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Dielectric reciprocity theorem analogous to the Betti-Maxwell theorem

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A reciprocity theorem is stated which applies to a dielectric medium at rest polarized by one or the other of two systems of charges. Its formulation is similar to the Betti-Maxwell reciprocity theorem for elastic media.

I. INTRODUCTION AND FORMULATION

For elastic media, a simple reciprocity theorem exists, known as the Betti-Maxwell theorem.[1–5] This theorem was applied to a particular case by Maxwell,[1] and was then formulated in a general way by Betti:[2] "Consider two equilibrium states of the same elastic body. The work of the external forces of the first state in the field of displacements of the second one, equals the work of the external forces of the second state in the field of displacements of the first one." So, this statement applies to the deformation of an elastic body alternately subjected to the stationary mechanical action of two other bodies X and Y.

It will be shown now that a similar statement exists for a dielectric medium which is alternately subjected to the stationary polarizing fields of two systems of charges. This statement is based on the hypothesis [hereafter referred to as (C)] of a linear dependence between the polarization and the electric field, and it may be formulated in the following way.

A dielectric medium N is considered, with arbitrary fixed shape, and maintained at a uniform temperature T. N need not be considered isotropic or homogeneous. Then for any charge systems X and Y, the interaction energy of X with the dielectric N polarized by Y alone, is equal to the interaction energy of Y with the dielectric polarized by X alone.

Denoting the interaction energy of two bodies A and B by $V[A,B]$, and adopting the notation $N(A)$ to represent the dielectric N in its stationary polarization state induced by A alone, the reciprocity relation is written

$$V[X,N(Y)]=V[Y,N(X)]\ . \tag{1}$$

II. DERIVATION OF THE RECIPROCITY RELATION (1)

(i) Let us consider first the simplest case of a thin dielectric slab, with perpendicular uniform polarization $\vec{P}_X$ or $\vec{P}_Y$, produced by a uniform external field $\vec{E}_X$ or $\vec{E}_Y$. Let $\vec{E}=\vec{E}_X+\vec{E}_N$ be Maxwell's total electric field, sum of the fields created by X, and the dielectric. In this case, the electric displacement $\vec{D}=\epsilon\vec{E}=\epsilon(\vec{E}_X+\vec{E}_{N(X)})$ divided by ϵ_0 coincides with the external field $\vec{E}_X$, so that we have a linear relation $\vec{P}_X=(\epsilon_0-\epsilon_0^2/\epsilon)\vec{E}_X$ [or similarly, $\vec{P}_Y=(\epsilon_0-\epsilon_0^2/\epsilon)\vec{E}_Y$] between the polarization vector and the external field. Relation (1) is easily obtained in this case, since

$$V[X,N(Y)]=-\int_N \vec{E}_X\cdot\vec{P}_Y\,dv$$
$$=-\int_N (\epsilon_0-\epsilon_0^2/\epsilon)\vec{E}_X\cdot\vec{E}_Y\,dv \tag{2}$$

is symmetrical with respect to X and Y.

In the general case considered in the Introduction, where the dielectric N has an arbitrary shape, the relation $\vec{P}_X=(\epsilon_0-\epsilon_0^2/\epsilon)\vec{E}_X$ is no longer valid, and no other general relation links the local values of $\vec{P}_X$ and $\vec{E}_X$. Nevertheless, relation (1) may be established in the following manner.

(ii) It is known[6,7] that the free energy of a dielectric N having an arbitrary shape, and any stationary polarization $N(A)$, induced by A, is

$$F_{N(A)}=F_{N(0)}-\tfrac{1}{2}V[A,N(A)]\ , \tag{3}$$

where $F_{N(0)}$ is the free energy of N nonpolarized. Though relation (3) remains valid[6] when N is only part of all the dielectrics present—which means that the body A may contain dielectrics— we will restrict (1) to the case where X and Y are systems of fixed charges.

To establish the reciprocity relation, we do not consider the dielectric N in the presence of one or the other of the charge systems X and Y, but in the simultaneous presence of both X and Y; let us denote by $X\cup Y\cup N$ the composite system consisting of X and Y and the dielectric N. Among the infinite number of fictitious polarization states ξ conceivable for N, the true polarization state $N(X\cup Y)$ induced by X and Y, corresponds to the lowest value of the free energy $F\equiv F_{X\cup Y\cup N}$ of the whole system.

Specifically, we will state that the total free energy $F(\xi_1)$ corresponding to the true polarization state $\xi_1=N(X\cup Y)$ is lower than the free energy $F(\xi_\lambda)$ corresponding to the fictitious polarization state ξ_λ which would result if the charges in X were multiplied by λ. A suitable notation for the polarization state ξ_λ is $N(\lambda X\cup Y)$, where λX represents the charge system in which the charges occupy the same positions as in X, but are multiplied by λ.

Thus, for any choice of λ

$$F(\xi_1)\leq F(\xi_\lambda)\ ,$$

and consequently

$$\left[\frac{\partial}{\partial\lambda}F(\xi_\lambda)\right]\Bigg|_{\lambda=1}=0\ . \qquad (4)$$

In (4), the total free energy $F\equiv F_{X\cup Y\cup N}$ may be broken down into the sum of the internal free energies F_X,F_Y,F_N of the three bodies, and the mutual interaction free energies, giving

$$F(\xi_\lambda)=F_{X\cup Y}+F[X\cup Y,N]+F_N\ ,$$
$$=F_X+F[X,Y]+F_Y+F[X\cup Y,N]+F_N\ ,$$

or more precisely

$$F(\xi_\lambda)=F_X+F_Y+F_{N(\lambda X\cup Y)}+F[X,Y]+F[X\cup Y,N(\lambda X\cup Y)] \qquad (5)$$

since N is considered in the polarization state $N(\lambda X\cup Y)$.

Because X and Y are fixed systems of charges, there is no correlation entropy between them ($S_{X,Y}=0$), and they have no correlation entropy with the dielectric ($S_{X\cup Y,N}=0$). Thus, the interaction free energies $F=V-TS$ in (5) reduce to the interaction energies V

$$F(\xi_\lambda)=F_X+F_Y+F_{N(\lambda X\cup Y)}+V[X,Y]+V[X\cup Y,N(\lambda X\cup Y)]\ . \qquad (6)$$

$F_{N(\lambda X\cup Y)}$ is the free energy of the dielectric when its polarization is generated by the systems λX and Y. Therefore, its value is given by Eq. (3), when the charge system A is taken to be $\lambda X\cup Y$ in this relation. Thus

$$F_{N(\lambda X\cup Y)}=F_{N(0)}-\tfrac{1}{2}V[N(\lambda X\cup Y),\lambda X\cup Y]\ . \qquad (7)$$

The interaction energies occurring in $F(\xi_\lambda)$ and $F_{N(\lambda X\cup Y)}$ may be broken down according to the principle (C) of superposition of polarizations:

$$V[X\cup Y,N(\lambda X\cup Y)]=\lambda V[X\cup Y,N(X)]+V[X\cup Y,N(Y)] \qquad (8)$$

and

$$V[N(\lambda X\cup Y),\lambda X\cup Y]=\lambda^2V[N(X),X]+\lambda V[N(X),Y]+\lambda V[N(Y),X]+V[N(Y),Y]\ . \qquad (9)$$

When $F_{N(\lambda X\cup Y)}$ is replaced in (6) by its value resulting from (7) and (9), and the value (8) of $V[X\cup Y,N(\lambda X\cup Y)]$ is used, the following expression of the total free energy $F(\xi_\lambda)$ is obtained:

$$F(\xi_\lambda)=F_X+F_Y+F_{N(0)}+V[X,Y]+\lambda(1-\lambda/2)V[X,N(X)]+\tfrac{1}{2}V[Y,N(Y)]+(1-\lambda/2)V[X,N(Y)]+(\lambda/2)V[Y,N(X)]\ . \qquad (10)$$

Replacing λ by 1 in the derivative

$$\frac{\partial}{\partial\lambda}F(\xi_\lambda)=(1-\lambda)V[X,N(X)]-\tfrac{1}{2}V[X,N(Y)]+\tfrac{1}{2}V[Y,N(X)]$$

calculated from (10), (4) is expressed as

$$-\tfrac{1}{2}V[X,N(Y)]+\tfrac{1}{2}V[Y,N(X)]=0\ ,$$

which is (1).

We notice that, apart from several direct applications of the linear hypothesis (C), this demonstration requires only relation (3) [for the obtaining of which hypothesis (C) has also been used through a Güntelberg's charging process].

III. CONCLUSION

The similarity between (1) and the Betti-Maxwell theorem is not surprising, when it is recalled that the Betti-Maxwell theorem also originates from a linear hypothesis, similar to (C). This hypothesis postulates the linear dependence of the stress tensor $\overleftrightarrow{T}$ and the deformation tensor $\overleftrightarrow{E}$ and is used in the derivation of the Clapeyron's theorem—which could be compared to (2)—through which Betti-Maxwell's is often demonstrated.[4,8]

Property (1) can even be applied to a single polarizing system of charges X, moving slowly: An elementary displacement of X may be considered as the result of the superposition on $X(t)$ of a tiny additional system δX of charges. Application of (1) to the bodies X and δX leads to the relation

$$-\int_N \vec{P}\cdot\delta\vec{E}_X\,dv=-\int_N \delta\vec{P}\cdot\vec{E}_X\,dv\ , \qquad (11)$$

between the polarization[9,10] $\vec{P}$ and the field $\vec{E}_X$ created by X. Formula (11) yields again[11] the known relation[12]

$$\delta(F_{X\cup N}-F_X)=-\int_N \vec{P}\cdot\delta\vec{E}_X\,dv\ , \qquad (12)$$

since (3) implies

$$\delta(F_{X\cup N}-F_X)=\delta(\tfrac{1}{2}V[X,N])\equiv\tfrac{1}{2}\left[-\int_N \vec{P}\cdot\delta\vec{E}_X\,dv-\int_N \delta\vec{P}\cdot\vec{E}_X\,dv\right]\ .$$

Theorem (1) is useful whenever a system $M=X\cup Y\cup\cdots$ of charges, consisting of several parts $X,Y,\ldots$ interacts with a neighboring dielectric N: In the computation of the interaction energy, relation (1) may then be used to simplify the sum

$$V[X,N(Y)]+V[Y,N(X)]$$

occurring in $V[M,N]$, to a single term $2V[X,N(Y)]$. For instance, there already exist applications of (1) to a molecule $M=X\cup Y\cup\cdots$ surrounded by a dielectric solvent N, or to a molecule X surrounded by a solvent N placed in an external field $\vec{E}_Y$.

[1]J. C. Maxwell, Philos. Mag. **27**, 294 (1864).

[2]E. Betti, Nuovo Cimento (Ser.2) **7**, 69 (1872).

[3]Lord Rayleigh, *Scientific Papers* (Cambridge University Press, Cambridge, 1899), Vol. 1, pp. 232 and 305.

[4]S. Timoshenko, *Strength of Materials* (Van Nostrand, Princeton, 1941).

[5]S. Timoshenko and J. N. Goodier, *Theory of Elasticity* (McGraw-Hill, New York, 1951).

[6]B. Blaive and J. Metzger, Physica **119A**, 553 (1983).

[7]In Ref. 6, the dielectric was assumed to be isotropic, but clearly this hypothesis has not been used for the obtaining of the present Eq. (3).

[8]J. Rivaud, *Cours de Mécanique* (Hermann, Paris, 1973).

[9]In a rigorous calculation, the interaction energy $-\int \vec{E}_X \cdot \vec{P}_N dv$ should be complemented by terms representing the interaction between the distributions of multipolar moments of order $n \geq 2$ in the dielectric (Ref. 10), and the derivatives of the external potential ϕ_X.

[10]B. Blaive and J. Metzger, J. Phys. (Paris) **42**, 1533 (1981).

[11]Using hypothesis (C) twice, it appears that $\delta\vec{P}$ in (11) is induced by $\delta\vec{E}_X$ alone, and that $\delta\vec{E}_X$ and $\delta\vec{P}$ are not necessarily small in (11). Thus (11) is equivalent to (1). This suggests another derivation of (1), requiring both (3) and (12) as a starting point.

[12]L. Landau and E. Lifshitz, *Electrodynamique des Milieux Continus* (Mir, Moscow, 1969).