Stationary, harmonic, and pulsed operations of an optically bistable laser with saturable absorber. I

Paul Mandel
Service de Chimie-Physique II, Université Libre de Bruxelles, Campus de la Plaine, Code Postal No. 231, B-1050 Brussels, Belgium

T. Erneux
Department of Applied Mathematics and Engineering Sciences, The Technological Institute, Northwestern University, Evanston, Illinois 60201
(Received 12 March 1984)

We study the semiclassical equations for a laser with a saturable absorber in the mean-field limit, assuming homogeneously broadened two-level atoms, for a set of parameters where the system displays optical bistability and time-periodic solutions. In the first part the bifurcation diagram for stationary and periodic solutions is obtained by numerical integration. Two different classes of stable periodic solutions arise: small-amplitude solutions and passive Q switching. We observe hysteresis domains involving up to three solutions (stationary and/or periodic). We also discuss the validity of some standard approximations and show that even in the absence of detuning the phases play an important role. We also discuss the influence of the initial conditions whose symmetry properties induce important modifications of the bifurcation diagram. In the second part we introduce an alternative adiabatic elimination scheme which allows us to construct the small-amplitude periodic solutions over nearly their whole range of existence. We then study these solutions near the Hopf bifurcation from which they emerge and derive analytic conditions for their stability. When they are stable, we also give the conditions under which a secondary Hopf bifurcation will occur, leading to quasiperiodic solutions.

I. INTRODUCTION

A laser with a saturable absorber (LSA) is an example of an active system displaying optical bistability. The first suggestion of its use to obtain optical bistability is due to Lasher[1] who proposed to couple two semiconductor lasers in a single cavity to form a LSA. This original scheme has been extensively used and improved in recent years using either semiconductor lasers[2] or gas lasers.[3]

There is much confusion about the theoretical description of a LSA (see Ref. 4 for a forthcoming review). Limiting ourselves to the simplest description in which each atomic species is modeled by a set of two-level atoms with homogeneous broadening and assuming a single-running-mode cavity configuration, the semiclassical theory yields in the mean-field limit eight coupled equations.[5] Neglecting phases reduces the problem to five coupled ordinary differential equations. Application of the standard adiabatic elimination scheme[6] leads to a set of three coupled rate equations[7,8] which can be further reduced to two equations.[9] The neglect of phases has never been justified in any sensible manner. Furthermore, the successive adiabatic eliminations of atomic variables require strong inequalities between the various cavity and atomic decay rates. Until now there seems to be no paper which gives the whole set of these decay rates for a given experimental setup (although this goal was nearly fulfilled in Ref. 3). Nevertheless the usual model is often rejected on the basis that the three rate equations fail to produce a satisfactory picture of the observed phenomena. Although there are experimental situations in which this model is manifestly inadequate,[3,10] the argument of disregarding this model because of the rate-equations failure is not acceptable in general.

The LSA has two interesting modes of operation: optical bistability and time-periodic intensity modulation. They can occur separately or simultaneously. Since optical bistability refers to an overlapping property of stationary solutions, it is independent of the adiabatic elimination procedures but may be sensitive to the neglect of phases. On the other hand, the emergence of periodic solutions is related to an instability (i.e., a bifurcation point) of the stationary solutions. The properties of the bifurcating branch of the solution critically depend on the nonlinearities in the differential equations, which are modified by the adiabatic elimination schemes and by the neglect of the phases. Hence a critical examination of the various levels of description for the LSA seems to be necessary.

The analytic description of the solutions of the LSA equations is usually restricted to stationary solutions because the problem then reduces to a set of algebraic equations. In our previous papers[11,12] on the LSA theory, we have constructed analytically periodic and quasiperiodic solutions. These solutions are obtained by a perturbation expansion valid only in the vicinity of the bifurcation points. In this paper we propose a different asymptotic analysis of the periodic solutions valid for arbitrary values of the bifurcation parameter. The new perturbation method is based on the limit $d=O(\bar{d})\rightarrow\infty$ and allows us to construct $O(1)$ harmonic periodic solutions for all values of the bifurcation parameter. Then we are able to describe the behavior of these periodic solutions over their whole range of existence. In particular we show how the period of the oscillations increases with the bifurcation parameter and predict the conditions for a secondary bifurcation from these periodic solutions to quasiperiodic solutions.

This paper is divided into three sections. In Sec. II we obtain numerically the bifurcation diagram of the eight, five, and three LSA equations using always the same fixed parameters. Section III is devoted to the analytic description of the harmonically modulated time-dependent solution.

II. ADIABATIC BIFURCATION DIAGRAMS

We start with the semiclassical equations derived in Ref. 5 for the LSA:

$$[i(\partial_{t'}+\kappa)-\nu]\langle\beta\rangle = Ng^*\langle\alpha\rangle+\bar{N}\bar{g}^*\langle\bar{\alpha}\rangle ,$$
$$[i(\partial_{t'}+\gamma_\perp)-\omega]\langle\alpha\rangle = -gD(t')\langle\beta\rangle ,$$
$$i(\partial_{t'}+\gamma_{\|})D = i\gamma_{\|}\sigma+2g\langle\alpha\rangle^*\langle\beta\rangle-2g^*\langle\alpha\rangle\langle\beta\rangle^* ,$$
$$[i(\partial_{t'}+\bar{\gamma}_\perp)-\bar{\omega}]\langle\bar{\alpha}\rangle = -\bar{g}\bar{D}(t')\langle\beta\rangle ,$$
$$i(\partial_{t'}+\bar{\gamma}_{\|})\bar{D} = i\bar{\gamma}_{\|}\bar{\sigma}+2\bar{g}\langle\bar{\alpha}\rangle^*\langle\beta\rangle-2\bar{g}^*\langle\bar{\alpha}\rangle\langle\beta\rangle^* .$$

All notations are explained in Ref. 5. These equations relate the electric field $\langle\beta\rangle$ to the atomic polarization $\langle\alpha\rangle$ ($\langle\bar{\alpha}\rangle$) and inversion D ($\bar{D}$) of the amplifying (absorbing) atoms. We introduce the following new variables and parameters:

$$\langle\beta\rangle = \left[\frac{\gamma_{\|}\gamma_\perp}{4|g|^2}\right]^{1/2}[x(t)-iy(t)]e^{-i\nu t/\kappa} ,$$
$$g^*\langle\alpha\rangle = \sigma\left[\frac{|g|^2\gamma_{\|}}{4\gamma_\perp}\right]^{1/2}[u(t)+iv(t)]e^{-i\nu t/\kappa} ,$$
$$\bar{g}^*\langle\bar{\alpha}\rangle = \bar{\sigma}\left[\frac{\gamma_{\|}\gamma_\perp}{|g|^2}\right]^{1/2}\frac{|\bar{g}|^2}{2\bar{\gamma}_\perp}[\bar{u}(t)+i\bar{v}(t)]e^{-i\nu/\kappa} ,$$
$$D=\sigma F,\quad \bar{D}=\bar{\sigma}\bar{F},\quad t=\kappa t',\quad A=\frac{|g|^2N\sigma}{\kappa\gamma_\perp} ,$$
$$\bar{A}=\frac{|\bar{g}|^2\bar{N}\bar{\sigma}}{\kappa\bar{\gamma}_\perp},\quad a=\frac{|\bar{g}|^2\gamma_{\|}\gamma_\perp}{|g|^2\bar{\gamma}_{\|}\bar{\gamma}_\perp},\quad \Delta=\frac{\nu-\omega}{\gamma_\perp},\quad \bar{\Delta}=\frac{\nu-\bar{\omega}}{\bar{\gamma}_\perp} ,$$
$$d=\gamma_\perp/\kappa,\quad \bar{d}=\bar{\gamma}_\perp/\kappa,\quad d_{\|}=\gamma_{\|}/\kappa,\quad \bar{d}_{\|}=\bar{\gamma}_{\|}/\kappa ,$$

in terms of which the LSA equations become

$$\dot{x} = -x+Av+\bar{A}\bar{v} , \quad (1a)$$
$$\dot{y} = -y+Au+\bar{A}\bar{u} , \quad (1b)$$
$$\dot{v} = d(-v+\Delta u+Fx) , \quad (1c)$$
$$\dot{u} = d(-u-\Delta v+Fy) , \quad (1d)$$
$$\dot{F} = d_{\|}(-F+1-uy-vx) , \quad (1e)$$
$$\dot{\bar{v}} = \bar{d}(-\bar{v}+\bar{\Delta}\bar{u}+\bar{F}x) , \quad (1f)$$
$$\dot{\bar{u}} = \bar{d}(-\bar{u}-\bar{\Delta}\bar{v}+\bar{F}y) , \quad (1g)$$
$$\dot{\bar{F}} = \bar{d}_{\|}(-\bar{F}+1-a\bar{u}y-a\bar{v}x) . \quad (1h)$$

In this section we shall fix the parameters appearing in Eqs. (1) as follows:

$$d=10,\quad \bar{d}=2,\quad d_{\|}=\bar{d}_{\|}=0.1,\quad \Delta=\bar{\Delta}=0 ,$$
$$a=5,\quad \bar{A}=-3.4375. \quad (2)$$

This choice is suggested by the recent results published by Arimondo *et al.*[3]

Since the two detuning parameters Δ and $\bar{\Delta}$ vanish, one is naturally led to neglect the phases, i.e., y, u, and $\bar{u}$, in Eqs. (1). This assumption then produces the five equations

$$\dot{x} = -x+Av+\bar{A}\bar{v} , \quad (3a)$$
$$\dot{v} = d(-v+Fx) , \quad (3b)$$
$$\dot{F} = d_{\|}(-F+1-vx) , \quad (3c)$$
$$\dot{\bar{v}} = \bar{d}(-\bar{v}+\bar{F}x) , \quad (3d)$$
$$\dot{\bar{F}} = \bar{d}_{\|}(-\bar{F}+1-a\bar{v}x) . \quad (3e)$$

A more widely used approximation is the rate equations which are justified in the limit $d,\bar{d}\to\infty$. Hence in the long-time limit the atomic polarization is related only to the instantaneous value of the field (x) and reduced atomic inversion (F or $\bar{F}$) through $v=Fx$ and $\bar{v}=\bar{F}x$; this gives

$$\dot{x} = x(-1+AF+\bar{A}\bar{F}) , \quad (4a)$$
$$\dot{F} = d_{\|}(-F+1-Fx^2) , \quad (4b)$$
$$\dot{\bar{F}} = \bar{d}_{\|}(-\bar{F}+1-a\bar{F}x^2) . \quad (4c)$$

For the sake of clarity we shall consider first the properties of Eqs. (3), then of Eqs. (1), and finally of Eqs. (4), always with the values of the parameters given by (2).

A. The five LSA equations

The stationary intensity $I=x^2$ of Eqs. (3) is easily found to be

$$I_0=0 ,$$
$$I_\mp=(1/2a)(a(A-1)-1+\bar{A}\mp\{[a(A-1)-1+\bar{A}]^2-4a(1-A-\bar{A})\}^{1/2}) . \quad (5)$$

These solutions are plotted on Fig. 1 and show that the choice of parameters (2) leads to optical bistability. The solution I_- is always unstable.[5] As shown earlier[11,13,14] the trivial solution I_0 loses its stability via a Hopf bifurcation at $A=A_2$, where

$$A_2=(1-\bar{A})\frac{\bar{d}(\bar{d}+1)}{d(d+1)}+\frac{\bar{d}+1}{d+1}(1+d+\bar{d})=3.788\ldots , \quad (6a)$$

provided that

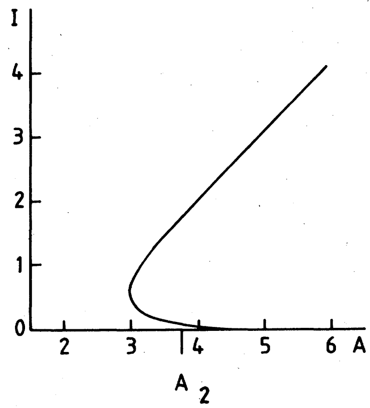

FIG. 1. Stationary intensity vs pump parameter of the amplifying atoms as given by (2) and (5).

$$\bar{A} < \bar{A}_c = \frac{\bar{d}(d+1)}{\bar{d}-d} . \tag{6b}$$

No analytic result has been published for the stability of I_+ in this context.

In order to go beyond this stationary bifurcation diagram, we have solved numerically Eqs. (3) using the following adiabatic procedure: (i) We start with $A < A_2$ and the initial conditions $x=0.01$ and $v=\bar{v}=F-1=\bar{F}-1=0$; (ii) Eqs. (3) are integrated until a stable state is reached; and (iii) A is modified by a small increment (typically 0.01) and Eqs. (3) are again integrated until a new stable state is reached, using as initial conditions the previous stable final state. The results of this adiabatic analysis are summarized in Table I.

Three types of time-dependent solutions have been observed. The first kind of solution is the small-amplitude harmonically modulated intensity which emerges from the Hopf bifurcation at $A=A_2$. A typical graph of this solution is displayed in Figs. 2 for $A=4.02$. Figure 2(a) shows the intensity versus time whereas Fig. 2(b) shows the intensity versus the reduced population difference $F(t)$. As the pump parameter A (which is our bifurcation parameter) is increased adiabatically, there appears a very small domain of existence for a different solution which is displayed in Fig. 3, again for $A=4.02$. This solution is still a small-amplitude harmonically modulated intensity but it differs from the previous one by the frequency, which is exactly one-third of the frequency of the previous periodic solution. The third periodic solution is a pulsed solution which describes passive Q switching. In Fig. 4 we show the pulsed intensity versus time and versus $F(t)$, still for $A=4.02$. Note that the peak intensity of the pulses is 3 orders of magnitude larger than the peak intensity of the harmonic solutions. Although Fig. 3(a) displays a neat pulse structure, there is a fine structure in the domain where the intensity nearly vanishes, between

TABLE I. Stability domains for the solutions of the five LSA equations.

I_0	$A \le 3.788$
Harmonic (f)	$3.788 \le A \le 4.020$
Harmonic $(f/3)$	$4.020 \le A \le 4.0215 \pm 0.0005$
Pulses	$3.975 \pm 0.005 \le A \le 4.445 \pm 0.005$
I_+	$4.2005 \pm 0.0005 \le A \le 6.0$ (at least)

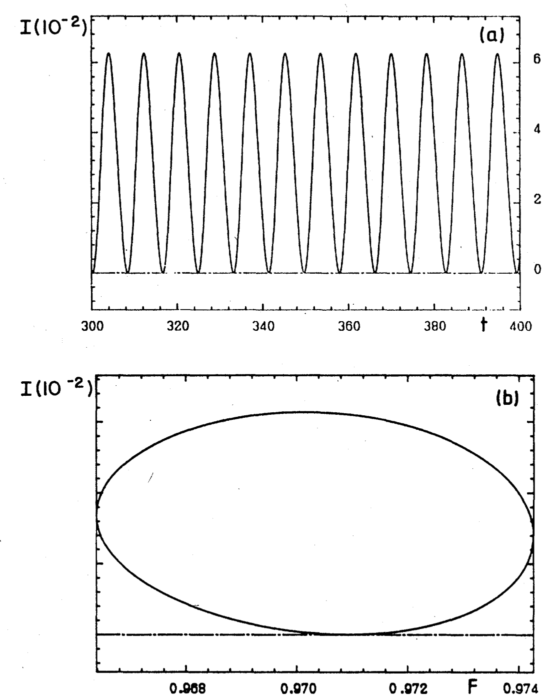

FIG. 2. Harmonically modulated intensity (a) vs time and (b) vs reduced amplifying atoms population difference for $A=4.02$.

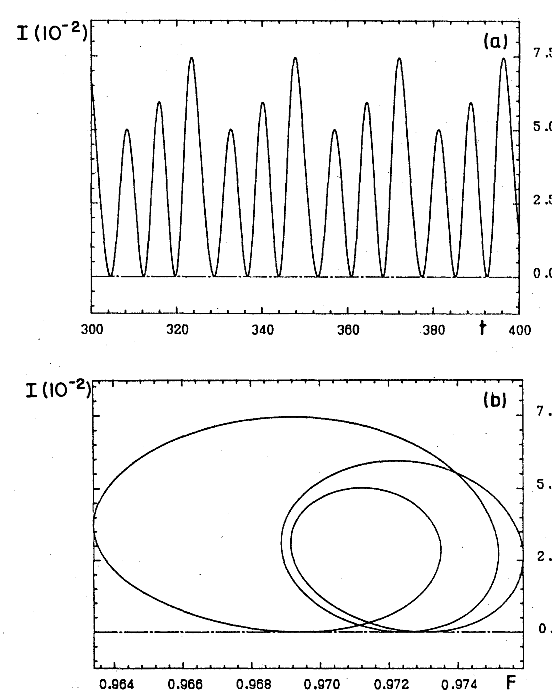

FIG. 3. Harmonically modulated intensity $(f/3)$ (a) vs time and (b) vs reduced amplifying atoms population difference for $A=4.02$.

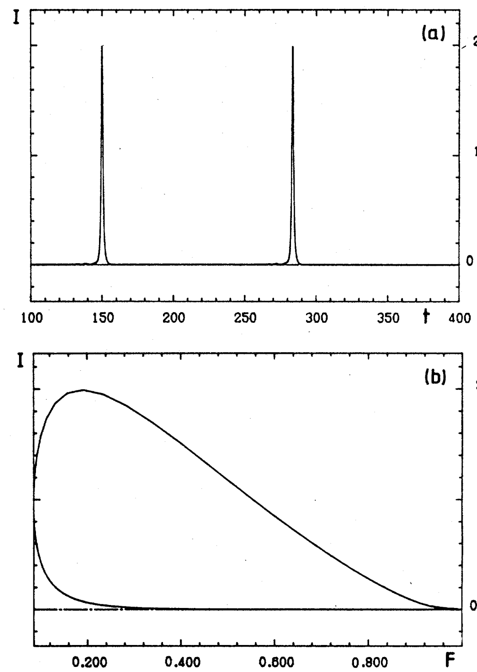

FIG. 4. Pulsed intensity (a) vs time and (b) vs amplifying atoms reduced population difference for $A=4.02$.

two consecutive pulses. This is more clearly visible in Fig. 5, which consists of blowups of Fig. 4(a). They indicate that the pulsed intensity quickly falls after reaching its peak, then the intensity again begins to rise exponentially. Both the decrease and increase of the intensity between the pulses are modulated by small oscillations whose frequency is constant but different from the frequency of the harmonic solutions at the same value of A.

A glance at Table I indicates that there are overlaps between the domains of existence for the three periodic solutions. We refer to these overlapping domains as birhythmicity if two stable periodic solutions coexist and trirhythmicity when the three periodic solutions coexist. Figures 2—4 have been selected to illustrate a situation of trirhythmicity in which the only difference is the choice of initial conditions.

B. The eight LSA equations

In this section we shall study Eqs. (1) with $\Delta=\bar{\Delta}=0$. The stationary solutions are still given by (5) and A_2 corresponds to a destabilization of the trivial solution I_0 via a degenerate Hopf bifurcation.[13] The new feature brought in by the phases is the existence of a very special type of periodic solution. Indeed, it is easy to verify[15,16] that

$$x(t)=\tilde{I}^{1/2}\cos\Omega_{\pm}t \,, \quad y(t)=\tilde{I}^{1/2}\sin\Omega_{\pm}t \qquad (7)$$

is an exact periodic solution of Eqs. (1), where

$$\tilde{I}=f(a,b) \,, \quad \Omega_{\pm}=\pm d f^{1/2}(b,a) \,, \qquad (8)$$

with

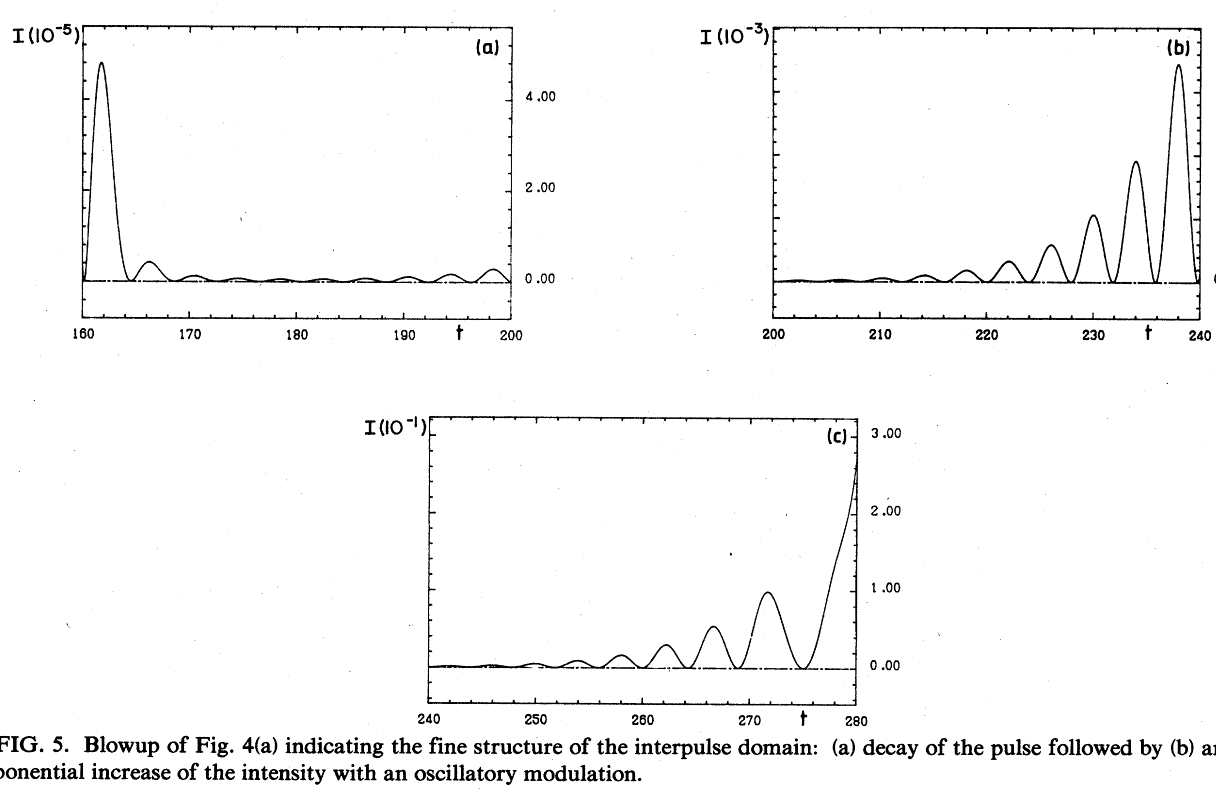

FIG. 5. Blowup of Fig. 4(a) indicating the fine structure of the interpulse domain: (a) decay of the pulse followed by (b) and (c) an exponential increase of the intensity with an oscillatory modulation.

$$f(a,b)=\frac{1}{a-b}\left[b-1+(\bar{d}-d)\left[\frac{\bar{A}}{\bar{d}(d+1)}+\frac{Ab}{d(\bar{d}+1)}\right]\right],$$

and $b=(d/\bar{d})^2$.

These new solutions have some interesting properties. Because they have a single frequency of oscillation in a pure sinusoidal mode, the corresponding intensity $(x^2+y^2=\tilde{I})$ is stationary. Had we used a polar rather than a Cartesian decomposition of the field, the solutions corresponding to (7) would have appeared as a new stationary state. Our choice of using a Cartesian decomposition was motivated by computer-related considerations. Another unexpected feature of these solutions is that they correspond to a detuned field, although we set $\Delta=\bar{\Delta}=0$. As previously discussed[17] the nonlinear losses induce in this case a dispersive response of an otherwise absorptive system.

The intensity $\tilde{I}$ emerges from I_0 precisely at A_2. It increases linearly with A until it reaches I_- at A_3, given by

$$A_3=\frac{d(\bar{d}+1)}{a}\left[\frac{1-a}{\bar{d}-d}-\frac{\bar{A}}{\bar{d}(d=1)}\right]=3.9375\ldots \tag{9}$$

Here again we have performed an adiabatic analysis of the bifurcation diagram induced by Eqs. (1). It turns out that the initial conditions play a more critical role in the five LSA equations case. We define a symmetric solution of Eqs. (1) by the relations

$$x=y,\quad u=v,\quad \bar{u}=\bar{v}\,. \tag{10}$$

Clearly if the initial conditions are symmetric, the solutions of Eqs. (1) will remain symmetric for all times. On the other hand, if the initial conditions are asymmetric, they will remain asymmetric for all times. This distinction between symmetric and asymmetric initial conditions leads to two completely different bifurcation diagrams.

Using the adiabatic analysis described in Sec. II A and the symmetric initial conditions

$$x=y=0.01,\quad u=v=\bar{u}=\bar{v}=F-1=\bar{F}-1=0$$

we recover the bifurcation diagram given in Table I except for the disappearance of the third harmonic $(f/3)$ solution. The equivalence between the two bifurcation diagrams means that for the identical values of all parameters, the intensity [which equals x^2 for Eqs. (3) and x^2+y^2 for Eqs. (1)] is identical in both cases. Therefore this suggests that the third harmonic solution is destabilized by the phases.

If we use an asymmetric initial condition, another bifurcation diagram is generated. It is summarized in Table II. The main differences are the appearance of the $\tilde{I}$ "stationary" solution and the disappearance of the third harmonic solution. Because the new $\tilde{I}$ solution has a domain of stability[11] we observe a modification of the stability boundaries for the two time-periodic solutions. Apart from these modifications, the main features of the harmonic and pulsed solutions (such as peak intensity and frequency) remain qualitatively similar to those derived in the five LSA equations approximation.

TABLE II. Stability domains for the asymmetric solutions of the eight LSA equations.

I_0	$A\le 3.788$
$\tilde{I}$	$3.788\le A\le 3.895\pm0.005$
Harmonic (f)	$3.86\pm0.01\le A\le 3.94\pm0.01$
Pulses	$3.92\pm0.01\le A\le 4.45\pm0.01$
I_+	$4.2005\pm0.0005\le A\le 6.0$ (at least)

C. The three LSA equations

Although the set of parameters (2) does not allow for an adiabatic elimination of the atomic polarizations, we shall nevertheless study the three LSA rate equations to mimic a too-often performed procedure. Equations (4) have a very simple bifurcation diagram corresponding to Fig. 1 with the Hopf bifurcation at A_2 removed. Hence the trivial solution I_0 loses its stability at $a=1-\bar{A}=4.4375$.

Integrating numerically Eqs. (4) did not lead to any stable periodic solution. When A is slightly greater than $1-\bar{A}$, we observe a jump to the stable I_+ branch. The domain of stability for I_+ has a lower bound A^* given by 4.09 ± 0.01. When $A=A^*$ there is a Hopf bifurcation leading apparently to an unstable solution; below A^* we observe a jump from I_+ to the trivial solution I_0.

D. Analysis

The choice of parameters (2) has been made in order to display a number of properties which are typical not only in the LSA but also in other nonlinear systems such as, e.g., optical bistability in a passive cavity and in a laser with injected signal.

(i) One often finds in the literature that a Hopf bifurcation signals the occurrence of passive Q switching, i.e., the spontaneous onset of pulses. This statement is wrong in general. The Hopf bifurcation signals the emergence of a periodic solution which is harmonically modulated in the vicinity of the bifurcation. If this bifurcation is supercritical, the small-amplitude periodic solution will be stable, in which case the bifurcation does not correspond to the onset of pulses. If the bifurcation is subcritical or vertical in the vicinity of the bifurcation point, no conclusion can be drawn on the nature of the solutions beyond the critical point. A well-known example is the single-running-mode laser second threshold which is a subcritical Hopf bifurcation leading to chaos. Two other examples are given in Table I. The trivial solution I_0 has a supercritical Hopf bifurcation at $A=A_2$ leading to a stable small-amplitude periodic intensity which has a harmonic modulation [see Fig. 2(a)]. On the other hand, the I_+ branch displays a subcritical Hopf bifurcation leading to pulses [see Fig. 4(a)] in the case of Eqs. (1) or (3) but leading to a jump towards I_0 in the case of Eqs. (4).

(ii) Another question is the relation between harmonically modulated solutions and pulses. In the present study there is a first-order transition between the two types of solutions. In other words there is an abrupt transition between them with an overlap implying birhythmicity. Another possibility is a smooth transition, without bifurcation, from harmonic to pulsed solutions. This case was illustrated for the LSA in our previous study of quasiperiodic solutions.[12] One clear difference between harmonic and pulsed solutions is the dynamical reponse of the absorbing atoms reduced population difference $\bar{F}(t)$. In the case of harmonic solutions we observe that $\bar{F}(t)$ remains positive which means that no inversion is created in the absorbing part of the LSA. On the other hand, in the pulsed regime $\bar{F}(t)$ becomes negative for a very small time corresponding to the peak of $I(t)$, implying a transient inversion in the absorbing cell. This phenomenon was already noticed by Antoranz *et al.*[18]

(iii) The pulsed regime displays a fine structure which is shown in Fig. 5. The small-amplitude oscillations have a frequency and an amplitude which are not related to the harmonic solution. A plausible explanation is the following. Between two consecutive pulses, the intensity is vanishingly small. Therefore we expect the system to be influenced by the properties of the trivial solution I_0. But the domain in which pulses occur is bounded from above and from below by two bifurcation points of I_0 (i.e., $A=A_2$ and $A=1-\bar{A}$). In this domain the linear stability analysis of I_0 yields three negative roots and one pair of complex roots with positive real parts. We believe that the fine structure seen in Fig. 5 is a manifestation of these two unstable roots. To test this interpretation we have analyzed pulses in another domain of the parameter space where the bifurcation point A_2 has disappeared. In this case the pulses did not show any fine structure and the only unstable root of I_0 was a real positive root.

(iv) A rather unexpected feature of the full set of eight equations is the critical dependence on the initial conditions. Depending on whether these initial conditions are symmetric or asymmetric, two different bifurcation diagrams are generated. The failure to realize this dependence on the symmetry of the initial conditions lead Antoranz *et al.*[18] to propose a bifurcation diagram which in essence is a superposition of the two separate diagrams. The two main differences brought in by the asymmetry are the appearance of a new class of solutions (7) and a drastic reduction of the domain of existence for the harmonic solution. However, when the harmonic and pulsed solutions arising from symmetric and asymmetric initial conditions are compared for identical values of all parameters, they lead to intensities having comparable frequencies and peak values (that is to say, the corresponding graphs are superposable).

III. THE HOPF BIFURCATION

A. The perturbation scheme

In this section we analyze the LSA equations (3) and show why the rate equations (4) are inadequate to describe the bifurcation diagram of the periodic solutions. This bifurcation diagram can, however, be obtained in the asymptotic limit $d\to\infty$ and $\bar{d}=O(d)$ provided that the adiabatic elimination of the variable v and $\bar{v}$ is appropriately replaced by a different perturbation method.

The LSA equations (3) admit a basic state $I_0=0$. From its linear stability analysis we know that if $\bar{A}<\bar{A}_c$, there exists a Hopf bifurcation point defined by

$$A=A_2,\quad x=v=\bar{v}=F-1=\bar{F}-1=0\ , \tag{11}$$

where $\bar{A}_c$ and A_2 are given by (6). The choice $\bar{A}=-3.4375$ corresponds to $\bar{A}=\bar{A}_c(1+0.25)$. As

$$\bar{d}=ld,\quad d\to\infty \tag{12}$$

$\bar{A}_c$ and A_2 approach the limits

$$\bar{A}_c=\simeq\frac{ld}{l-1}=O(d) \tag{13}$$

and if $\bar{A}=O(d)<\bar{A}_c$,

$$A_2\simeq-\bar{A}l^2+ld(1+l)=O(d)\ . \tag{14}$$

These results suggest that in order to analyze the Hopf bifurcation we must first rescale A and $\bar{A}$ as

$$A=dA',\quad \bar{A}=d\bar{A}'\ , \tag{15}$$

where A' and $\bar{A}'$ are $O(1)$ quantities. Then we rewrite Eqs. (3) in terms of A', $\bar{A}'$, and l:

$$\dot{x}=d(A'v+\bar{A}'\bar{v}-d^{-1}x)\ , \tag{16a}$$
$$\dot{v}=d(-v+Fx)\ , \tag{16b}$$
$$\dot{\bar{v}}=ld(-\bar{v}+\bar{F}x)\ , \tag{16c}$$
$$\dot{F}=d_{||}(-F+1-vx)\ , \tag{16d}$$
$$\dot{\bar{F}}=\bar{d}_{||}(-\bar{F}+1-a\bar{v}x)\ . \tag{16e}$$

Since $\dot{x}$, $\dot{v}$, and $\dot{\bar{v}}$ are proportional to d in Eqs. (16) we expect that x, v, and $\dot{v}$ will initially evolve on a fast time scale $T=td$. However, we shall see that this behavior persists in the long-time limit because the rapid evolution of x, v, and $\bar{v}$ corresponds to undamped periodic oscillations. This explains why the variables v and $\bar{v}$ cannot be eliminated adiabatically from Eqs. (16).

To determine the periodic solutions of Eqs. (16) we propose a perturbation scheme. We first define a fast time by

$$T=td \tag{17}$$

and rewrite Eqs. (16) with T considered as our basic time scale:

$$x_T=A'v+\bar{A}'\bar{v}-\epsilon x\ , \tag{18a}$$
$$v_T=-v+Fx\ , \tag{18b}$$
$$\bar{v}_T=l(-\bar{v}+\bar{F}x)\ , \tag{18c}$$
$$F_T=\epsilon d_{||}(-F+1-vx)\ , \tag{18d}$$
$$\bar{F}_T=\epsilon\bar{d}_{||}(-\bar{F}+1-a\bar{v}x)\ , \tag{18e}$$

where $\epsilon=d^{-1}$ and $f_T\equiv df/dT$. Then we seek 2π-periodic solutions of (18) of the form

$$\begin{bmatrix} x \\ v \\ \bar{v} \\ F \\ \bar{F} \end{bmatrix}=\sum_{j=0}^{\infty}\epsilon^j\begin{bmatrix} x_j(T') \\ v_j(T') \\ \bar{v}_j(T') \\ F_j(T') \\ \bar{F}_j(T') \end{bmatrix}\ , \tag{19}$$

where

$$T' \equiv \sigma(\epsilon)T = (\sigma + \epsilon\sigma_1 + \cdots)T \tag{20}$$

and $\sigma(\epsilon)$ is the unknown frequency of the oscillations which must be determined by the perturbation analysis. Introducing (19) and (20) into (18) and equating to zero the coefficients of each power of ϵ, we obtain a sequence of problems for the unknowns $x_j, v_j, \ldots, \bar{F}_j$. The $O(1)$ problem is given by

$$\sigma x_{0T'} = A'v_0 + \bar{A}'\bar{v}_0 , \tag{21a}$$

$$\sigma v_{0T'} = -v_0 + x_0 F_0 , \tag{21b}$$

$$\sigma \bar{v}_{0T'} = l(-\bar{v}_0 + x_0 \bar{F}_0) , \tag{21c}$$

$$F_{0T'} = \bar{F}_{0T'} = 0 . \tag{22}$$

From (22) we conclude that

$$F_0 = f, \quad \bar{F}_0 = \bar{f} , \tag{23}$$

where f and $\bar{f}$ are two constants. Then we observe that (21) is a linear system. This system admits 2π-periodic solutions if and only if $\omega = \pm i\sigma$ are imaginary eigenvalues of the following characteristic equation:

$$\omega^3 + (1+l)\omega^2 + (l - fA' - \bar{f}\bar{A}'l)\omega - l(A'f + \bar{A}'\bar{f}) = 0 . \tag{24}$$

The analysis of (24) leads to the conditions

$$-l(l+1) + fA' + l^2\bar{f}\bar{A}' = 0 , \tag{25}$$

$$\sigma^2 = -\frac{l(A'f + \bar{A}'\bar{f})}{1+l} > 0 . \tag{26}$$

Provided that these conditions are verified, Eqs. (21) admit the following solution:

$$\begin{bmatrix} x_0 \\ v_0 \\ \bar{v}_0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ p \\ q \end{bmatrix} e^{iT'} + \text{c.c.} , \tag{27}$$

where p and q are given by

$$p = \frac{f}{1+i\sigma}, \quad q = \frac{\bar{f}l}{l+i\sigma} , \tag{28}$$

α is an unknown amplitude and c.c. denotes complex conjugate. Thus the solution (27) represents periodic oscillations in the fast time scale T. To determine the amplitude α of these oscillations, we must consider the next order of our perturbation analysis. The $O(\epsilon)$ problem for F_1 and $\bar{F}_1$ is given by

$$\sigma F_{1T'} = d_{\parallel}(-F_0 + 1 - x_0 v_0) , \tag{29a}$$

$$\sigma \bar{F}_{1T'} = \bar{d}_{\parallel}(-\bar{F}_0 + 1 - a x_0 \bar{v}_0) . \tag{29b}$$

Introducing (23) and (27) into (29), we observe that F_1 and $\bar{F}_1$ remain bounded functions of T' (or T) if and only if $f, \bar{f}$, and $\alpha\alpha^*$ verify the following solvability conditions:

$$1 - f - (p + p^*)\alpha\alpha^* = 0 , \tag{30a}$$

$$1 - \bar{f} - a(q + q^*)\alpha\alpha^* = 0 . \tag{30b}$$

Then the solution of (29) is given by

$$F_1 = f_1 - \left[\frac{\alpha^2 p}{2i\sigma} d_{\parallel} e^{2iT'} + \text{c.c.} \right] , \tag{31a}$$

$$\bar{F}_1 = \bar{f}_1 - \left[\frac{\alpha^2 q a}{2i\sigma} \bar{d}_{\parallel} e^{2iT'} + \text{c.c.} \right] , \tag{31b}$$

where f_1 and $\bar{f}_1$ are two new constant coefficients to be determined.

At this stage of the perturbation analysis the solution is given in first approximation by (23) and (27), where the amplitudes f, $\bar{f}$, and α may be determined by the conditions (25), (26), and (30). In Sec. III B we analyze these conditions in detail and determine the bifurcation diagram of the periodic solutions.

B. The amplitude equations

Using the definition of p and q, we can rewrite the three amplitude equations as

$$-l(l+1) + fA' + l^2\bar{f}\bar{A}' = 0 , \tag{32a}$$

$$1 - f - \frac{2f}{1+\sigma^2}\alpha\alpha^* = 0 , \tag{32b}$$

$$1 - \bar{f} - \frac{2a\bar{f}l^2}{l^2+\sigma^2}\alpha\alpha^* = 0 . \tag{32c}$$

Equations (32) are the *bifurcation equations:* to each solution of (32) corresponds a periodic solution of the LSA equations (3) described by (23) and (27). The solution of (32) is easily found to be

$$\alpha\alpha^* = \frac{l-1}{2l(al^2-1)}(A' - A_2') > 0 , \tag{33}$$

$$f = 1 - \frac{A' - A_2'}{A'(1-al^2)} , \tag{34a}$$

$$\bar{f} = 1 + \frac{a(A' - A_2')}{\bar{A}'(1-al^2)} , \tag{34b}$$

where

$$A_2' = l(l+1) - l^2\bar{A}' \tag{35}$$

corresponds to the Hopf bifurcation point $A = A_2$ in the limit $d = O(\bar{d}) \to \infty$ [see Eq. (14)]. From (33) we note that the amplitude $|\alpha|$ of the periodic oscillations increases like $|A' - A_2'|^{1/2}$ as $|A' - A_2'|$ increases. This is in agreement with the Hopf bifurcation theorem. Moreover, we also note from (33) that the transition to periodic solutions is defined only for $A' > A_2'$ or $A' < A_2'$ if

$$al^2 < 1 \tag{36}$$

or

$$al^2 > 1 \tag{37}$$

(as in Sec. II, we choose $\bar{d} < d$, whence $l < 1$). Hence the condition (36) determines the *direction of the bifurcation:* the bifurcation is supercritical (or subcritical) when (36) [or (37)] is verified. As we shall see in Sec. III C, this property has important consequences on the stability of the periodic solutions.

The frequency σ of the periodic solutions can be obtained from (26) and is given by

$$\sigma^2=\frac{la(l-1)}{1-al^2}(A'-A_3')>0\ ,\qquad (38)$$

where

$$A_3'=\frac{\bar{A}'(1-l)+l(1-a)}{a(1-l)}\ .\qquad (39)$$

Thus the frequency is also a function of the bifurcation parameter A'. If (36) holds, σ^2 may vanish as $|A'-A_3'|$ vanishes, i.e., the period of the oscillations becomes large. This type of behavior is in agreement with previous theoretical work on the LSA.[11] However, the situation $\sigma \to 0$ corresponds to a singularity of the perturbation expansion and a different series for the periodic solution should be proposed in the vicinity of $A'=A_3'$. We shall not examine this problem here.

C. The stability analysis

In this section we concentrate on the stability properties of the periodic solutions. Although their stability could be studied for all values of A', we shall limit our analysis to the case where $A'=A_2'+O(\epsilon)$. As we shall demonstrate, supercritical periodic solutions ($A'>A_2'$) are stable near $A'=A_2'$ but may change stability at a larger value $A'=A_c'>A_2'$. Hence A_c' is called a *secondary bifurcation point* and corresponds to the emergence of quasiperiodic solutions in our problem.

When $|A'-A_2'|=O(\epsilon)$, the expansion (19) of the periodic solutions becomes nonuniform and a new expansion valid near A_2' must be proposed. The detailed analysis is tedious but is similar to the problem studied in Ref. (12). It leads to simple conclusions, so we first summarize the principal steps of the analysis and then present our bifurcation results.

The nonuniformity of (19) near A_2' suggests a new expansion of the form

$$\begin{bmatrix} x \\ v \\ \bar{v} \end{bmatrix} = \sum_{j=0}^{\infty} \epsilon^{j+1/2} \begin{bmatrix} x_j \\ v_j \\ \bar{v}_j \end{bmatrix}\ ,\qquad (40a)$$

$$\begin{bmatrix} F \\ \bar{F} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sum_{j=0}^{\infty} \epsilon^{j+1} \begin{bmatrix} F_j \\ \bar{F}_j \end{bmatrix}\ .\qquad (40b)$$

Then by expanding $A'-A_2'$ as

$$A'-A_2'=\epsilon a_1+O(\epsilon^2)\qquad (41)$$

and analyzing two orders of the perturbation series, we find the following results:

$$\begin{bmatrix} x \\ v \\ \bar{v} \end{bmatrix} = \epsilon^{1/2}\left[\alpha(t)\begin{bmatrix} 1 \\ p \\ q \end{bmatrix} e^{i\sigma T}+\text{c.c.}\right]+O(\epsilon^{3/2})\ ,\qquad (42)$$

$$F=\epsilon f(t)+O(\epsilon^2),\quad \bar{F}=\epsilon\bar{f}(t)+O(\epsilon^2)\ ,\qquad (43)$$

where $T=td$ and p, q, and σ are defined by

$$p=\frac{1}{1+i\sigma},\quad q=\frac{l}{l+i\sigma}\ ,$$
$$\sigma^2=l[\bar{A}'(l-1)-l]>0\ .\qquad (44)$$

Defining r and θ by $\alpha=r\exp(i\theta)$, the amplitudes r, f, and $\bar{f}$ are determined by the solutions of the three differential equations,

$$r_t=rG\left[\frac{1-l}{l}[a_1-(1+l+l^2+l^2m^2)] + (1+l^2m^2)f-l^2(1+m^2)\bar{f}\right]\ ,\qquad (45)$$

$$f_t=d_{\|}\left[-f-r^2\frac{2}{1+l^2m^2}\right]\ ,\qquad (46)$$

$$\bar{f}_t=\bar{d}_{\|}\left[-\bar{f}-r^2\frac{2a}{1+m^2}\right]\ ,\qquad (47)$$

where $G=G(l,m)>0$ and m is defined by

$$m^2=\frac{\bar{A}'-\bar{A}_c'}{\bar{A}_c'}>0,\quad \bar{A}_c'=\frac{-l}{1-l}\ .$$

Equations (45)—(47) are the bifurcation equations. Their steady-state solutions correspond to the periodic solutions of the LSA equations. These steady-state solutions are given by

(i) $r=f=\bar{f}=0$; (48)

$$\text{(ii)}\ f=-r^2\frac{2}{1+l^2m^2},\quad \bar{f}=-r^2\frac{2a}{1+m^2}\ ,$$
$$r^2=\frac{1-l}{2l(1-al^2)}[a_1-(1+l+l^2+l^2m^2)]\ .\qquad (49)$$

The solution (48) corresponds to the basic state I_0. The solution (49) gives the amplitude of the periodic solution. It can be verified that the expression for r^2 in (49) exactly corresponds to the expression for $\alpha\alpha^*$ given by (33) using (41). The bifurcation is thus supercritical (or subcritical) if (36) [or (37)] is verified. We now examine the linear stability of the solutions (48) and (49). The basic state is stable (unstable) if

$$a_1<a_{1c}\quad (a_1>a_{1c})\qquad (50)$$

where

$$a_{1c}=1+l+l^2+l^2m^2\ .\qquad (51)$$

On the other hand, the steady-state solution (49) is stable if (36) is verified and if

$$-(d_{\|}+\bar{d}_{\|})d_{\|}\bar{d}_{\|}-4r^2G(d_{\|}^2-\bar{d}_{\|}^2l^2a)<0\ .\qquad (52)$$

Since $G>0$, this condition is always verified if

$$d_{\|}^2-\bar{d}_{\|}^2l^2a>0\ .\qquad (53)$$

However, if

$$d_{\|}^2-\bar{d}_{\|}^2l^2a<0\qquad (54)$$

the bifurcation equations [(45)—(47)] admit a new Hopf bifurcation to periodic amplitudes. This Hopf bifurcation is defined at the critical amplitude $r^2=r_c^2$,

$$r_c^2=\frac{(d_{\|}+\bar{d}_{\|})d_{\|}\bar{d}_{\|}}{4G(\bar{d}_{\|}^2al^2-d_{\|}^2)}>0\ .\qquad (55)$$

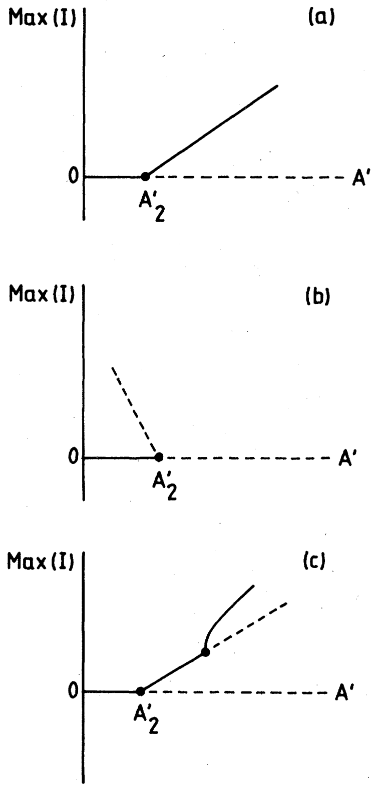

FIG. 6. Bifurcation diagram for the harmonic solutions. (a) $al^2-1<0$ and $d_{\parallel}^2-\bar{d}_{\parallel}^2 al^2>0$: supercritical bifurcation towards stable periodic solution. (b) $al^2-1>0$ and $d_{\parallel}^2-\bar{d}_{\parallel}^2 al^2 \lessgtr 0$: subcritical bifurcation towards unstable periodic solution. (c) $al^2-1<0$ and $d_{\parallel}^2-\bar{d}_{\parallel}^2 al^2<0$: supercritical bifurcation followed by a secondary bifurcation towards quasiperiodic solutions.

These periodic solutions of (45)—(47) can be studied either numerically or by a new perturbation expansion. A similar situation has been analyzed in Ref. 12. The important point to realize is that this new branch of solutions corresponds to *quasiperiodic solutions* of the LSA problem (3). Indeed, by our perturbation analysis we have constructed time-dependent solutions of the form (42) and (43) where the amplitudes r, f, and $\bar{f}$ are now time-periodic functions on the slow time scale t. In this case (42) and (43) describe quasiperiodic solutions characterized by two basic frequencies: the frequency of the periodic amplitudes and $\sigma\epsilon^{-1}$. Because $\epsilon \ll 1$ they are in general not commensurable and the behavior of these quasiperiodic solutions will appear completely irregular. Note that this secondary bifurcation to quasiperiodic solutions is possible only if (36) and (54) are simultaneously verified. This implies

$$d_{\parallel} \neq \bar{d}_{\parallel} \ . \tag{56}$$

Thus the inequality of $d_{\parallel}$ and $\bar{d}_{\parallel}$ is an important source of instability in the LSA. In a future work we intend to explore this question in more detail. Figure 6 gives a summary of the possible bifurcation diagrams.

ACKNOWLEDGMENTS

The authors are Senior Research Associate and Senior Research Assistant, respectively, with the Fonds National de la Recherche Scientifique (Belgium). This work has been carried out in the framework of an Operation launched by the Commission of the European Community, under the experimental phase of the European Community Stimulation Action (1983—1985) and with the help of North Atlantic Treaty Organization (NATO) Research Grant No. 0348/83. This research was also supported by the U.S. Air Force Office of Scientific Research under Grant No. AE OSR80-0016 and the U.S. Department of Energy under Grant No. DE-AC02-78ERO-4650.

[1]G. J. Lasher, Solid State Electron. **7**, 707 (1964).
[2]C. Harder, K. Y. Lau, and A. Yariv, IEEE J. Quantum Electron. **QE**-18, 1351 (1982).
[3]E. Arimondo, F. Casagrande, L. A. Lugiato, and P. Glorieux, Appl. Phys. B **30**, 57 (1983).
[4]N. B. Abraham, P. Mandel, and L. M. Narducci, in *Progress in Optics,* edited by E. Wolf (North-Holland, Amsterdam, in press).
[5]L. A. Lugiato, P. Mandel, S. T. Dembinski, and A. Kossakowski, Phys. Rev. A **18**, 238 (1978).
[6]L. A. Lugiato, P. Mandel, and L. M. Narducci, Phys. Rev. A **29**, 1438 (1984).
[7]E. Hofelich-Abate and F. Hofelich, J. Appl. Phys. **39**, 4823 (1968).
[8]N. G. Basov, IEEE J. Quantum Electron. **QE**-4, 855 (1968).
[9]E. Hofelich-Abate and F. Hofelich, Phys. Lett. **26A**, 426 (1968).
[10]Yu. V. Brzhazovskii, L. S. Vasilenko, S. G. Rautian, G. S. Popova, and V. P. Chebotaev, Zh. Eksp. Teor. Fiz. **61**, 500 (1971) [Sov. Phys.—JETP **34**, 265 (1972)].
[11]T. Erneux and P. Mandel, Z. Phys. B **44**, 353 (1981); **44**, 365 (1981).
[12]T. Erneux, P. Mandel, and J. F. Magnam, Phys. Rev. A **29**, 2690 (1984).
[13]P. Peplowski, Physica (Utrecht) **6D**, 364 (1983).
[14]J. C. Antoranz and M. Velarde, Opt. Commun. **38**, 61 (1981).
[15]P. Mandel, Phys. Rev. A **21**, 2020 (1980).
[16]S. T. Dembinski, A. Kossakowski, P. Peplowski, L. A. Lugiato, and P. Mandel, Phys. Lett **68A**, 20 (1978).
[17]P. Mandel, Phys. Lett. **83A**, 207 (1981).
[18]J. C. Antoranz, L. L. Bonilla, J. Gea, and M. G. Velarde, Phys. Rev. Lett. **49**, 35 (1982).