Angular momentum analysis of interactions between spin-$\frac{1}{2}$ particles

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In the interaction of N spin-$\frac{1}{2}$ particles, the possible types of dependence of the cross section on polarization and momentum three-vectors and/or on two-spinor components may be defined with the use of angular-momentum-coupling trees with $j=1$ and/or $j=\frac{1}{2}$ external labels, respectively. We show, using integrity-basis theory, that all such trees with spin-$\frac{1}{2}$ labels only may be expressed as sums of products of Kronecker delta functions and $2jm$ symbols in the spinor component labels; this is related to Kramers's method of spinor invariants. The integrity-basis method also identifies the nature of inter-relationships between the spinor invariants. Explicit reductions of angular momentum trees of degree $2N$ in spin-$\frac{1}{2}$ labels are given for $N\leq 4$. As simple illustrations of the application of such trees, we discuss the Dirac and Anderson spin Hamiltonians of nonrelativistic exchange theory in this context and also develop a concise description of the polarization dependence of Møller scattering in QED.

I. INTRODUCTION

The quantum theory of angular momentum plays an important role in the analysis of the polarization dependence of particle reactions in many fields of physics. When the associated interactions are rotationally invariant (e.g., $e\vec{A}\cdot\vec{p}/m$, $iep_\mu A^\mu$) the Wigner-Eckart theorem for SO(3) may be applied to their matrix elements. The geometrical dependence of the total cross section then reduces to an angular-momentum-coupling tree, i.e., a product of $3j$ (or—as they are sometimes called[1]—$3jm$) symbols with internal suffices summed in pairs, whose external suffices (corresponding to terminal legs of the corresponding angular-momentum-coupling diagram[2]) are contracted, e.g., with polarization vectors and/or spinors for the participating particles.

For a given number of particles (N, for example) several trees may be obtained, each with different topology and/or different angular momentum (j) labels for the intermediate states (internal lines). The linearly independent trees, contracted with polarization vectors, etc., enumerate all possible dependencies of the cross section of geometrical properties, such as choice of polarization.

In the case of spin-1 particles such as photons interacting with each other or with unpolarized matter (including rotationally invariant scatterers such as fluids), the geometrical dependence is specified entirely by the polarization vectors in the nonrelativistic $E1$ coupling limit and (because of gauge invariance) even in the relativistic limit for simple processes such as Thomson or Compton scattering.[3] The angular-momentum-coupling trees corresponding to such processes then have $2N$ external spin-1 legs which are contracted against each photon polarization vector or its conjugate. In a more general problem, contractions also involve photon wave vectors.

With the use of integrity-basis theory, Minard *et al.*[4] proved that such trees may be reduced to the sums of products of Kronecker delta symbols in the polarization vector component labels and gave explicit reductions for $N\leq 4$. The results have proved their usefulness in a wide variety of topics. The possible polarization dependencies of a variety of nonlinear or field-dependent optical processes have been analyzed, using the results of Ref. 4, for circular dichroism and optical activity,[5] natural Raman scattering,[6] field-induced Raman scattering,[7] and hyper-Raman scattering.[8] Similar techniques indicated an error in the "standard" cross section for Compton scattering and indeed its correct form.[3]

The integrity-basis formalism defines the number and type of a minimal set of (rotational) invariants, out of which all possible invariants must be constructed. In the analysis of Minard *et al.*[4] the elements used in the construction are components of a given set of three-vectors; not surprisingly, the types of invariants are found to correspond to scalar and triple-scalar products. In indicating the minimum set, the integrity-basis approach reveals the existence of inter-relationships between invariants, e.g., relations at fifth and eighth order in vector components between certain combinations of scalar products of vectors.[4]

In Sec. II we develop a parallel formalism for the number and type of spinor invariants, i.e., for invariants built from the components of two-spinors. In agreement with the pioneering work of Kramers (see Brinkman[9]) we find all spinor invariants to be bilinears in spinor components, corresponding to scalar and alternating spinor products. We also find inter-relationships between such products, commencing at fourth order. This proves that angular-momentum-coupling trees with $j=\frac{1}{2}$ external labels may be reduced to Kronecker delta functions and $2jm$ symbols, as was previously proved for $j=1$.[4] The corresponding reductions and inter-relationships are given for lowest-order trees in Sec. III.

The results are relevant to a wide variety of problems, as indeed are those for spin-1 trees. In the nonrelativistic limit, they are relevant to exchange theory. The two terms in the classical Dirac-Heisenberg-Serber—Van

Vleck exchange operator[10] $\mathscr{P}^{\sigma}_{12}=\frac{1}{2}+2\vec{s}_1\cdot\vec{s}_2$ correspond to the two angular-momentum-coupling trees obtainable from the permutation operator. In relativistic quantum mechanics the polarization dependence of such processes as Møller, Bhahba, or Compton scattering involving polarized fermions may be expressed in terms of matrix elements between two-spinors, using spin-$\frac{1}{2}$ tree manipulations as an alternative to Pauli matrix manipulations for intermediate steps in the calculation. Such methods of computing the polarization dependence of such processes can thus be made much more easily than earlier calculations suggest and form an alternative to the standard trace techniques of quantum electrodynamics. These two examples (exchange and Møller scattering) are discussed in more detail in Sec. IV.

II. INTEGRITY-BASIS APPROACH TO SPINOR INVARIANTS

Integrity-basis theory[11] may be used to determine the degrees of polynomial functions in all spinor components that may occur in independent invariants. The $j=1$ results of Minard *et al.*[4] (see also Bader and Rodriguez[12]) are here extended to $j=\frac{1}{2}$.

Consider first one Pauli spinor $\psi=u\xi+v\eta$ ($\xi\equiv|\uparrow\rangle$, $\eta\equiv|\downarrow\rangle$) and ask what invariants may be constructed from powers of u and v. The nth-order polynomial $P^n_j=u^jv^{n-j}$ ($j=0,1,\ldots,n$) has a character

$$\chi_{nj}(\theta)=\exp[\tfrac{1}{2}i\theta(2j-n)]$$

under a rotation θ about the z axis [since $u\to\exp(\frac{1}{2}i\theta)u$, $v\to\exp(-\frac{1}{2}i\theta)v$].[13] Hence the character of the set of polynomials $\{P^n_j\,|\,j=0,1,\ldots,n\}$, is

$$\chi_n(\theta)=\sum_{j=0}^{n}\chi_{nj}(\theta)=\sin[(n+\tfrac{1}{2})\theta]/\sin(\tfrac{1}{2}\theta)\ .$$

Parenthetically, we mention that we recognize this result as the character $\chi^{(n)}(\theta)$ of the irreducible representation $j=n$ of SO(3) under rotation by θ. For example, at $n=1$, $[\frac{1}{2}\otimes\frac{1}{2}]_+=1$, not 0 (0 appears only in the antisymmetric product). More generally, in plethysm language[14] $\frac{1}{2}\otimes\{n\}=\frac{1}{2}n$. In the standard spherical basis, $P^{2n}_j/\sqrt{(2n-j)!j!}\sim\psi^n_{n-j}$.[13] Each of these three alternative statements shows that *no* invariant may be formed from u,v.

The integrity-basis method confirms this result as follows. We define a character generating function:

$$F_\theta(a)=\sum_{n=0}^{\infty}\chi_n(\theta)a^n$$
$$=1/[1+a^2-2a\cos(\tfrac{1}{2}\theta)]\ .$$

The powers of a (and thus the degree of any polynomial corresponding to an independent invariant) are given through the character orthogonality theorem by the a dependence of the function [$\langle\cdots\rangle$ denotes the SO(3) average]

$$\chi^{(0)}(a)=\langle\chi^{(0)}(\theta)F_\theta(a)\rangle$$
$$=\frac{1}{\pi}\int_0^{2\pi}\sin^2(\tfrac{1}{2}\theta)F_\theta(a)d\theta\ . \tag{1}$$

The integration may be performed by using the orthogonality relations between Chebyshev polynomials of type 1 (Ref. 15) or alternatively, as in Minard *et al.*,[4] by converting to a contour integral in the Argand plane and using Cauchy's residue theorem. The choice $\lambda=\exp\frac{1}{2}i\theta$ for the complex variable is necessary to render the integrand meromorphic for the spinor case; however, the limits of integration must then be extended to $(-2\pi,2\pi)$ in order to close the contour, using the fact that the integrand is an even function of θ. We have (C being the unit circle)

$$\chi^{(0)}(a)=\frac{-1}{4\pi i}\oint_C\frac{(\lambda^2-1)^2d\lambda}{\lambda^2(1-a\lambda)(\lambda-a)}$$
$$=1\ . \tag{2}$$

Since no powers of a occur, no invariants are possible, as before.

Still considering one spinor only, let us now consider the possibility of forming invariants from powers of u,v and their complex conjugate u^*,v^*, with equal numbers of u and v combined as of their conjugates. This leads us to consider the set of polynomials

$$\{Q^{2n}_{j_1j_2}\}=\{P^n_{j_1}(P^n_{j_2})^*\,|\,j_1,j_2=0,1,\ldots,n\}\ .$$

This set will have a character

$$\Xi_{2n}(\theta)=[\chi_n(\theta)]^2 \tag{3}$$

with an associated generating function

$$G_\theta(A)=\sum_{n=0}^{\infty}\Xi_{2n}(\theta)A^n$$
$$=(1+A)/[(1-A)(1+A^2-2A\cos\theta)]\ .$$

Replacing $F_\theta(a)$ by $G_\theta(A)$ in Eq. (1) gives

$$\chi^{(0)}(A)=1/(1-A)\ . \tag{4}$$

The single power of A in the denominator of this expression indicates that all possible invariants are functions of a single invariant occurring at $n=1$.[4] This invariant is obviously the spinor normalization factor:

$$\int\psi^*\psi\equiv u^*u+v^*v\ . \tag{5}$$

Indeed, Eq. (3) corresponds to the Kronecker square of the irreducible representation $j=n$, $n\otimes n=0+1+2+\cdots+2n$; this also indicates that an invariant occurs at each n [each power of A occurs once in the expansion of Eq. (4)]. However, the result that only one is independent requires the integrity-basis analysis.

Since u^*,v^* transform contragrediently to u,v they transform as $v,-u$.[13] It follows that the alternating product of two spinors[9]

$$(\psi_1\wedge\psi_2)\equiv u_1v_2-u_2v_1 \tag{6}$$

is an invariant. In terms of angular momentum coupling this corresponds to $[\psi_1\psi_2]_0^0$ or $[\frac{1}{2}\otimes\frac{1}{2}]_-=0$. The spinors must differ for the antisymmetric part not to vanish.

Integrity-basis theory may be used to establish Eqs. (5) and (6) as the only possible independent invariant functions of two-spinor components. The product set of polynomials

$$\{P_{j_1}^{n_1}(u_1,v_1)P_{j-j_1}^{n-n_1}(u_2,v_2)\,|\,j_1=0,1,\ldots,n_1\,;\; j_2=0,1,\ldots,n-n_1;\; n_1=0,1,\ldots,n\}$$

has characters given by the product generating function[4]

$$F_\theta(a,b)=F_\theta(a)F_\theta(b) \tag{7}$$

which gives from Eq. (1) an integrity-basis character function

$$\chi^{(0)}(a,b)=1/(1-ab)\ . \tag{8}$$

Again the denominator indicates just one independent invariant, bilinear in ψ_1 and ψ_2 components, i.e., the alternating product. All higher-order polynomials in u_1,v_1,u_2,v_2 which are invariant may be reduced to polynomials in the alternating product.

For three spinors we obtain similarly

$$\chi^{(0)}(a,b,c)=1/[(1-ab)(1-bc)(1-ca)]\ .$$

The three alternating products $(\psi_1\wedge\psi_2),(\psi_2\wedge\psi_3),(\psi_3\wedge\psi_1)$ thus define all possible invariants in this case.

For four spinors we obtain [with a little help from REDUCE, the computer algebra program (Anthony C. Hearn, The Rand Corporation, Santa Monica, CA 90406)]

$$\chi^{(0)}(a,b,c,d)=(1-abcd)/[(1-ab)(1-bc)(1-cd)\times(1-ac)(1-bd)(1-ad)]\ .$$

Again the denominators indicate that the six possible alternating products among four spinors define all possible invariants. The subtractive term in the numerator indicates that not even these are independent[4] and that the relation linking them is quadrilinear in the components of the four spinors. This is readily seen to be the relation

$$(\psi_1\wedge\psi_2)(\psi_3\wedge\psi_4)+(\psi_1\wedge\psi_3)(\psi_4\wedge\psi_2)+(\psi_2\wedge\psi_3)(\psi_1\wedge\psi_4)=0\ . \tag{9}$$

Now, including the complex conjugate components, consider two spinors. As before, the generating function is

$$G_\theta(A,B)=G_\theta(A)G_\theta(B)\ ,$$

$$\chi^{(0)}(A,B)=1/[(1-A)(1-B)(1-AB)]\ .$$

The term A in one denominator corresponds to an invariant bilinear in the first spinor, i.e., $\int\psi_1^*\psi_1$, and the B term similarly corresponds to the other spinor. Invariants such as $\int\psi_1^*\psi_2$ are excluded by the constraint [Eq. (7)] on the form of the polynomials that equal powers of ψ and ψ^* be present. (Relaxing this constraint gives a much more tedious algebraic problem.) As a result, the denominator term AB appears apparently as an independent invariant; in fact this corresponds simply to $|\int\psi_1^*\psi_2|^2$. In a similar if more complicated manner, all the terms in the expansion of the character function for three spinors

$$\chi^{(0)}(A,B,C)=\frac{1-A-B-C+(1-ABC)(AB+BC+CA+ABC)}{(1-A)^2(1-B)^2(1-C)^2(1-AB)(1-BC)(1-CA)}$$

may be identified as parametrizing such invariants.

All these results indicate that the only possible forms for independent spinor invariants are the alternating product $(\psi_1\wedge\psi_2)$ and the scalar product $\int\psi_1^*\psi_2\equiv u_1^*u_2+v_1^*v_2$; these two forms may be related by defining the conjugate spinor

$$\bar\psi=v^*\xi-u^*\eta\ , \tag{10}$$

since then $\psi_1\wedge\bar\psi_2=-\int\psi_2^*\psi_1$.

A general proof that all rotational invariants formed from spinors must have these forms is given by Brinkman,[9] who used Kramers's method of spinor invariants. The method of integrity bases has shown, however, that these invariants are not independent. Relations of the form of Eq. (9) exist and define all dependent invariants up to the order of complexity of the above examples.

Let us compare this conclusion with that for vectors, where the possible invariants include triple-scalar products as well as ordinary scalar products. Higher-order combinations of each (or both) of these invariants may not be independent.[4] We now use the spinor-vector correspondence[9,13] to relate all these invariants and their inter-relationships to spinor invariants and their inter-relationships.

Any spinor $\psi=(u,v)$ defines a vector

$$\vec{N}(\psi)=\left\{\frac{u^{1+m}v^{1-m}}{[(1+m)!(1-m)!]^{1/2}}\,\middle|\,m=1,\ 0,\ -1\right\}=(u^2/\sqrt{2},\ uv,\ v^2/\sqrt{2})\ .$$

This is a null vector: $\vec{N}\cdot\vec{N}\equiv\sum_m N_m(-1)^{1-m}N_{-m}=0$.

A more general vector requires two spinors for its definition:

$$\vec{V}(\psi_1,\psi_2)=(u_1u_2/\sqrt{2},\ \tfrac{1}{2}(u_1v_2+v_1u_2),\ v_1v_2/\sqrt{2})\ .$$

If $\vec{V}$ is real [i.e., $V_m=(-1)^{1-m}V_{-m}^*$] we may choose $\psi_2=\bar\psi_1$.

The vector invariants, including the triple-scalar product, may be reduced to spinor invariants. For example,

$$\vec{V}(\psi_1,\psi_2)\cdot\vec{V}'(\psi_1',\psi_2')=\tfrac{1}{6}[2(\psi_1'\wedge\psi_1)(\psi_2'\wedge\psi_2)-(\psi_1\wedge\psi_2)(\psi_1'\wedge\psi_2')]\ , \tag{11}$$

$$\vec{U}(\phi,\phi)\cdot\vec{V}(\chi,\chi)\times\vec{W}(\psi,\psi)=(\phi\wedge\chi)(\chi\wedge\psi)(\psi\wedge\phi)\ .$$

The higher-order relations among these invariants then follow straightforwardly from Eq. (9).

III. ANGULAR-MOMENTUM-COUPLING TREES

A. Summary

As in previous work[2–8,16] we write a $3j$ (or $3jm$) symbol in the form of Fig. 1(a), i.e., three lines meeting at a

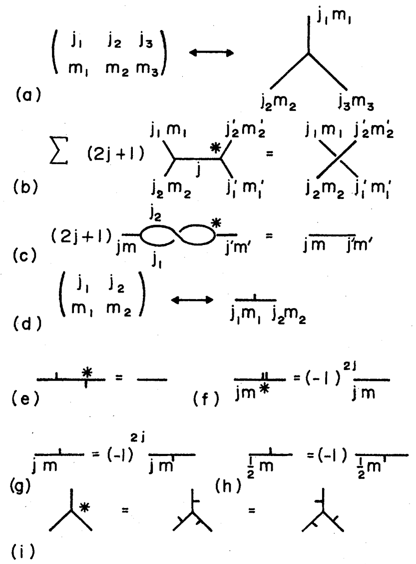

FIG. 1. Basic components of the diagram representation. (a) Diagram representation of a $3jm$ symbol. (b) and (c) Unitarity relations for $3jm$ symbols. (d) $2jm$ symbol. (e) Unitarity relation for a $2jm$ symbol. (f) $2j$ phase. (g) and (h) Role of the $2j$ phase in reversing stubs. (i) Derome-Sharp lemma, relating complex conjugation to a $2jm$ transformation.

node. The lines are labeled by j and m labels; the rotational invariance of the diagram reflects the cyclic permutation symmetry of the $3jm$ symbol. The unitarity relations

$$\sum_{j,m}(2j+1)\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m\end{bmatrix}^* \begin{bmatrix} j_1' & j_2' & j \\ m_1' & m_2' & m\end{bmatrix} = \delta_{j_1j_1'}\delta_{j_2j_2'}\delta_{m_1m_1'}\delta_{m_2m_2'}\,, \tag{12}$$

$$\sum_{m_1,m_2}(2j+1)\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m\end{bmatrix}^* \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m'\end{bmatrix} = \delta_{jj'}\delta_{mm'}$$

take the diagram forms of Figs. 1(b) and 1(c), respectively. Omitted labels in any diagram are automatically summed.

The special case $j_3=m_3=0$ defines a $2jm$ symbol

$$\begin{bmatrix} j_1 & j_2 \\ m_1 & m_2\end{bmatrix} \equiv \sqrt{2j_1+1}\begin{bmatrix} j_1 & j_2 & 0 \\ m_1 & m_2 & 0\end{bmatrix} \tag{13}$$

with the diagram form of Fig. 1(d). From Eq. (12) we have the $2jm$ unitarity condition of Fig. 1(e). [For simplicity we omit external labels on all figures from Fig. 1(e) onwards where possible; corresponding external legs on each side of an equation are understood to have identical labels.] Since integer and half-integer representations of SO(3) are orthogonal and symplectic, respectively,[1] we have the result of Fig. 1(f) and thus [using Fig. 1(e)] those of Figs. 1(g) and 1(h). Complex conjugation amounts to a $2jm$ transformation,[1,17] as in Fig. 1(i). We use a spherical [$\mathrm{SO}(3)\supset\mathrm{SO}(2)$] basis for $j=\frac{1}{2}$, in which

$$\begin{bmatrix} j_1 & j_2 \\ m_1 & m_2\end{bmatrix} = (-1)^{j_1-m_1}\delta_{j_1j_2}\delta_{m_1,-m_2}\,,$$

and a Cartesian basis [$\mathrm{SO}(3)\supset D_4$] for $j=1$, in which

$$\begin{bmatrix} 1 & 1 \\ \alpha & \beta\end{bmatrix} = \delta_{\alpha\beta}\,,$$
$$\begin{bmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma\end{bmatrix} = \frac{1}{\sqrt{6}}\epsilon_{\alpha\beta\gamma}\,. \tag{14}$$

All $2jm$ symbols are real. Open and filled bars denote symmetrizers and antisymmetrizers, respectively, as in Fig. 2.[18]

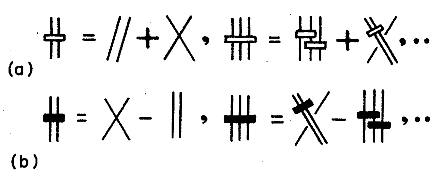

FIG. 2 (a) and (b) Symmetrizers and antisymmetrizers, respectively.

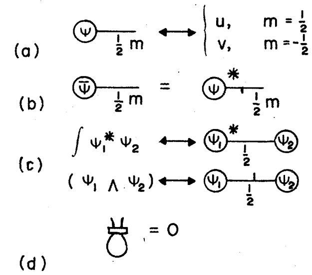

FIG. 3. (a) Spinor components. (b) Spinor conjugates. (c) Scalar and alternating spinor products. (d) An elementary relation.

B. Spinor components

Components of spinors are given the diagram representation of Fig. 3(a). The conjugate of Eq. (10) is given [us-

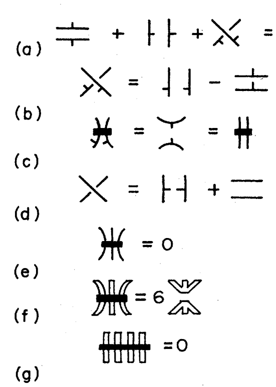

FIG. 4. (a)–(d) Four different forms of the interrelationship between alternating products of spinors given in Eq. (9). (e) Vanishing antisymmetrizer for $j=\frac{1}{2}$. (f) $j=1$ analog of (c). (g) $j=1$ analog of (e).

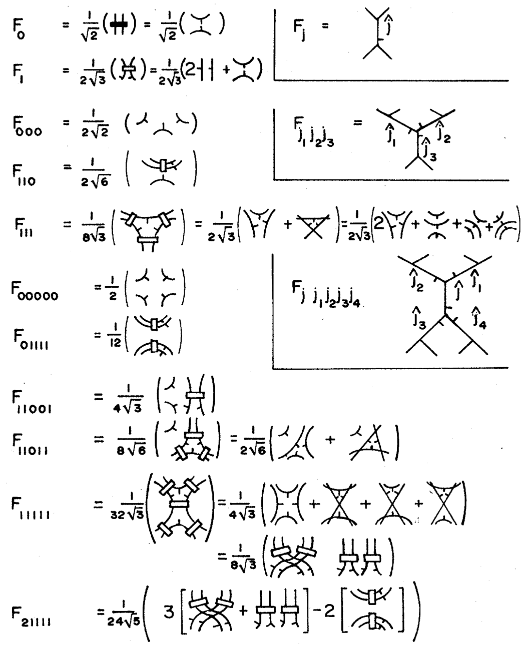

FIG. 5. Reduction of lowest-order spin-$\frac{1}{2}$ angular-momentum-coupling trees; $\hat{j} \equiv (2j+1)^{1/2}$.

ing Fig. 1(d)] by Fig. 3(b). The scalar and alternating products [Eqs. (5) and (6)] thus have the form of Fig. 3(c). Note that in the definition of the alternating product the antisymmetry is associated with that of the $2jm$ symbol [Fig. 1(h)]. Corresponding to this antisymmetry, we have Fig. 3(d).

The quadratic relation [Eq. (9)] between alternating products can be given various diagram forms, e.g., Figs. 4(a)—4(d); these are inter-related by the use of Fig. 1(e), 1(h), etc. Combining Figs. 2(a) and 4(c) we have Fig. 4(e). This is a special case of a result given, for example, by Penrose:[18] An antisymmetrizer over $2(j+1)$ labels of spin j vanishes. For $j=1$, the analog of ($j=\frac{1}{2}$) Fig. 4(e) is Fig. 4(f) (we use double lines to denote $j=1$ labels), while the analog of Fig. 4(c) is Fig. 4(g).

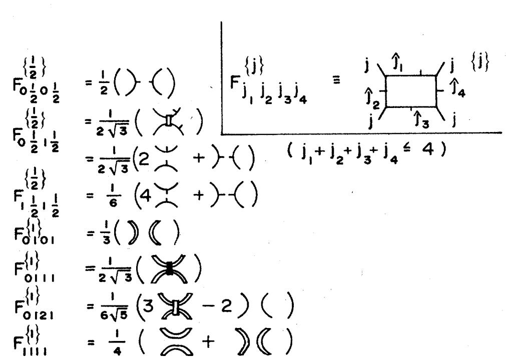

FIG. 6. Reduction of some lowest-order box diagrams with $j=\frac{1}{2}$ or $j=1$ labels. The latter are denoted by a double line.

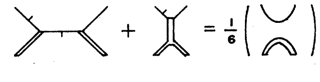

FIG. 7. A relation between trees with mixed ($j=\frac{1}{2}$ and 1) labels.

C. Coupling trees

We find explicit reductions of angular-momentum-coupling trees with $j=\frac{1}{2}$ labels, as with the $j=1$ trees,[4] by iterating the results of Sec. III B. The results are given in Fig. 5. Ring or box diagrams are considered in Fig. 6 for both $j=\frac{1}{2}$ and $j=1$ labels (the latter not having been published before). One may also consider mixed trees, i.e., trees with both $j=\frac{1}{2}$ and $j=1$ labels. For example, the relation of Fig. 7 is equivalent [from Fig. 8(b) and Eq. (14)] to the relation

$$\sigma^k \sigma^l = i\epsilon^{klm}\sigma^m + \delta^{kl}\mathbb{1} \qquad (15)$$

between Pauli matrices. Similarly, the reduction of the tree F_1 in Fig. 5 is equivalent to the relation

$$\sigma^k_{mm'}\sigma^k_{nn'} = 2\delta_{mn'}\delta_{m'n} - \delta_{mm'}\delta_{nn'} \, . \qquad (16)$$

IV. EXAMPLES OF APPLICATIONS

A. Exchange theory: the Dirac-Heisenberg-Serber—Van Vleck coupling

As shown by Dirac,[19] for example, the permutation operator $\mathscr{P}^{\sigma}_{12}$ for the labels of two spin-$\frac{1}{2}$ particles may be expanded in terms of spin operators $\vec{s}_1, \vec{s}_2$ acting on these particles:

$$\mathscr{P}^{\sigma}_{12} = \tfrac{1}{2} + 2\vec{s}_1 \cdot \vec{s}_2 \, . \qquad (17)$$

This expansion is fundamental to the formulation of exchange and superexchange interactions in multielectronic systems.[10]

Dirac[19] has already discussed the rotational invariance of the operator $\mathscr{P}^{\sigma}_{12}$. This invariance underlies the following expansion of this operator in coupling trees.

Consider the matrix elements of $\mathscr{P}^{\sigma}_{12}$ between states $|j_1 m_1 j_2 m_2\rangle$ ($j_1 = j_2 = \frac{1}{2}$):

$$\langle j_1 m_1 j_2 m_2 | \mathscr{P}^{\sigma}_{12} | j_1' m_1' j_2' m_2' \rangle = \delta_{m_1 m_2'} \delta_{m_2 m_1'} (\delta_{j_1 j_2'} \delta_{j_2 j_1'}) \, .$$

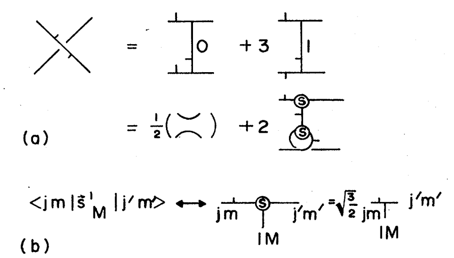

FIG. 8. (a) Expansion of a permutation operator in terms of coupling trees and its simplification. (b) Relation between a matrix element of a spin operator and a $3jm$ vertex, used in (a).

The right side may be expanded using Eq. (12), as in Fig. 8(a); matrix elements of the spin operator may be reduced via the Wigner-Eckart theorem to a multiple of a Clebsch-Gordan symbol, as in Fig. 8(b). The two terms in Eq. (17) then correspond precisely to the two diagrams on the right side of Fig. 8(a).

The relation of Eq. (9) results in a variety of forms for F_1 (Fig. 5) and so for $\mathscr{P}^{\sigma}_{12}$, as in the different results of Figs. 4(a)—4(d). In particular, the left side of Fig. 4(d) can be identified as $\mathscr{P}^{\sigma}_{12}$ and the last term on the right side as the unit operator, so that the remaining diagram in Fig. 4(d) can be identified with the operator

$$\begin{aligned}\mathscr{D}_{12}&=\mathscr{P}^{\sigma}_{12}-1\\&=2\vec{s}_1\cdot\vec{s}_2-\tfrac{1}{2}\ .\end{aligned}\tag{18}$$

This corresponds to the Anderson spin operator, i.e., to the shift of singlet rather than triplet pair levels.[20] The simple diagram form of Eq. (18) given in the first term on the right side of Fig. 4(d) shows that the matrix elements of the Anderson operator may be factorized into two terms, each depending on just one of the two relevant particle states (bra or ket):

$$\begin{aligned}&\langle \tfrac{1}{2}m_1,\tfrac{1}{2}m_2\,|\,(2\vec{s}_1\cdot\vec{s}_2-\tfrac{1}{2})\,|\,\tfrac{1}{2}m_1',\tfrac{1}{2}m_2'\rangle\\&\qquad=(-1)^{m_2'-m_1}\delta_{m_1,-m_2}\delta_{m_1',-m_2'}\ .\end{aligned}\tag{19}$$

Factorizability can have important consequences in exchange theory;[21] one such property of the Anderson operator has already been noted and used.[22] However, the factorization of Eq. (19) seems not to have been noted before and could have a similar practical interest.

In multielectronic atoms it is of course a standard technique to use angular-momentum-coupling diagrams for an analysis of parentage, spin-orbit coupling effects, etc. The forms of exchange Hamiltonians under such recoupling and projection onto a ground manifold are of considerable practical interest.[22] Angular momentum tree reductions such as those given in Sec. III may be expected to assist in the formulation of these Hamiltonians.

B. Relativistic particle scattering

The standard technique for computing cross sections in quantum electrodynamics is that of tracing products of γ matrices and polarization projection operators. Variations of this technique are possible, although not well explored. For example, it is possible to calculate the *amplitudes* in terms of momentum and polarization vectors by trace techniques, which may subsequently be avoided in computing the cross section;[23] this is particularly useful when computing general polarization dependences. Traces of products of γ matrices in fact correspond to angular-momentum-coupling diagrams.[3,24] This suggests that increased use of relatively humble angular-momentum-coupling techniques could simplify the evaluation of cross sections in quantum electrodynamics.

However, the situation is not as straightforward as for the scattering of massless vector particles. The requirements of gauge invariance cannot be invoked (as in Stedman and Pooke[3]) to simplify the dependence of the cross section of the relevant spin and momentum three-vectors. Also, it is standard practice to eliminate spinors from the answer by introducing the polarization (four- or three-) vectors of the fermions. In fact it is often feasible to write a more compact expression for the amplitude and also for the cross section using two-spinors to parametrize the polarization states; we shall show this as an illustration of a relativistic application of spin-$\frac{1}{2}$ trees below. This may have some intrinsic interest in view of the recent interest in the analysis of relativistic effects in Compton scattering from bound polarized electrons[25] and in view of recent discoveries[26] of the value of two-spinors and Pauli matrices in some elegant solutions to problems in relativistic quantum mechanics; we note the occurrence of Pauli matrices as fundamental components of the angular-momentum-coupling diagrams in the relativistic wave-equation theory of Biritz.[27]

We consider then the case of Møller scattering. The invariant amplitude corresponding to the Feynman diagrams of Fig. 9(a) is

$$\frac{-e^2}{t}\bar{U}_3\gamma^{\mu}U_1\bar{U}_4\gamma_{\mu}U_2+\frac{e^2}{s}\bar{U}_4\gamma^{\nu}U_1\bar{U}_3\gamma_{\nu}U_2\ ,\tag{20}$$

where $p_i=(E_i,\vec{p}_i)$ is the four-momentum of particle i, $s=(p_1-p_4)^2$, $t=(p_1-p_3)^2$, and U_i is the (Dirac, four-) spinor for particle i. We write this in terms of two-spinors ϕ and Pauli matrices $\vec{\sigma}$, using[28]

$$U_i=\left[\frac{F_i}{2m}\right]^{1/2}\begin{bmatrix}\phi_1\\ \dfrac{\vec{\sigma}\cdot\vec{p}_i}{F_i}\phi_i\end{bmatrix}\ ,\tag{21}$$

where $F_i=E_i+m$, and in the Weyl representation

$$\begin{aligned}&\sigma^{\mu}=(\mathbb{1},\vec{\sigma}),\quad \tilde{\sigma}^{\mu}=(\mathbb{1},-\vec{\sigma})\ ,\\&\gamma^{\mu}=\begin{bmatrix}0&\tilde{\sigma}^{\mu}\\ \sigma^{\mu}&0\end{bmatrix}\ .\end{aligned}\tag{22}$$

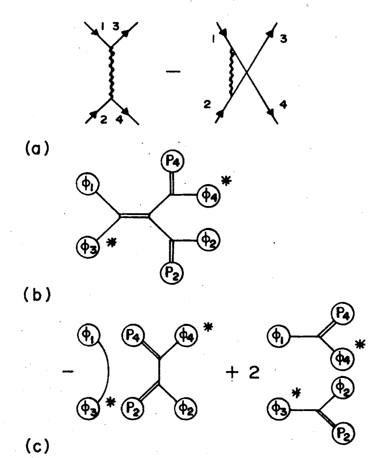

FIG. 9. (a) Feynman diagrams for Møller scattering. (b) Angular-momentum-coupling structure of the second term in Eq. (23). (c) Reduction of (b) using angular-momentum-coupling-tree analysis.

Substituting Eqs. (21) and (22) into Eq. (20) gives a series of terms, each of which may be simplified using our earlier results. As an illustration of this, one combination of terms occurring in this expansion is

$$X=(\phi_3^\dagger\phi_1)(\phi_4^\dagger\vec{\sigma}\cdot\vec{p}_4\vec{\sigma}\cdot\vec{p}_2\phi_2)+(\phi_3^\dagger\sigma^k\phi_1)(\phi_4^\dagger\vec{\sigma}\cdot\vec{p}_4\sigma^k\vec{\sigma}\cdot\vec{p}_2\phi_2)\ . \qquad (23)$$

This can be reduced to the form

$$2(\phi_3^\dagger\vec{\sigma}\cdot\vec{p}_2\phi_2)(\phi_4^\dagger\vec{\sigma}\cdot\vec{p}_4\phi_1)\ .$$

This is because the second term in Eq. (23) has the structure of Fig. 9(b), which reduces, using F_1 (Fig. 5), to the form of Fig. 9(c), the first term in which cancels the first term in Eq. (23). These manipulations are equivalent to employing algebraic identities on the Pauli matrices [Eqs. (15) and (16)]; since the whole problem is based on a (Feynman) diagram technique, it is helpful to see the role of (angular momentum) diagram manipulations in simplifying the answer in this way. In proceeding on these lines we find that the amplitude for Møller scattering for particles of general polarization may be written in the form

$$\frac{e^2}{2m^2}\left[\frac{1}{s}+\frac{1}{t}\right]\{(F_1F_2F_3F_4)^{1/2}[(41)(32)-(31)(42)]+(F_1F_2F_3F_4)^{-1/2}(P_{41}P_{32}-P_{42}P_{31})\}$$
$$+\frac{e^2}{2m^2}(F_1F_2F_3F_4)^{1/2}\left[-\frac{1}{t}\left[\frac{1}{F_2F_4}p_{23}^\dagger p_{41}+\frac{1}{F_3F_1}p_{32}p_{14}^\dagger\right]+\frac{1}{s}\left[\frac{1}{F_3F_2}p_{24}^\dagger p_{13}+\frac{1}{F_4F_1}p_{42}^\dagger p_{13}\right]\right]\ , \qquad (24)$$

where $(ij)\equiv\int\phi_i^*\phi_j$, $p_{ij}\equiv\int\phi_i^*\vec{\sigma}\cdot\vec{p}_i\phi_j$,

$$P_{ij}\equiv\int\phi_i^*(\vec{\sigma}\cdot\vec{p}_i)(\vec{\sigma}\cdot\vec{p}_j)\phi_j=(ij)(\vec{p}_i\cdot\vec{p}_j)-\int\phi_i^*(\vec{\sigma}\cdot\vec{p}_i\times\vec{p}_j)\phi_j\ .$$

This expression for the polarization dependence of Møller scattering is novel and is much more compact than the expression given by Sarkar[29] for the final cross section (in terms of scalar products in all relevant four-vectors).

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