions will occur. Following Haberman's study,[6] there exist, however, two different classes of transition problems: (i) $z=z_s(A)$ is no more an exact solution of the evolution equations when $A=A(\epsilon t)$ where $\epsilon<1$, or (ii) $z=z_s(A)$ is still an exact solution of the equations when $A=A(\epsilon t)$. In the first case, Haberman has shown that the system quickly adopts a slowly varying regime of the form $z=z_s(A(\epsilon t))+O(\epsilon)$ which is linearly stable if $z_s(A)$ with A constant is linearly stable. It is only in a small neighborhood of the critical point $A=A_C$ that a different development of the solution may occur [Fig. 1(b)]. In the second case, however, the critical point $A=A_C$ does *not* correspond to the breakdown of the slowly varying solu-

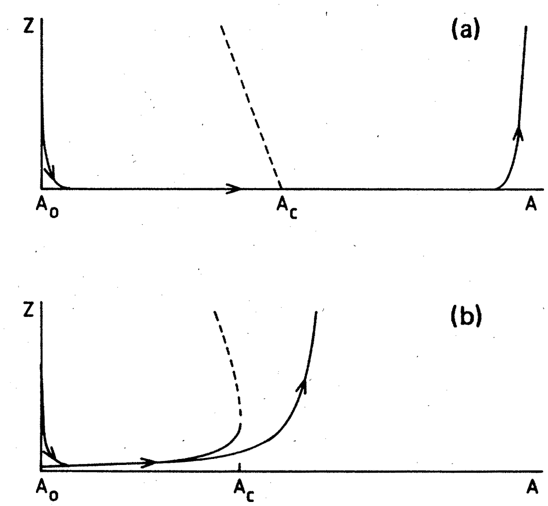

FIG. 1. Slow passage through criticality. Solid and dashed curves represent the branches of stable and unstable steady sates, respectively $[z=z_s(A)]$. The curve with arrows represents the time-dependent evolution of $z(t)$ when $A=A_0+\epsilon t$ and $\epsilon<1$. $A=A_C$ corresponds either to (a) a steady bifurcation point or (b) a limit point.

tion $z=z_s(A(\epsilon t))$ whose stability properties are very different from those of $z=z_s(A)$ with a time-independent A [as the zero solution in Fig. 1(a)]. Since the zero-intensity solution of the LSA equations remains an exact solution of these equations when parameters are slowly varying, we must analyze its stability in detail.

As a tutorial example which nevertheless captures the essentials of this class of problems, let us consider the simple first-order nonlinear differential equation

$$z_t=zA(t)+z^2,\quad z(0)=z_i \tag{1.1}$$

where

$$A(t)=A_0+\alpha t,\quad A_0<0,\ \alpha>0\ . \tag{1.2}$$

When A is constant the linear-stability analysis of $z=0$ indicates that $A=0$ corresponds to a steady bifurcation point. When $\alpha\neq 0$, $z=0$ is still an exact solution of (1.1) and (1.2). In order to determine its linear stability, or more precisely to describe $z(t)$ when z_i is small, we analyze the linearized equation, which is

$$z_t=zA(t),\quad z(0)=z_i\ . \tag{1.3}$$

Equation (1.3) has the following solution:

$$z(t)=z_i\exp\left[\int_0^t A(s)ds\right]. \tag{1.4}$$

It is clear from (1.4) that $z(t)$ will grow exponentially when $t>t^*$, where t^* is defined by

$$\int_0^{t^*} A(s)ds=0\ . \tag{1.5}$$

Moreover, using (1.2) we observe from (1.5) that $A(t^*)>0$ is always larger than the steady bifurcation point reached for $t=\bar{t}$, i.e., $A(\bar{t})=0$. By rewriting (1.5) as

$$\int_0^{\bar{t}} A(s)ds+\int_{\bar{t}}^{t^*} A(s)ds=0\ , \tag{1.6}$$

we note that (1.6) expresses a balance between the "stability" accumulated from 0 to $\bar{t}$ [where $A(t)\leq 0$] and the "instability" produced from $\bar{t}$ to t^* [where $A(t)\geq 0$]. Hence the condition (1.5) for marginal stability will depend critically on the rates of damping and divergence of the system. This justifies that we refer to these new instabilities as dynamical instabilities as opposed to adiabatic instabilities (corresponding here to the condition $A_t=0$). A new feature brought in by the dynamical instabilities is a dependence on the initial value of A ($A=A_0$). Indeed, for our particular choice [(1.1) and (1.2)], we find from (1.5) that

$$A(t^*)=-A_0\ . \tag{1.7}$$

Thus, if we increase $|A_0|$, we increase $A(t^*)$, whatever α, the rate of change of $A(t)$. The transition to a new state may be considerably delayed if $|A_0|$ is sufficiently large. Our principal purpose is to show that a similar phenomenon may appear in the more complicated LSA equations.

This paper is divided into three sections. In Sec. II we discuss analytically the dynamical stability of the trivial solution (zero intensity) of the LSA where the first instability may be either a steady or a Hopf bifurcation. In particular, we shall concentrate on the cases leading to similar conclusions as the study of Eqs. (1.1) and (1.2). In Sec. III we show that the validity of our results strongly depends on the size of imperfections which are always present in the real LSA problem. Finally, Sec. IV presents numerical results obtained by integrating the eight LSA equations.

II. DYNAMICAL STABILITY OF THE ZERO-INTENSITY STATE

The time evolution of the LSA can be examined by studying the semiclassical equations.[1] These equations admit a trivial solution corresponding to the zero-intensity state. Following Ref. 1, its linear stability can be determined by the solutions of the following three ordinary differential equations:

$$\begin{aligned} x'&=-x+Av+\bar{A}\bar{v}\ ,\\ v'&=d(-v+x)\ ,\\ \bar{v}'&=\bar{d}(-\bar{v}+x)\ , \end{aligned} \tag{2.1}$$

where $'$ denotes the derivative with respect to the time t. The three variables x and $v(\bar{v})$ are related to the electric field amplitude and the polariation of the amplifying (absorbing) atoms. A ($\bar{A}$) and d ($\bar{d}$) correspond to the pump parameters and the atomic decay rates of the amplifying (absorbing) atoms. A and $\bar{A}$ are the control parameters. In this paper we consider the initial conditions

$$x(0)=\eta\ll 1,\ v(0)=\bar{v}(0)=0\ . \tag{2.2}$$

Our purpose is to analyze Eqs. (2.1) and (2.2) under the condition that

$$A=A(\tau)=A_0+\alpha\tau\ , \tag{2.3}$$

where τ is a slow time defined by

$$\tau=\epsilon t,\quad 0<\epsilon\ll 1\ . \tag{2.4}$$

To this end, it is mathematically convenient to consider the third-order equation in x instead of (2.1). This equation can be obtained by successively eliminating the variables v and $\bar{v}$ from (2.1). Then, after introducing the new time variable (2.4), we obtain

$$\begin{aligned} &\epsilon^3x'''+\epsilon^2x''[d+\bar{d}+1+O(\epsilon)]\\ &\quad+\epsilon x'[\bar{d}-Ad-\bar{A}\bar{d}+d\bar{d}+d+O(\epsilon)]\\ &\quad+x[d\bar{d}(1-A-\bar{A})+O(\epsilon)]=0\ , \end{aligned} \tag{2.5}$$

where x is subject to the conditions

$$\begin{aligned} &x(0)=\eta,\quad \epsilon x'(0)=-\eta\ ,\\ &\epsilon^2x''(0)=\eta(1+A_0d+\bar{A}\bar{d}) \end{aligned} \tag{2.6}$$

and $'$ now denotes the time derivative with respect to τ. The $O(\epsilon)$ quantities in Eq. (2.5) represent terms proportional to A' or A''. Equations (2.5) and (2.6) can be solved by the WKB method. They admit the solution

$$x(\tau,\epsilon)=\sum_{j=1}^{3}c_j\exp\frac{1}{\epsilon}\left[\int_0^{\tau}\omega_j(s)ds+O(\epsilon)\right], \tag{2.7}$$

where $\omega=\omega_j$ $(j=1,2,3)$ satisfies the characteristic equation

$$\omega^3+\omega^2(d+\bar{d}+1)+\omega(\bar{d}-Ad-\bar{A}\bar{d}+d\bar{d}+d)+d\bar{d}(1-A-\bar{A})=0 \quad (2.8)$$

with $A=A(\tau)$ given by (2.3). The unknown amplitudes c_j must be determined by the initial conditions (2.6). Assuming that $\mathrm{Re}(\omega_j)<0$ at $\tau=0$, it is clear from (2.7) that x will decay (grow) exponentially when $\tau<\tau^*$ $(\tau>\tau^*)$ as soon as one of the roots of (2.8) verifies

$$\int_0^{\tau^*}\omega_j(s)ds=0\ . \quad (2.9)$$

The expression (2.9) represents the new condition for marginal stability. We now concentrate on Eq. (2.8) and determine its solutions. Two different situations must be considered.

(1) When A is a constant, the first bifurcation point is a Hopf bifurcation point $A=A_H$ to time-periodic solutions (i.e., $\omega=\pm i\Omega$).

(2) When A is constant, the first bifurcation point corresponds to a steady bifurcation point $A=A_S$ (i.e., $\omega=0$).

We shall examine each case separately.

A. Hopf bifurcation

Defining $\mu>0$ by

$$\mu\equiv(\bar{A}-\bar{A}_c)/\bar{A}_c\ ,\quad \bar{A}_c\equiv\bar{d}(1+d)/(\bar{d}-d),\quad d>\bar{d} \quad (2.10)$$

the Hopf bifurcation point is located at

$$A=A_H\equiv A_c(1+\mu\bar{d}^2/d^2),\quad A_c\equiv d(1+\bar{d})/(d-\bar{d})\ . \quad (2.11)$$

$A=A_H$ corresponds to the first bifurcation, i.e., the basic state $x=v=\bar{v}=0$ is stable (unstable) when $A<A_H$ $(A>A_H)$. We now consider $A=A(\tau)$, given by

$$A=A_H+\lambda(\tau)\ , \quad (2.12)$$

where $\lambda(\tau)$ is obtained from (2.3) and is given by

$$\lambda(\tau)=(A_0-A_H)+\alpha\tau\ . \quad (2.13)$$

Then using (2.10)–(2.12), the characteristic equation (2.8) transforms into

$$\omega^3+(1+d+\bar{d})\omega^2+(\mu\bar{d}^2-\lambda d)\omega+\mu\bar{d}^2(1+d+\bar{d})-\lambda d\bar{d}=0\ . \quad (2.14)$$

Our analysis of Eq. (2.14) will be asymptotic, depending on the smallness of the two parameters λ and μ. If $\mu=O(1)$ and $|\lambda(\tau)|<1$, the coefficients of (2.14) are in first approximation independent of λ and there is no effect of the time variation of A. On the contrary, if $\mu<1$ and $\lambda(\tau)=\mu\Lambda(\tau)+O(\mu^2)$, or equivalently if

$$A_0-A_H=\mu A_1+O(\mu^2),\quad \alpha=\mu\alpha_1+O(\mu^2) \quad (2.15)$$

and thus $\Lambda(\tau)=A_1+\alpha_1\tau$, the three solutions of (2.14) are given by

$$\omega_1=-(1+d+\bar{d})+O(\mu)<0\ , \quad (2.16)$$

$$\omega_2=+i\mu^{1/2}|\Omega|+\mu\frac{d(1+\bar{d})\Lambda(\tau)}{2(1+d+\bar{d})^2}+O(\mu^{3/2})\ ,$$
$$\omega_3=-i\mu^{1/2}|\Omega|+\mu\frac{d(1+\bar{d})\Lambda(\tau)}{2(1+d+\bar{d})^2}+O(\mu^{3/2}) \quad (2.17)$$

if

$$\Omega^2(\tau)\equiv\bar{d}^2-\frac{d\bar{d}\Lambda(\tau)}{1+d+\bar{d}}>0 \quad (2.18)$$

or

$$\omega_2=+\mu^{1/2}\left[\frac{d\bar{d}\Lambda(\tau)}{1+d+\bar{d}}-\bar{d}^2\right]^{1/2},$$
$$\omega_3=-\mu^{1/2}\left[\frac{d\bar{d}\Lambda(\tau)}{1+d+\bar{d}}-\bar{d}^2\right]^{1/2} \quad (2.19)$$

if

$$\bar{d}^2-\frac{d\bar{d}\Lambda(\tau)}{1+d+\bar{d}}<0\ .$$

At $\tau=0$ (2.18) is verified since $\Lambda=A_1<0$, and assuming that $\Omega^2(\tau)$ remains positive, we conclude from (2.17) that x will grow in amplitude if

$$\int_0^{\tau}\Lambda(s)ds>0\ . \quad (2.20)$$

Otherwise, we must study the transition problem when Ω changes its sign, i.e, we have a turning-point problem in the WKB method. We do not analyze this problem here. We obtain the critical time τ^* and $\Lambda(\tau^*)$ from the condition

$$\int_0^{\tau^*}\Lambda(s)ds=0\ ,$$

i.e.,

$$\tau^*=-2\frac{A_1}{\alpha_1},\quad \Lambda(\tau^*)=-A_1\ . \quad (2.21)$$

Then from (2.12) and (2.15) we note that the exponential growth will appear at a critical deviation $A(\tau^*)-A_H$ which, in first approximation, is equal to the distance A_H-A_0. In other words, for larger values of $|A_0|$ (the initial position of A), we obtain a larger value of $A(\tau^*)$.

B. Steady bifurcation

When μ, defined by (2.10), is negative, the first bifurcation point of the zero-intensity state is a steady bifurcation point given by

$$A=A_s\equiv A_c-\mu\bar{A}_c,\quad \mu<0\ . \quad (2.22)$$

In order to analyze the effect of this steady bifurcation, we first redefine A as

$$A=A_s+\lambda(\tau)\ , \quad (2.23)$$

where $\lambda(\tau)$ is obtained from (2.3) and is given by

$$\lambda(\tau)=(A_0-A_s)+\alpha\tau\ . \quad (2.24)$$

Then the characteristic equation (2.8) can be rewritten as

$$\omega^3+(1+d+\bar{d})\omega^2+[-\mu\bar{d}(1+d)-d\lambda]\omega-d\bar{d}\lambda=0 . \quad (2.25)$$

Again, our analysis of (2.25) will be asymptotic, depending on the smallness of two parameters μ and λ. If $\mu=O(1)$ and $|\lambda(\tau)|<1$, the solutions of (2.25) can be found without difficulty. Defining ν and expanding α as

$$\nu\equiv A_s-A_0<1,\quad \alpha=\nu\alpha_1+O(\nu^2) , \quad (2.26)$$

we find that $\lambda=\nu\Lambda+O(\nu^2)$ where $\Lambda(\tau)=-1+\alpha_1\tau$. Using these results we obtain the solutions of (2.25):

$$\omega_1=O(1)<0,\quad \omega_2=O(1)<0 , \quad (2.27)$$

$$\omega_3=-\nu\frac{d\Lambda(\tau)}{\mu(1+d)}+O(\nu^2) . \quad (2.28)$$

Thus x will grow in amplitude when the integral of ω_3 becomes positive. This leads to the condition (2.20). Examining (2.20) with the new expression for $\Lambda(\tau)$ gives the same result as for the previous case: $A(\tau^*)-A_s$ is in first approximation equal to A_s-A_0. If now $\mu=O(\nu)$, a similar asymptotic analysis of the solutions of (2.25) indicates that the bifurcation point $A=A_s$ is a turning point between a stable (and oscillatory) slowly varying state and an unstable state. We do not analyze this turning-point problem.

Hence we see that the nature of the bifurcation point drastically modifies the dynamical response of the system. When

$$|\mu|=O(1),\quad |\lambda(\tau)|\ll 1 ,$$

there will be a significant delay for a stationary bifurcation and no delay, in first approximation, for a Hopf bifurcation. But when

$$|\mu|\ll 1,\quad |\lambda(\tau)|=O(|\mu|) ,$$

there is a significant delay for a Hopf bifurcation and a very small delay for a steady bifurcation. In both cases a significant delay means $A(t^*)-A_b=A_b-A_0$ in first approximation where $A_b=A_s$ or A_H.

III. IMPERFECTIONS

In a real LSA experiment, the sharp transitions corresponding to a bifurcation rarely occur. Small imperfections tend to smooth these transitions. In the LSA the imperfections are associated with spontaneous emission, noise of different origins, impurities, or other inhomogeneities. They are particularly complex to describe and their experimental study is difficult. Fortunately, the effect of these imperfections is generally limited to the vicinity of the smoothed bifurcation points.[8] However, since the bifurcation point no longer corresponds to a change of stability of the slowly varying zero-intensity state, different conclusions can be expected with a time-dependent bifurcation parameter $A(t)$. In this section we present a general theory for imperfect transition problems. Our principal purpose is to study the influence of two small parameters: the first parameter characterizes the size of the imperfection and the second parameter represents the rate of change of $A(t)$. As we shall demonstrate, the behavior of the system critically depends on the relative magnitude of these two quantities.

To analyze the perturbation of bifurcations produced by small imperfections, we assume that the imperfect LSA problem can still be modeled by a system of ordinary differential equations of the form

$$\underline{z}_t=F(\underline{z},A(t))+\delta G(\underline{z},A(t)) . \quad (3.1)$$

In these equations $\underline{z}_t=F(\underline{z},A)$ represents the usual LSA equations obtained by the semiclassical theory. $\underline{z}=0$ corresponds to the zero-intensity solution and satisfies the condition

$$F(0,A(t))=0 . \quad (3.2)$$

Its dynamical stability was studied in Sec. II. $G(z,A(t))$ is an $O(1)$ quantity which represents the global effect of the imperfections. The new parameter $\delta>0$ characterizes the magnitude of these imperfections. We assume for the simplicity of the theory that $A=A_s$ is a steady bifurcation point of $z=0$. The analysis can, however, be applied when a Hopf bifurcation is the first bifurcation ($A=A_H$). Furthermore, we assume that the steady bifurcation is subcritical, as shown in Fig. 1(a). The condition for subcritical bifurcation in the LSA is given in Ref. 9. It is also the condition for optical bistability.

We first study the small-amplitude steady-state solutions when $\delta<1$ and A is a constant. They are sketched in Figs. 1(a) and 1(b) for $\delta=0$ and $\delta\neq 0$, respectively. When $\delta\to 0$ the perturbed branches $\underline{z}_s(A,\delta)$ approach the bifurcation branches $\underline{z}_s(A,0)$. By using the method of matched asymptotic expansions, we can find the steady-state solutions when $\delta<1$.[8] In the vicinity of $A=A_s$ they are approximated by the following expansions:

$$A-A_s=\delta^{2/3}A_1+O(\delta),$$
$$z_s(A,\delta)=\delta^{1/3}\beta\underline{u}+O(\delta^{2/3}) , \quad (3.3)$$

where $\underline{u}$ is the solution of the linearized equations (2.1) evaluated at $A=A_s$. The amplitude β is related to A_1 by

$$\beta(A_1+b\beta^2)+c=0 . \quad (3.4)$$

It is possible to give specific sufficient conditions on the operators F and G to imply (3.4). However, we shall not present them. In (3.4), which we call the imperfect bifurcation problem, the coefficients b and c are determined by inner products of the derivative of F and G on the mode $\underline{u}$. The assumption that the basic state $\underline{z}=0$ admits a subcritical bifurcation requires that $b>0$. On the other hand, c is positive or negative depending on the properties of $G(0,A_s)$. In this paper we only consider the case $c>0$. From the amplitude equation (3.4) we observe that there exist two different branches of steady positive solutions provided that $A_1<-\frac{3}{2}(2bc^2)^{1/3}$. Thus, there exists a limit point $A=A_L$ for the steady states defined by

$$A_L(\delta)=A_s-\delta^{2/3}\tfrac{3}{2}(2bc^2)^{1/3}+O(\delta) . \quad (3.5)$$

We now assume that $A=A(\epsilon t)$ ($\epsilon<1$) is a slowly varying bifurcation parameter. We choose $A(\epsilon t)$ to be a smooth monotonically increasing function of the form

$$A = A_0 + \epsilon t \ . \quad (3.6)$$

Our analysis thus involves two small parameters δ and ϵ. We shall consider the extreme cases: (i) $\delta \ll \epsilon$ and (ii) $\delta \gg \epsilon$. They lead to two different developments of the time-dependent solution.

Case (i): $\delta \ll \epsilon$. The analysis of this case indicates that the initial condition plays an important role. We assume that this condition is given by

$$\underline{z}(0,\delta,\epsilon) = \epsilon \underline{z}_i = O(\epsilon) \ . \quad (3.7)$$

Moreover, we also assume that $\delta = \epsilon^p \delta_p + O(\epsilon^{p+1})$ where $p \gg 1$. Then, we seek a regular expansion of the solution of the form

$$\underline{z}(t,\tau,\epsilon) = \epsilon \underline{z}_1(t,\tau) + \epsilon^2 \underline{z}_2(t,\tau) + O(\epsilon^3) \ . \quad (3.8)$$

After introducing (3.8) into (3.1), we find that the leading-order solution satisfies Eqs. (2.1) with $\underline{z}_1(0) = \underline{z}_i$, which is the linearized problem without imperfection studied in Sec. II. In Sec. II we have found that the unstable behavior appears above a critical value $A = A(t^*)$ and that $A(t^*)$ does not correspond to the bifurcation point $A = A_s$ but rather depends on the deviation $A_s - A_0$, where A_0 is the initial position of A. Furthermore, the analysis of the $O(\epsilon^2)$ problem for simpler equations exhibiting the same type of steady bifurcation[6] suggests that the expansion (3.8) becomes nonuniform only in a small vicinity of $A = A(t^*)$. Figure 1(a) gives a typical evolution of the slowly varying solution.

Case (ii): $\delta \gg \epsilon$. Since δ is larger than ϵ, we analyze this case by first seeking a regular expansion of the time-dependent solution of the form

$$\underline{z}(t,\tau,\delta,\epsilon) = \underline{z}_0(t,\tau,\delta) + \epsilon \underline{z}_1(t,\tau,\delta) + O(\epsilon^2) \ ; \quad (3.9)$$

where τ is defined by $\tau = \epsilon(t - t_0)$ and $\epsilon t_0 = A_L(\delta) - A_0$. Thus $A(\tau) = A_L(\delta) + \tau$. This specifies $\tau = 0$ to be the instant at which the limit point (3.5) is reached. After introducing (3.9) into (3.1), we obtain the following results. The system quickly adopts a slowly varying solution which, to leading order, has the same form as the static solution, i.e.,

$$\underline{z}_0(t,\tau,\delta) \approx \underline{z}_s(A(\tau),\delta) \ \text{ as } t \to \infty \ . \quad (3.10)$$

Knowing $\underline{z}_0$, we determine $\underline{z}_1$. By analyzing the asymptotic behavior of $\underline{z}_0$ and $\underline{z}_1$ when $t \to \infty$ and $\tau \to 0$, we find that the regular expansion (3.9) becomes nonuniform as τ approaches zero. We therefore expect a different behavior for the slowly varying solution near the limit point $A_L(\delta)$. Similar situations have been studied in Refs. 6, 7, and 10. Therefore, we summarize the principal results. The asymptotic analysis indicates that the deviation of the slowly varying solution from the lower branch of steady states goes from $O(\epsilon)$ to $O(\epsilon^{1/3})$. Moreover, the analysis of the solution near the limit point reveals that the rapid jump appears at an $O(\epsilon^{2/3})$ distance from the limit point. Figure 1(b) presents a typical evolution of the slowly varying solution. When δ becomes smaller, the distance between the jump transition and the limit point tends to increase. This suggests that larger deviations may be found as $\delta \to 0$. Indeed, the analysis of case (i) ($\delta \ll \epsilon$) indicated that $O(1)$ deviations between the jump solution and the bifurcation point can be observed.

In conclusion, the analysis of these two extreme cases reveals that quite different behaviors can be observed with a time-dependent bifurcation parameter. When the size of the imperfection δ is larger than the rate of change ϵ of A, the jump occurs at a small distance ($O(\epsilon^{2/3})$) from the limit point. However, if δ is smaller than ϵ, the jump appears only at an $O(1)$ distance from the bifurcation point and does not depend on ϵ.

IV. NUMERICAL ANALYSIS

Sections II and III were devoted to an analytical study of the LSA equations with a time-dependent pump parameter. Due to the difficulty of solving the equations, we had to resort to perturbation methods. In this section we integrate numerically the LSA equations to see if the trends given by the analytical results remain valid beyond their domain of application. Furthermore, in order to avoid the problems related to the various approximation schemes,[1] we shall integrate the full set of eight LSA equations without detuning:

$$\begin{aligned}
x' &= -x + Av + \bar{A}\bar{v} \ , \\
y' &= -y + Au + \bar{A}\bar{u} \ , \\
v' &= d(-v + Fx) \ , \\
u' &= d(-u + Fy) \ , \\
F' &= d_{\|}(-F + 1 - uy - vx) \ , \\
\bar{v}' &= \bar{d}(-\bar{v} + \bar{F}x) \ , \\
\bar{u}' &= \bar{d}(-\bar{u} + \bar{F}y) \ , \\
\bar{F}' &= \bar{d}_{\|}(-\bar{F} + 1 - a\bar{u}y - a\bar{v}x) \ .
\end{aligned} \quad (4.1)$$

The initial conditions for the eight dependent variables are

$$x = 0.001, \ \ y = u = v = \bar{u} = \bar{v} = F - 1 = \bar{F} - 1 = 0 \ , \quad (4.2)$$

corresponding to a small perturbation of the trivial solution. The fixed parameters are

$$\begin{aligned}
&d = 10, \ \ \bar{d} = 2, \ \ d_{\|} = \bar{d}_{\|} = 0.1 \ , \\
&a = 5, \ \ \bar{A} = \bar{A}_c(1 - \epsilon) \ ,
\end{aligned} \quad (4.3)$$

where $\bar{A}_c$ is defined in (2.10). In our first paper[1] we choose $\epsilon = -0.25$ so that $\bar{A} = -3.4375$. We shall begin with this value of $\bar{A}$ so that a comparison can be made between the adiabatic analysis performed in Ref. 1 and the analysis of the same equations with a time-dependent A. Let

$$\begin{aligned}
&A(t) = 3.5 + 5 \times 10^{-3} t \\
&3.5 \le A(t) \le 5.5 \ .
\end{aligned} \quad (4.4)$$

The results of the numerical integrations of Eqs. (4.1)–(4.4) are presented in Fig. 2. It clearly displays two

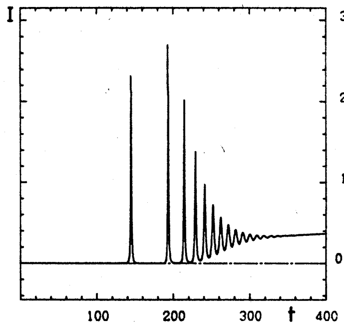

FIG. 2. Intensity versus time when the pump parameter A increases linearly in time ($\epsilon=-0.25$).

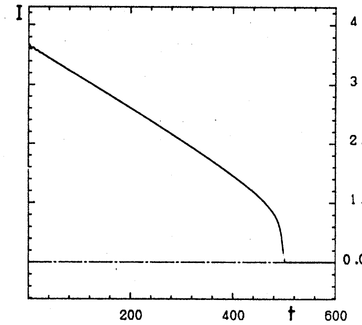

FIG. 4. Intensity versus time when the pump parameter A decreases linearly in time.

distinct regimes. We first observe passive Q-switching characterized by high peak-intensity pulses. This peak intensity then decreases as time increases until the system reaches the stable I_+ solution which is followed adiabatically. However, a blowup of the domain $t \leq 100$, shown in Fig. 3, indicates that the system went first through another regime corresponding to stable small-amplitude time-periodic solutions ($t \leq 65$), followed by a transition domain in which the intensity increases until it reaches the pulsed regime. This increase is periodically modulated. The three regimes described in Figs. 2 and 3 correspond to the three regimes which were characterized by our adiabatic analysis.[1]

A surprising result appears when we sweep across the same domain but with decreasing values of A. Let

$$A(t)=5.5-5\times 10^{-3}t \tag{4.5}$$

and we take as initial conditions the state of the system reached previously for $A(t)=5.5$ (i.e., the state corresponding to $t=400$ in Fig. 2). The result is shown in Fig. 4. The main features are that (i) neither time-periodic solutions nor pulses appear and (ii) the figure faithfully reproduces the stationary I_+ solution even in the domain where the linear-stability analysis predicts an unstable state. Dividing the sweeping velocity by 10 did not change the result. Hence, we have an example of a dynamical stabilization of a state which is unstable according to the adiabatic theory. We showed in Ref. 1 that I_+ is unstable to small perturbations at constant A for $A<4.2$. The value $A=4.2$ is reached in Fig. 4 for $t=t_H=260$. This stabilization effect is particularly surprising. According to our discussion in the Introduction and since I_+ is not an exact solution of the LSA equations with $A=A(t)$, we expect that the jump transition will occur in a small vicinity of the Hopf bifurcation point. Although no explicit analytical results have been published on the linear stability of I_+ for the eight or five LSA equations, we have observed numerically that the rate of divergence of I_+ below the bifurcation point is very small and remains very small well beyond this bifurcation. The jump to the trivial solution $I=0$ only occurs after the limit point has been reached. In future work we intend to analyze the slow passage through the critical Hopf bifurcation in more detail.

The choice $\epsilon=-0.25$ is inconvenient for a discussion of some aspects of the problem because the first instability of the zero-intensity solution is a Hopf bifurcation to stable small-amplitude periodic solutions which always requires blowups to be detected. Hence, we shall now consider the case

$$\epsilon=0.25, \quad \bar{A}=-2.0625 \tag{4.6}$$

for which the first instability of the zero intensity is a subcritical steady bifurcation at $A=A_s=3.0625$. Beyond this bifurcation, there is a domain of Q-switching which is stable for $3.0625 \leq A \leq 3.155 \pm 0.005$ whereas the I_+ steady solution is stable for $A > 3.0595 \pm 0.0005$.

We have integrated numerically Eqs. (4.1), using the in-

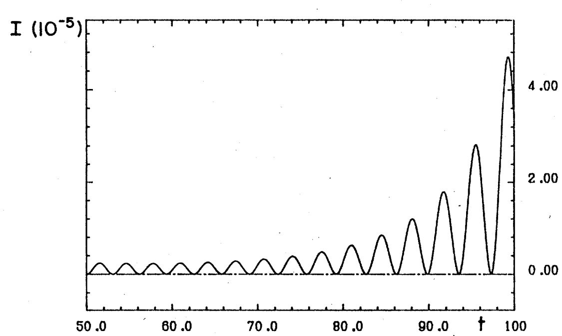

FIG. 3. Blowup of Fig. 2 to show the fine structure in the short-time domain.

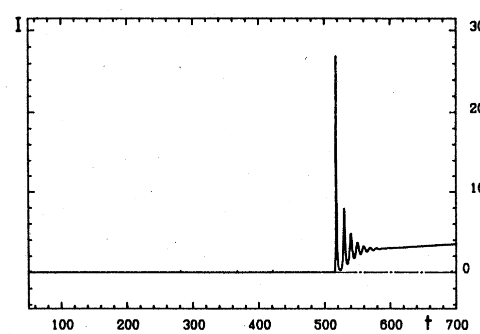

FIG. 5. Intensity versus time when the pump parameter A increases linearly in time ($\epsilon=0.25$). Initial value: $A(0)=1.5$.

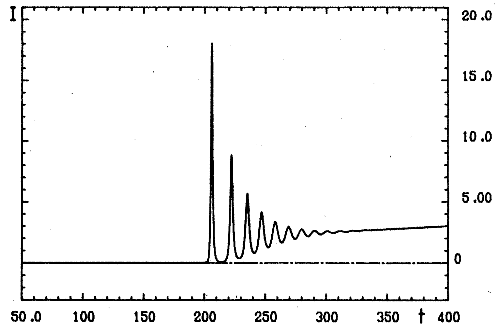

FIG. 6. Same as Fig. 5 but with $A(0)=2.5$.

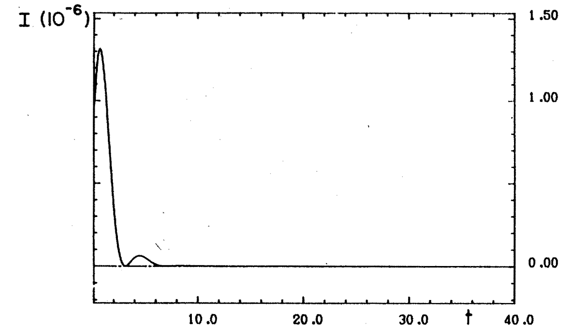

FIG. 8. Intensity versus time for an imperfect LSA with an input field amplitude $e=10^{-5}$.

itial conditions (4.2) and the parameters (4.3) and (4.6) with a time-dependence law for A which is

$$A(t)=A(0)+5\times10^{-3}t \tag{4.7}$$

In Fig. 5 we have plotted the intensity versus time for $A(0)=1.5$. The steady bifurcation predicted by the linear-stability analysis at constant A occurs at $A_s=3.0625$, i.e., for $t_s=312.5$. We notice that the dynamical bifurcation occurs at $t^*=510$, i.e., $A(t^*)=4.05$. The delay observed is qualitatively understood in terms of the arguments developed in the Introduction and in Sec. II. To further confirm this interpretation, we have plotted in Fig. 6 the result of a numerical integration which differs from the previous one (Fig. 4) only by the initial condition, which is now $A(0)=2.5$. The steady bifurcation is reached for $t_s=112.5$ whereas the dynamical bifurcation corresponds to $t^*=205$ and $A(t^*)=3.525$. Quite clearly there is an important delay linked to the dynamical nature of the bifurcation. This delay is best characterized by $D=[A(t^*)-A_s]/[A_s-A(0)]$, which is plotted on Fig. 7 versus $A(0)$. This function naturally diverges as $A(0)$ tends to A_s but the delay $A(t^*)-A_s$ is fairly small. On the contrary, when $A(0)$ is much smaller than A_s, the function D varies very little.

To gain further insight into the properties of $A(t^*)$, we have integrated (4.1) with (4.2), (4.3), (4.6), and $A(t)=2.5+10^{-2}t$ but changing each time one parameter. The following observations were made.

(i) Dividing or multiplying the sweeping velocity by a factor of 5 did not modify $A(t^*)$.
(ii) Increasing d or $\bar{d}$ increased $A(t^*)$.
(iii) Varying $d_{\|}$ or $\bar{d}_{\|}$ has no effect on $A(t^*)$.
(iv) Decreasing ϵ increased $A(t^*)$ but decreased D.

Hence the main correction to the perturbative result $D\simeq1$ derived in Sec. II is a dependence on d and $\bar{d}$.

Let us now analyze numerically the influence of an "imperfection" on the dynamical properties of the LSA. We shall consider an LSA with an injected signal.[11] This amounts to replacing the first of Eqs. (4.1) by

$$x'=-x+Av+\bar{A}\bar{v}+e\ , \tag{4.8}$$

where e is the amplitude of the injected signal. We still use (4.2), (4.3), and (4.6) with $A(t)$ given by

$$A(t)=2.5+10^{-2}t\ . \tag{4.9}$$

Hence the correspondence with the parameters δ and ϵ used in Sec. III is

$$\delta=e,\quad \epsilon=10^{-2}\ . \tag{4.10}$$

The integration of (4.1) modified by (4.8) leads to the following results. As e is varied from 0 to 10^{-5} the delay persists but decreases slowly with increasing e. For instance, with $e=10^{-7}$, 10^{-6}, and 10^{-5}, the delay is $A(t^*)=3.45$, 3.40, and 3.35, respectively, whereas $A_c=A_s=3.01$. For $e=10^{-5}$, Fig. 8 displays the short-

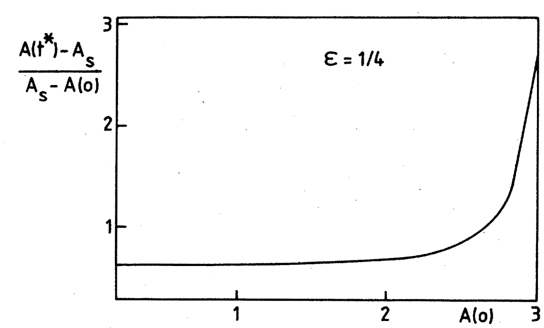

FIG. 7. Plot of the relative delay as a function of the initial value of the pump parameter.

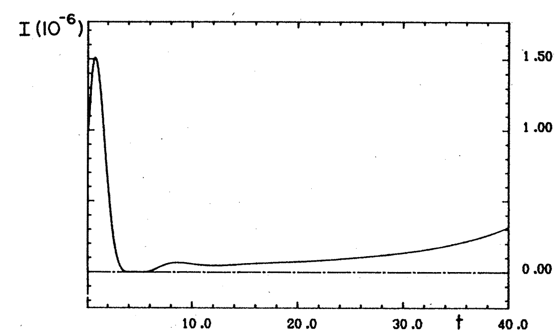

FIG. 9. Intensity versus time with the same parameters as Fig. 8 except for $e=10^{-4}$.

time behavior which is typical for the range $10^{-7} \leq e \leq 10^{-5}$: after a brief transient ($0 < t < 10$), the system follows the zero-intensity state until $A(t^*)$ is reached. As e is further increased, the delay still diminishes but the short-time behavior is modified. Starting with $e = 5\times 10^{-5}$ the solution follows a *nonzero* state even for small times. Figure 9 displays the short-time behavior for $e = 10^{-4}$. From these observations we can locate the transition region which separates the two extreme cases ($\delta \gg \epsilon$ and $\delta \ll \epsilon$) discussed in Sec. III and characterize it by the condition

$$\delta = O(\epsilon^2) \; . \tag{4.11}$$

We have not been able to determine whether this result holds in general or whether it is specific to the particular type of imperfection considered here.

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[1]P. Mandel and T. Erneux, preceding paper [Phys. Rev. A **30**, xxx (1984)].
[2]R. Ruschin and S. H. Bauer, Chem. Phys. Lett. **66**, 100 (1979).
[3]A. Jacques and P. Glorieux, Opt. Commun. **40**, 455 (1982).
[4]E. Arimondo, F. Casagrande, L. A. Lugiato, and P. Glorieux, Appl. Phys. B **30**, 57 (1983).
[5]E. Arimondo and B. M. Dinelli, Opt. Commun. **44**, 277 (1983).
[6]R. Haberman, SIAM J. Appl. Math. **37**, 69 (1979).
[7]A. K. Kapila, SIAM J. Appl. Math. **41**, 29 (1981).
[8]B. J. Matkowsky and E. L. Reiss, SIAM J. Appl. Math. **33**, 230 (1977).
[9]T. Erneux, P. Mandel, and J. F. Magnan, Phys. Rev. A **29**, 2690 (1984).
[10]T. Erneux and P. Mandel, Phys. Rev. A **28**, 896 (1983).
[11]P. Mandel, Z. Phys. B **33**, 205 (1979).